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Prospective Middle School Mathematics Teachers' Covariational Reasoning for Interpreting Dynamic Events During Peer Interactions

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Abstract This study investigated the covariational reasoning abilities of prospective middle school mathematics teachers in a task about dynamic functional events involving two simultaneously changing quantities in an individual process and also in a peer interaction process. The focus was the ways in which prospective teachers' covariational reasoning abilities re-emerge in the peer interaction process in excess of their covariational reasoning. The data sources were taken from the individual written responses of prospective teachers, transcripts of individual comments, and transcripts of conversations in pairs. The data were analyzed for prospective teachers in terms of the cognitive and interactive aspects of individual behavior and also interaction. The findings revealed that prospective teachers at different levels working in pairs benefited from the process in terms of developing an awareness of their own individual and also a pair's understanding of covarying quantities. Furthermore, the prospective teachers had opportunities to develop their knowledge on the connection between variables, rate of change, and slope. The prospective teachers' work in pairs provided salient explanations for their reasoning about the task superior to their individual responses.

Keywords Covariational reasoning · Dynamic functional events · Peer learning

Introduction

Thompson (1994) identified dynamic functional relationships between dependent and independent variables by focusing on rate concept, which involves the change in some

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quantity, the coordination of two quantities, and the simultaneous covariation of two quantities. Hereon, understanding covariation was described as "holding in mind as sustained image of two quantities' values simultaneously" (Saldanha & Thompson, 1998, p. 298) in a developmental structure from the coordination of two quantities to the continuous coordination of the quantities for some duration of time.

The covariant aspect of functions has been emphasized through contextualized dynamic functional situations (i.e. how speed varies with time or how the height of water in a bottle varies with volume) (Carlson, 1998; Confrey & Smith, 1991, 1995). In this way, students can consider how one variable changes by visualizing the changes in the other variable. At this point, graphs of dynamic functional situations play an important role for representing and interpreting simultaneous changes of variables because they provide great convenience while explaining increasing, decreasing, maxima, minima, and inflection points. Thus, constructing and interpreting the graph of a dynamic functional relationship gives learners opportunity to show their reasoning about relative changes of input and output, and the direction of these changes (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Carlson, Oehrtman & Engelke, 2010).

Carlson et al. (2002) developed a comprehensive covariational reasoning framework to examine undergraduate students' reasoning about quantities that covary when they interpret models of dynamic events. In their framework, covariational reasoning is defined as "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Similar to Saldanha and Thompson's view (1998), Carlson et al. (2002) considered that the images of covariation are developmental. They used the term "developmental" in the Piagetian sense, which involves defining covariation by the levels which emerge in a successive order. In their model, Carlson et al. (2002) clearly described the five covariational reasoning levels (CRLs) and supported mental actions (MAs) categorized in those levels. In this study, we utilized this framework in order to investigate prospective teachers' covariational reasoning. The details of the framework are given in the methodology part of the paper.

In recent years, mathematics education researchers have had an interest in a covariational approach that is critical for obtaining a better conceptualization of the rate of change concept (Carlson et al., 2002; Confrey & Smith, 1994; Johnson, 2012; Strom, 2006). In this regard, it is strongly recommended that dynamic functional situations are used to obtain a connection between students' understanding about the rate of change and covariant aspect of functions (Carlson, 1998; Confrey & Smith, 1991, 1995; Koklu & Jakubowski, 2010). While understanding functional relationships in dynamic situations is fundamental in comprehending major concepts in calculus, the studies mostly conducted with undergraduate students revealed that students had a weak understanding of functions (Monk, 1992; Monk & Nemirovsky, 1994) and other concepts in calculus such as limit (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas & Vidakovic, 1996) and derivative (Zandieh, 2000). Even students who are high achievers had difficulty in modeling dynamic functional situations including quantities that covary (Carlson, 1998; Carlson et al., 2002; Koklu, 2007; Monk & Nemirovsky, 1994). Moreover, they were challenged especially while interpreting and constructing graphs of dynamic functional situations (Brasell & Rowe, 1993; Kieran, 1992; Monk, 1992). The difficulties were associated with undergraduate students' thought in that the shape of the graph should reflect the visual aspect of the given condition presented in the graph the graph (Monk, 1992; Kaput, 1992).

Related literature also indicates that although there are many studies investigating undergraduate students' covariational reasoning abilities, few studies are conducted in order to understand teachers' covariational reasoning abilities (Portnoy, Heid, Lunt & Zembat, 2005; Şen-Zeytun, Çetinkaya & Erbaş, 2010). For instance, Şen-Zeytun et al. (2010) focused on secondary school math teachers' covariational reasoning abilities and the teachers' prediction about students' covariational reasoning abilities by using model-eliciting activities. They came to conclusion that teachers had a poor understanding of covariation because teachers thought functions as correspondence rather than covariation.

Interestingly, little is known about prospective middle school mathematics teachers' reasoning about the covariant aspects of function graphs in dynamic event context. Yet, students in middle school start to learn covariational reasoning by analyzing patterns of change in various contexts (Common Core State Standards Initiative [CCSSI], 2010; Ministry of National Education [MoNE], 2013; National Council of Teachers of Mathematics [NCTM], 2000). Furthermore, they learn ratio and rate of change concepts, the slope of line graphs, and linear equations in middle school mathematics. In this respect, constructing a strong foundation for students' covariational reasoning in middle school helps to prevent the formation of misconceptions and errors about function in the coming years (Confrey & Smith, 1995; Koklu, 2007). To do this, teachers' knowledge of content becomes crucial since this affects the way in which they represent the nature of knowledge within the area of content to their students (Ball, 1990; Even, 1993). From this point of view, prospective teachers' covariational reasoning processes primarily should be explored because the difficulties they have or a low level of understanding can directly influence their future students' reasoning and instruction. In the present study, we aimed to examine prospective middle school teachers' covariational reasoning abilities while they were interpreting a dynamic situation. In particular, we concentrated on two prospective teachers' covariational reasoning processes in peer interaction.

Peer Interaction

The interaction between students is an important requirement for learning since it provides opportunities for them to formulate ideas, reveal their own understanding, and reflect on their thoughts. Peer interaction studies have given the importance of collaborative discussion on a mathematical task in problem solving in terms of controlling the solutions and reaching the correct solutions (Eizenberg & Zaslavsky, 2003). Besides, social studies conducted with children suggested that students in pairs produce more ideas by transacting on each other's ideas in the interactions for different discussion styles (Kruger, 1993). Specifically, in the interactions between children in the different developmental levels, the different types of conversations between pairs influence the students' recognition in the context of conversation (Psaltis & Duveen, 2007). Producing meaningful peer explanations and having the opportunity to show learners' own understanding in cooperative environments were presented in order to enhance the improvement of individual performance (Webb, 1991; Webb & Farivar, 1999). For example, in one pair work, students considered and checked alternative approaches in attention, then they reacted to each other's questions in a dialogue, and the rate of participation was similar for both partners (Sela & Zaslavsky, 2007). Moreover, it has been proposed that low ability students benefit more in heterogeneous (e.g. high-low) pair working than by working with homogeneous (e.g. low-low) ability peers since low ability students may have a chance to improve their understanding by receiving explanations from students who are high achievers (Patchan, Hawk, Stevens & Shunn, 2013; Webb, Troper & Fall, 1995). Peer learning has been suggested for teachers (Leikin, 2004) as well as students. Prospective teachers can express their feelings in peer groups since peer groups can be seen as safe environments (Karlsson, 2013). A cooperative learning environment that provides peer learning for teachers and the need to collaborate during their attempts are emphasized for mathematics teachers' professional development in terms of their mathematical knowledge, pedagogical content knowledge, and curricular content knowledge (Leikin, 2004). Since there are few studies on teachers' peer learning, it seems a need to investigate the cognitive processes of a pair of teachers in the interaction. Furthermore, considering that group ability compositions might have different influences on learners, we decided to create a peer composition involving a low-performing and a high-performing prospective teacher. In such a composition, it was expected that both teachers may have the opportunity to hear multiple perspectives by stimulating discussion on covariational reasoning of two simultaneously varying quantities. In line with the purpose, we investigated the covariational reasoning abilities of both a high-performing and a low-performing prospective middle school mathematics teacher during individual and peer interaction processes while interpreting dynamic functional situations involving two simultaneously changing quantities. Considering the purpose, we answered the following research questions:

– How do prospective middle school mathematics teachers individually reason about coordinating variables in dynamic functional situations involving two simultaneously changing quantities? More specifically, we examined the covariational reasoning behaviors that the prospective teachers exhibit while interpreting the covarying quantities in the task involving filling bottles with water and the graphs of height as a function of volume.
– To what extent do prospective middle school mathematics teachers reason about the role of variables in dynamic functional situations involving two simultaneously changing quantities during the peer interaction process? How does that peer interaction process influence the teachers' covariational reasoning behaviors?

Method

Since this study aims to investigate deeply the nature of prospective teachers' reasoning processes, a qualitative approach is the most appropriate to use (Creswell, 2007). The design of the study is a case study that describes and analyzes a bounded system in depth (Yin, 2009) in which prospective middle school mathematics teachers enrolled in the third year of their teacher education program. The case was mainly about prospective teachers' covariational reasoning abilities in a peer interaction process. Therefore, two third grade prospective middle school mathematics teachers as an embedded unit of analysis were used as an explanatory example of the way in which peer interaction influenced their covariational reasoning behaviors.

Task: Bottles and Graphs

The dynamic situation of filling bottles with water continuously is used to analyze students' covariational reasoning abilities (Carlson et al., 2002). For this study, the task was adapted from the "looking at gradients" classroom material that includes six bottles and their graphs representing the change of height as function of volume as the water filled the empty bottle at a constant rate (Swan, Bell, Burkhardt, Janvier, & the Shell Center team, 1985, p. 94).

In the present task, we took six bottles with their height-volume graphs from the Shell Centre material and added two more bottles with two blank spaces that were intentionally left for the sketches of their height-volume graphs (indeed bottles D and H). Participants were asked to match the bottles with the given graphs, then draw two graphs for two bottles, and finally explain their justification for each match (see Fig. 1). While bottle H is cylindrical in shape, the bottles B, D, E, and F are formed of non-cylindrical parts, and the bottles A, C, and G have both a cylindrical part and a non-cylindrical part.

In the graph context, the task includes a concave down graph with decreasing rate (bottle E), a concave up graph with an increasing rate (bottle B), and a linear graph with a constant rate (bottle H). The graphs of bottles C and D include inflection points where the rate changes from a decrease to an increase. On the other hand, the graph of bottle F includes an inflection point where the rate changes from increasing an increase to a decrease. Besides, there are breaking points in the graphs of bottles A, C, and G, which are combination of linear graph and concave up/concave down graphs. In brief, the graphs provide various situations of increasing functions to examine students' ability to utilize covariational reasoning because of the different shapes of the bottles. As a result, this task is useful for investigating covariational reasoning ability, since it shows the processes of students' thinking for different graphs by collecting possible covariational situations. Participants' covariational reasoning was evaluated by using a covariational reasoning framework; the details of which are provided below.

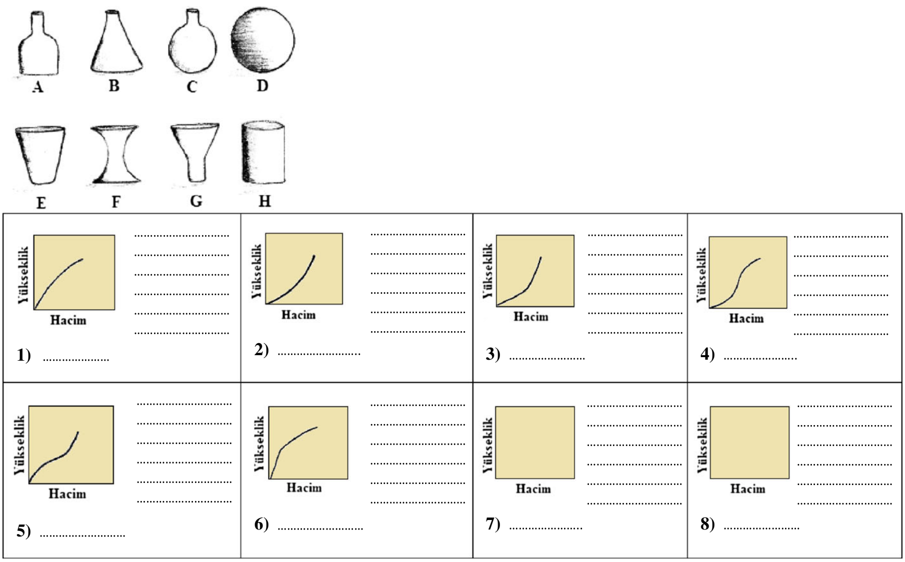


Fig. 1 Bottles and graphs in the task

Covariational Reasoning Framework

To provide a clear description of the covariational reasoning framework, five covariational reasoning levels (CRLs) and supported mental actions were summarized in Table 1. Expressions of common covariational reasoning behaviors in each mental action are followed with exemplary specific behaviors which the students exhibited when responding to a dynamic situation in graphical context called the Bottle Problem (Carlson, 1998).

In the framework, mental actions (MAs) enable us to classify students' behaviors as they interpret covariation tasks. From this point, researchers proposed five distinct CRLs (Carlson et al., 2002). The coordination level (L1) supports the mental action of coordinating the value of one variable with changes in the other. Specifically, students who exhibit behaviors in MA1 express one variable changes, the other variable changes (e.g. as volume changes, height changes). Namely, students label the coordinate axes by the awareness of only change rather than direction, amount, or rate of change. The direction level (L2) supports both MA1 and MA2 described as coordinating the direction of change of one variable with changes in the other variable. Students who are in L2 construct an increasing straight line by verbalizing the awareness of the direction of change of height (y-axis) while considering changes in volume (x-axis) (e.g. as more water is added, the level of the water in the bottle goes up.). The quantitative coordination level (L3) supports MA1, MA2, and the mental action of coordinating the amount of one variable with changes in the other variable (MA3). In this level, students separate x-axis into intervals of fixed lengths while considering the amount of change in height for each new interval of the volume. This is followed by plotting points/constructing secant lines by connecting the points on the graph. The average rate level (L4) supports MA1, MA2, MA3, and the mental action of coordinating the average rate of change of the function with uniform increments of change in the input variable (MA4). Students with L4 reasoning construct continuous secant lines for the domain by carrying out the mental computation or estimation of the slope of graph over equal amounts of water. Instantaneous rate level (L5) supports from MA1 through MA4 and the mental action of coordinating the instantaneous rate of change of the function with the continuous changes in the independent variable for the entire domain of the function. L5

Table 1 Mental actions and levels of covariational reasoning framework

*Mental Actions and Levels of Covariational Reasoning Framework**

Mental Action	Level 1 Coordination	Level 2 Direction	Level 3 Quantitative Coordination	Level 4 Average Rate	Level 5 Instantaneous Rate
(MA1)	Coordinating **the value** of one variable with changes in the other variable	Coordinating **the direction** of one variable with changes in the other variable	Coordinating **the amount** of one variable with changes in the other variable	Coordinating **the average rate of change** of the function with uniform increments of change in the input variable	Coordinating the **instantaneous rate of change** of the function with the continuous changes in the independent variable for the entire domain of the function
(MA2)					
(MA3)					
(MA4)					
(MA5)					

Adapted from Carlson et al., 2002, p. 357

reasoning requires constructing a smooth curve that involves concavity positions for the related bottle and considering the changing nature of rate while imagining the water changing continuously for smaller and smaller amounts of water.

In some situations, students appear to engage in a mental action, but they offer no evidence that they have the understanding which is necessary for performing the specific mental action in a meaningful way, which indicates the presence of pseudo-analytic behavior (Carlson et al., 2002; Vinner, 1997). Pseudo-analytic behaviors describe a behavior which might look like conceptual behavior, but which in fact is produced by mental processes which do not characterize conceptual behavior (Vinner, 1997, p. 100). For instance, even when students exhibit the behaviors of an MA5 appearance such as the construction of a smooth curve, it does not have the guarantee of being applied at an instantaneous level (L5) because the students' answer might rely on memorized facts without a rationale for his or her construction.

Participants

In the study, the participants were junior (third grade) mathematics education students in a middle grades mathematics education (grades 5–8) degree program in a large public university in Ankara, Turkey. The program offers content (mathematics, physics, and statistics) courses, education courses, and mathematics education courses. While mostly taking mathematics courses in the first 2 years, students take courses such as methods of teaching mathematics, school experience, and practice teaching in the following years. At the time of the collection of the data, the participants were students of the methods of teaching mathematics-II course offered in the sixth semester, which was a continuation of the methods of teaching mathematics course-I from the fifth semester.

In line with the purpose, the participants were two junior prospective middle school mathematics teachers who were selected through a two-stage procedure via a purposive sampling method. At first, 46 juniors responded to the task. Ten juniors who had varied performance levels, depending on the number of correct answers in the task, were selected for follow-up interviews. The interviews were conducted with the participants on the basis of the levels of responses in the task. Specifically, two female students with the pseudonyms Aysu and Buket, who were low-performing and high-performing in the task, were selected as the participants of the study. Aysu was selected as a low-performing student having only three correct answers among eight items in the task. Besides, Buket was selected as a high-performing student answering all the items correctly. In addition, Aysu was a low achiever and Buket was a high achiever in their mathematics and physics courses. In addition, we considered them as talkative participants since we knew them in person.

Data Collection Procedure

In the beginning, participants responded to the bottles and graphs task by writing the reasons for the matches that they made. After finishing the analysis of all the written responses by counting the number of correct answers, researchers conducted individual semistructured one-to-one interviews which took approximately 10 min with two selected participants who were designated as high-performing and low-performing. During the individual interviews, the interviewer (third author) asked questions such as "How did you match the bottles and the graphs?" and "Did you use primarily bottles or graphs during the matching process?” to the each participant. While answering the questions, the participants could examine their own written responses to the task whenever they wanted. However, at no time during the data collection process was information given about the correctness of the students’ written responses in order to avoid biased thoughts/opinions.

After the individual interviews, the interviewer conducted a paired interview which lasted about 45 min in which the peers’ covariational approach on the task was examined. The paired interview process involved two interviewees and one interviewer (one of the researchers) (Houssart & Evens, 2011). Conducting a paired interview after an individual interview is recommended and preferred as an alternative methodological approach to paired interview (Cooper, 2003; Fujii, 2003; Houssart & Evens, 2011; Zeidler, Walker, Ackett & Simmons, 2002). This approach is especially preferred by researchers when selecting pairs who have responded differently in the individual process because it provides several advantages for the participants such as comparing and contrasting views and exploring different viewpoints with each other and also with the interviewer (Houssart & Evens, 2011). In our study, we conducted, firstly, an individual interview then a paired interview, since we wished to understand the influence of peer interaction on both a high-performing and a low-performing teacher’s covariational reasoning behaviors by comparing their reasoning in individual processes.

The interviewer conducted the paired interview with at points a guiding role: directing pairs to the task in order to reveal the nature of covariational reasoning for different bottle-graph matching; clarifying the ways in which each participant utilized in the matching process; giving encouragement to each participant for criticizing the way/mathematical ideas each used in the matching process; making connections between ideas raised while working in pairs; and distributing equal participation time to each participant in order to encourage collaboration between the participants. To illustrate, the interviewer asked the participants, in order to orient them, if they could think in a different way with various mathematical concepts while interpreting. She also asked the participants why the slope was constant in order to clarify their methods of reasoning after their statements about constant slope. Finally, she asked if the participants agreed on their interpretations and how they had decided in order to encourage them and for them to make connections between their ideas. Other researchers as non-participant observers videotaped the whole process.

Data Analysis

In the analysis of the data from the individual process, each participant’s written responses were examined for the task in two steps. First, we determined the correctness of the graphs-bottles matches matching and the graphs constructed for the two remaining bottles. Then, we analyzed how the students explained their matches between graphs and bottles in the task. We determined their CRL, which supported the MAs considering their written explanations through the framework. Besides, oral explanations and drawings obtained from the one-to-one interview data were used to triangulate the participants’ mental action behaviors.

After the individual process, the peer interaction process was examined to identify the participants’ re-emerging MAs associated with their CRL (Carlson et al., 2002). The participants’ MA behaviors in the process were coded in the video transcript. The participants’ MAs were analyzed in the context of peer interaction by combining cognitive and interactive aspects of double-coding methodology (Tabach, Hershkowitz & Schwarz, 2006). The parallel analyses led to the identification of patterns of interactions between students logically and chronologically (Dreyfus, Hershkowitz & Schwarz, 2001). The peer interaction transcript included the name of the speaker (A for Aysu, B for Buket) with the sequence number (e.g. A1 and B1 in Fig. 2a) and the logical flow of the utterance, which was represented by the circles and arrows. Each statement was marked by a circle connected by arrow (or arrows) to a previous statement which generated it (Tabach et al., 2006). Vertical arrows indicate that the speaker continued with their previous statement, while diagonal arrows indicate that one subject was inspired by the other subjects' comments (Dreyfus et al., 2001) (see Fig. 2a, b).

In other words, the vertical arrows show that the source statement (starting arrow) reacts to her own statement (end arrow), whereas the diagonal arrows show that the source statement (starting arrow) reacts to the other participant's statement (end arrow). The pattern of arrows indicates whether the interaction is symmetrical or asymmetrical contribution (Tabach et al., 2006). In other words, if one participant dominates the discussion, it is called asymmetrical contribution. Otherwise, if the participants make a contribution symmetrically that means they participated in the conversation with a relatively equal number of statements. By the way, we analyzed the re-emerged mental actions in the peer interaction process considering the participants' contributions to the conversation.

Findings

The findings were presented into two parts. In the first part, each prospective middle school mathematics teacher's written responses and personal responses in the individual interview were analyzed in an attempt to determine the participants' covariational reasoning behaviors that they exhibit while interpreting the covarying quantities in the bottles and graphs task. In the second part, the peer interaction process was analyzed in order to reveal the changes of high-performing and low-performing teachers' covariational reasoning behaviors for interpreting dynamic functional situations.

The Case of Aysu

Aysu gave three correct answers (matches for the bottles A, G, and H) and five incorrect answers (matches for the bottles B, C, D, E, and F) in the bottles and graphs task. She gave the appropriate explanations for the graphs of bottle H and the cylindrical parts of bottles A and G. While her matches were correct for bottles A and G, her written responses were erroneous for the non-cylindrical parts of the bottles. In her written

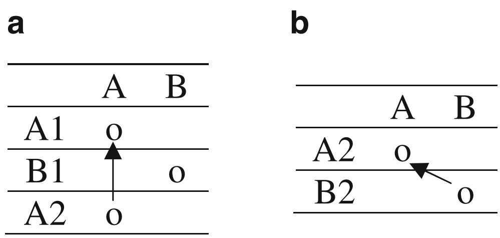


Fig. 2 **a** Speaker A continues on her previous statement. **b** Speaker B continues on Speaker A's statement

responses, she described the sketches as if they were volume-height graphs rather the height-volume graphs. That is, since she switched the roles of variables in the graphs, she considered the amount of change in volume per unit height and gave incorrect explanations.

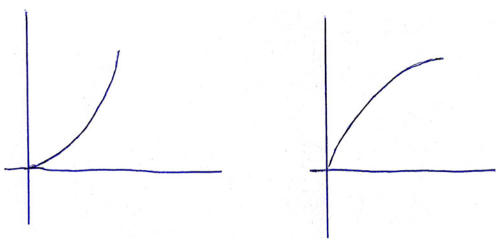
When we analyzed Aysu's mental actions in her written responses, it was seen that she coordinated the direction of change of volume while considering the changes in height (MA2). Furthermore, she was able to coordinate the amount of changes in one variable (volume) with the changes in another variable (height). Since she could coordinate the relative magnitudes of change in the x-variable with changes in the y-variable, her mental action was categorized as MA3. In the individual interview, in supporting her written responses, she explained that she had considered the amount of change of volume per unit height while matching the bottles and graphs. When the question of how she had learned increasing function graphs in high school or undergraduate course was asked in the interview, Aysu stated that while the first sketch represented an "increasingly increasing graph," the second sketch represented a "decreasingly increasing graph" (see Fig. 3). She only focused on the shape of the function graph rather than mentioning the role of dependent and independent variables in the graphs. Furthermore, she explained that she did not ever examine changes in dependent variable in the function graphs in relation to the unit per unit changes in the independent variable.

In some situations, although Aysu's behaviors appeared in upper mental actions, enough evidence was not gathered that she possessed an understanding which supported the behavior. For instance, the following response was one of the examples of that behavior which was classified as pseudo-analytic MA5: "When we look at bottle C, there is an inflection point at the middle of spherical part of the bottle. Therefore, this graph (the graph of bottle C) is neither of the graphs of bottles B, E, or H." This type of behavior consists of pseudo-analytical mental action because the teacher did not engage in MA5 conceptually. Even though she might seem aware of the inflection point where the rate of change changes from increasing to decreasing, or decreasing to increasing, she had only an image of what the graph should look like rather than the representation of how the variables changed. In this form of reasoning, the mental process did not apply a conceptual understanding that supports the instantaneous rate level (L5) reasoning. At the same time, while she named the graphs as concave up and concave down, she did not mention the rate of change or instantaneous rate of change. Consequently, she appeared to use the quantitative coordination level (L3) reasoning predominantly that supported MA1, MA2, and MA3.

The Case of Buket

Buket's matches were all correct and she clearly explained her rationale for matching the bottles to the graphs. She also drew acceptable graphs for bottles D and H. Her

Fig. 3 Aysu's construction of increasing function graphs



written responses appeared to be evidence of her understanding of the slope and changing rate for the fixed amount of water in unit time (MA4). For example, her response to the graph of bottle E was an indication of an image of average rate of change (MA4) because she imagined and adjusted the slope for different intervals in the graph while focusing on the rate of change of the height with respect to input for uniform increments of the time. However, she used time as an independent variable rather than volume, and she considered uniform increments of the volume in time. Additionally, her response for the graph of bottle D indicated an understanding of coordinating the instantaneous rate of change of height (with respect to time) with the changes in the volume in concave up and concave down constructions (MA5).

As the evidence of MA5 behavior, Buket was aware of the direction of concavities and interpreted the instantaneous rate of change on the slope of the tangent lines of the functions at any point. For example, see the following explanation for bottle E in the interview:

> Slope decreases in time. For example, on this point of tangency the slope is more; on that point of tangency the slope is less [compares the points of the graph] … So, volume increases and height also increases. But the important point is how the slope changes. In the graph, the slope of the graph decreases in time, that is, the function is decreasingly increasing.

In her justification for matching bottle E, it was clear that when volume increases in time, the instantaneous rate of change as the slope of the tangent lines decreases with respect to an equal interval of time. By this decrease, she meant that function graph increases at a decreasing rate.

At the same time, Buket's personal responses in the interview were parallel to her written responses. She interpreted the change in the inflection point of the graph as mid-level of bottle D. She emphasized the inflection point on the graph of bottle C as such: "It is the graph of Bottle C because there is an inflection point" (B58). Actually, she described the slope as rate that is "the speed of increment of height per unit time" (B28). She analyzed the instantaneous slopes of the function while deciding whether a function is increasing at an increasing rate or increasing at a decreasing rate. Thus, her behaviors showed the images categorized in level 5 instantaneous rate reasoning.

Peer Interaction Process

At the beginning of the pair work, the researcher (third author) initiated the process by directing the pair to examine their responses to the task. In order to elaborate their reasoning on covarying quantities in the task, the researcher asked the participants how they had interpreted the bottles and graphs. Buket made suggestions about the slope in the graph of bottle B, such as "the increment of height in unit volume increases continuously, which is related to the instantaneous slope," and required a validation of her own reasoning. Aysu produced validations that Buket needed to draw a steeper tangent line. At this point, Buket continued her previous explanations that there was more increment of height in time and so it was the graph of bottle B. Aysu suggested that something was incorrect in her responses. She then realized that while she had chosen the first graph for the bottle B (a mismatch) and the second graph for the bottle E (also a mismatch), Buket did the reverse of Aysu's matches.

Discussion About the Role of Variables in the Graphs

The pair started to compare and contrast the role of the variables after seeing the differences in their matches and their explanations about the sketches of the first two graphs. At that point, they tried to clarify and justify their explanations (see Fig. 4). Aysu offered an explanation as to why Buket considered the graph of bottle B as the graph of height as a function of volume instead of the graph of volume as a function of height (A18). In response, Buket drew horizontal lines to show equal amounts of changes in height in bottle B and its graph (see Fig. 5a). In particular, Buket focused on the amount of change in volume while considering fixed increases in the height of the bottle B (B17). However, Aysu continued to search for evidence for a validation. It seemed that Aysu focused on the amount of change in volume per unit height (MA3) by switching the roles of the independent variable and dependent variable (A18, A19). At that point, Buket emphasized the role of the variables in the graph as a new element in the dialogue (B18). Aysu specifically introduced the difference between her reasoning and that of Buket's (A20). Buket provided an explanation focusing on the slope and rate of changes for fixed amounts of height of water on the graph (B20). Then, Aysu posited her reasoning which showed that Aysu's response to the graph of bottle B contradicted the image that she had previously had of the increasingly increasing function graph (A21). Following Aysu's conflicting viewpoint, Buket explained the role of the input variable on the direction of the dependent variable as increasing or

	A R B
A18: I am not convinced about this point. I said that it is the [amount of change of] volume per unit height. Explain this to me, this is unit height *(she draws horizontal lines on bottle B)* and the [amount of change of] volume decreases here [the upper level of the bottle] [MA3].	
B17: This volume [below line 1, see Figure 5.a] is greater than the volume [above line 1] [MA3].	
A19: So, the volume decreasingly increases [pseudo-analytic MA4].	
B18: Yes, the volume decreasingly increases. But, this reasoning would be true if the x-axis referred to height, and the y-axis referred to volume [MA4]. Do you understand what I mean? The graph is the reason.	
R29: Don't change the labels of the axes in the graph	
B19: In this graph we consider the volume.	
A20: You consider the [amount of change of] height per unit volume. I consider the [amount of change of] volume per unit height.	
B20: OK. Let's consider the [amount of change of] volume per unit height. Here *(point y_1)* is unit height and here is the corresponding place *(x_1)*. The growth of volume is higher here *(x_1-0)* and less there*(x_2-x_1)*, isn't it? When we look at this unit height *(point y_2)*, the growth of volume *(point x_2)* has decreased even more [MA4]. Thus, the growth of volume should decrease [for bottle B] [MA3]. [see Figure 5.b]	
A21: In that case, you can't interpret the graph [in figure 5.b] as increasingly increases [suggested MA4].	
B21: I am already interpreting it as increase at an increasing rate based on height, but your interpretation is based on volume. That's why we are in disagreement. The [amount of change of] height increasingly increases, so the [amount of change of] volume decreasingly increases [MA4].	

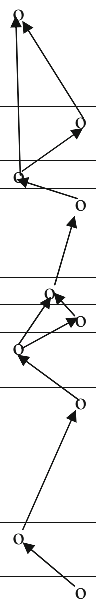


Fig. 4 Students' mental actions while establishing coordination between two quantities in the dyadic working (excerpt 1)

decreasing by comparing and contrasting her own viewpoint and Aysu's viewpoint (B21). Whereas Aysu could not understand the role of the independent variable (the x-axis of the graph) in the image of increase at an increasing rate function graph, she had just realized that she considered the independent variable in a different way to Buket.

Considering Aysu's explanations on the graphs of bottles B and E, the researcher asked if she had taken unit height into consideration in order to clarify her reasoning about the dependent and independent variables. Then, Aysu provided the following explanations (A26):

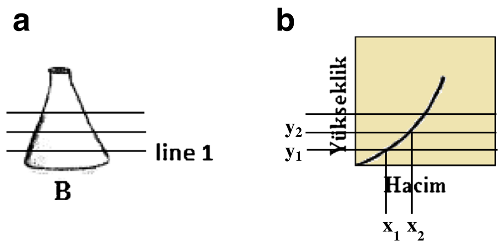
> Teacher, I have understood my mistake. Since I considered the amount of change of volume per unit height, I made an incorrect match [MA3]. I said that [the amount of change of height is] decreasingly increasing [per unit volume]. In fact, it would be increasingly increasing [per unit volume] [pseudo-analytic MA4].

Aysu's explanation showed an understanding of MA4 in the above since it seemed that having an awareness of the rate of change of height with respect to volume while considering uniform increments of volume. However, she justified the decision above like so: " …because of the interpretation that we made [on the graph of bottle B], my graph will be the opposite of the graph of bottle B. Thus, this must be the graph of bottle E (A27)." This statement indicated that Aysu did not seem to understand the differences in Buket's and her own reasoning about the average rate of change, since she focused on the appearances of the graphs. Aysu probably saw the influence of the variables on the interpretation of concavity for the graph of bottle B.

Consequently, the examination of different graphs and bottles in the peer interaction process helped Aysu in terms of realizing her mistake about the variables. However, there was no change in Aysu's concept images of the dependent and independent variables in the graph of bottle E (A29): "When I took the volume [as a fixed variable] I was confused. I couldn't set the volume. For example, if someone asks me [to examine the change in] height by fixing the volume I'm confused." After that admission, Buket proposed that Aysu could also draw the correct graph even if she considered height as an independent variable. Then, Aysu explained what her image of increasing graph was. Considering Buket's next explanation of what connects slope and the change of height, Aysu emphasized that her expressions were correct for volume-height graphs not for height-volume graphs.

When summing up the mental actions of covariational reasoning during this interaction process, Aysu's responses were usually classified as MA3 and those of Buket's were classified as MA4. Despite the fact that Aysu appeared to show image of average rate level, her expression of her opinion of what the graph should look like was

Fig. 5 **a** Buket's sketches on bottle B. **b** The graph of bottle B that was mentioned in B17 and B20 in Fig. 4



classified as pseudo-analytic MA4. While Buket individually considered volume as an independent variable in her responses, in the interaction with Aysu she interpreted the graph through unit height to explain the effect of the input variable. In this dialogue, the behaviors exhibited by Aysu and Buket suggested quantitative coordination (L3) and average rate (L4) covariational reasoning ability. During the discussion about the role of variables in the graphs, the pattern of arrows in the conversation indicated that one prospective teacher continued to the other's statements. This means that there was a symmetrical contribution of the prospective teachers in the discussion.

Discussion About the Interchangeable Role of the Variables for Bottle G and Its Graph

In order to assert the prospective teachers' mental actions in covariational reasoning from a different perspective, the researchers asked the participants to draw the graph for bottle G by reversing the labels on the axes. This is because the graph for bottle G includes a breaking point which shows the change from a linear to a nonlinear graph. Consequently, the participants began to discuss how to interpret the graph for bottle G if the x-axis was labeled as height and the y-axis was labeled as volume on the graph (see Fig. 6b, c). Aysu and Buket drew an acceptable sketch for the cylindrical part (first part) of bottle G, but their graphs differed for the non-cylindrical part (second part) of bottle G. For the first part of the graph, Aysu gave the correct explanation, which was that "the amount of volume in unit height was constant and there was no change in the speed of increment in volume as decreasing or increasing." This explanation indicated MA3 behavior. Besides, Buket connected the first part of the bottle to the linear graph and constant slope (MA4). This explanation indicated MA4 since she seemed to consider constant slope as average rate of change. After Buket's explanations, Aysu tried to explain the concept of slope as a ratio based on the amount of change in one variable with another variable for the linear part (MA3).

While Buket's graph for the second part of bottle G was concave up, that of Aysu's was concave down. After the participants had noticed that their sketches were different for bottle G, the researcher initiated a discussion about how they had interpreted the difference in their sketches. At this point, an interactional process occurred between the researcher, Buket, and Aysu, which can be seen in excerpt 2 (see Fig. 7).

Buket, being both a dominant character in the process and also showing higher covariational reasoning behaviors, responded with clear expression supported by MA4 even if the labels of axes on the graph were changed from height-volume to volume-

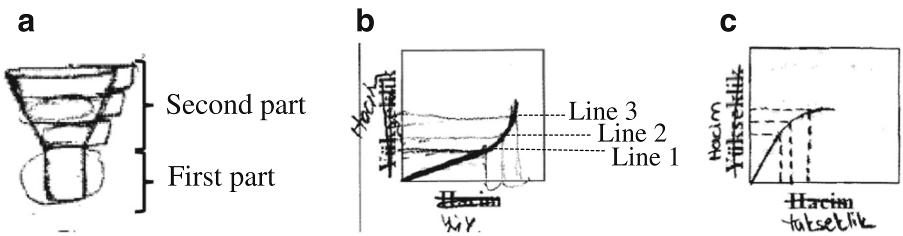


Fig. 6 **a** Buket's horizontal lines on bottle G drawn to show the fixed height. **b** Buket's graph for bottle G when the x-axis was height and the y-axis was volume. **c** Aysu's graph for bottle G when the x-axis was height and the y-axis was volume

	A	R	B
R49: Well, your answers vary. How can you explain this difference?		o	
B40: The graph should be increasing concave-up or increasing concave-down due to the increment after the breaking point [suggested MA5]. When we look at the increment of the volume per unit height, this increment *(from line 3 to line 2)* is greater than the increment *(from line 2 to line 1)* [MA4]. We expect that volume increases per unit height [MA3]. (see Figure 6.b). In this situation, I expect an increment [in the amount] of volume [MA3]. Then, I divide the x-axis unit by unit, so the increment of volume *(between x_2 and x_3)* is higher than the increment of volume *(between x_1 and x_2)* [MA4]. Thus, the graph of the function should be increasing at increasing rate because our explanations were on [the amount of change of] the volume [MA3].			o
R50: Let's look at Aysu's ideas.		o	
A40: It is necessary that the [amount of change of] height per unit volume decreases, isn't it [MA4]?	o		
B41: Yes.			o
A41: If I put the equal amount of volume *(above line 2)*, the height will be less than the interval between line 2 and line 1 [suggested MA4].	o		
R51: Which construction makes more sense to you?		o	
A42: I cannot sketch the second part of the graph. I have difficulty with that part of the graph.	o		
R52: Why is it difficult to think of height as an independent variable on the x-axes?		o	
A43: For once, I unitize the volume instead of the height. For example, suppose that I take the distance between two points on the y-axis unit by unit. I construct the graph of function as decreasingly increasing rather than increasingly increasing.. I have a problem.	o		
R53: Well, can you overcome this problem now?		o	
A44: Yes, I can. Buket's explanations sound reasonable, but I still have a problem with the construction of [a nonlinear function] graph.	o		

Fig. 7 Discussion on the graph of bottle G when the labels of the axes are changed (excerpt 2)

height (B39). She particularly emphasized that the volume was a dependent variable for this situation (B40) (see Fig. 6b). On the other hand, Aysu again switched the role of variables instead of considering volume per unit height (A40, A41). That is, she considered the amount of change in the variable on the y-axis whatever the independent variable was on the labels of the graph (MA3). As a result, she incorrectly thought that height increases at a decreasing rate per unit volume for the second part of bottle G. She, then, drew a concave down graph incorrectly. Besides, she realized her mistake after Buket had explained once more. That is because she treated one quantity as a dependent variable on the graph while she assigned the same quantity as an independent variable on the bottle (see A43 in Fig. 7). Therefore, she misinterpreted the non-cylindrical bottles and their nonlinear graphs.

In summary, Aysu had the awareness for interpreting and understanding the covarying quantities. Hence, the mental actions in Aysu's explanations rose from MA3 to MA4. For instance, after Buket had given explanations on slope, Aysu unusually said, "Function increases at an increasing rate." Besides, while Buket was explaining her reasoning to Aysu, she described the changing nature of a dynamic event in terms of MA3 and MA4 behavior. During the discussion about the interchanging role of variables for the graph of bottle G, the prospective teachers participated equally in the peer interaction process through the researcher's guiding questions.

Discussion and Conclusions

In the current study, we aimed to investigate prospective middle school mathematics teachers' covariational reasoning abilities on a task about dynamic functional events involving two simultaneously changing quantities in an individual process and also in a peer interaction process.

In the individual process, the results revealed that the high-level prospective teacher used implicitly "time" quantity as an independent variable instead of volume while coordinating the height and volume in the graphs. This understanding might be related to her experiences with time-dependent function graphs in physics courses at school or college. On the other hand, the low-level prospective teacher interpreted incorrectly the height-volume graphs by switching the role of the dependent and independent variables. Prospective teachers' interpretations of the dependent and independent variables that we observed in the individual process were consistent with the results of studies conducted to examine teachers' covariational reasoning abilities (Carlson, Larsen & Lesh, 2003; Keene, 2007; Kertil, 2014; Şen-Zeytun et al., 2010).

In the peer interaction process, the prospective teachers' different viewpoints on the role of variables helped to stimulate and elaborate the discussion on the disagreements that they had about thinking about dependent and independent variables in reverse order. The low-level teacher developed an awareness of her misunderstanding and errors originating from the misplacement of the dependent and independent variables. The high-level teacher corrected herself by considering volume rather than time as an independent variable. We concluded that these disagreements on the role of the variables helped them to propose and defend their own ideas, and to ask their peer to clarify and justify their graph-bottle matches rather than simply reaching consensus on an agreed answer (De Lisi & Golbeck, 1999; Goos, Galbraith & Renshaw, 2002; Moshman & Geil, 1998). Even if low-performing prospective teacher seemed willing to reconsider the high-performers explanations about the role of inverse variables, she remained, however, unsuccessful in interpreting the variables on the nonlinear graphs. Therefore, we inferred that being involved in pair work might not be possible to tarnish the incorrect strict image of the role of inverse variables in a prospective teacher's mind.

It is obviously crucial to overcome "resistance to change in incorrect assertion" for prospective teachers because they will become in-service teachers in the future. It is inevitable that their inflexible thinking could influence their students' learning by leading them to misconceptions about rate, ratio, and covariation. At this point, we recommended that providing the coordination between the structure of calculus courses and educational courses in prospective teachers' education programs may be useful for developing learners' understanding of the role of variables and the dynamic view of functional dependency. To do this, courses can be designed to involve tasks in which there are the dynamically and simultaneously changing variables in both linear and nonlinear functions.

Another remarkable point in the current study was related to the changing nature of "mental actions behaviors" in the peer interaction process. While the low-level teacher coordinated the amount of change in one variable with changes in another variable in an individual process, she started to consider the slope and changing rate for a fixed amount of height in unit height (MA4) in the peer interaction process. Although the high-level teacher fully exhibited the behaviors associated with average rate and instantaneous rate of change individually, she frequently displayed behaviors considering the peer's response and spoke at a level (i.e. MA3 and MA4) that the peer understood. This situation indicated that the participants displayed common behaviors when interpreting covarying quantities. Furthermore, we noticed that by virtue of the common behaviors in the interaction, the low-level teacher improved her reasoning from perceptually conceptualizing the rate of change to unit per unit comparison in the change of the variables. This situation supported the results of the studies related to the way in which low ability learners benefit from the ideas of high ability learners while working in pairs (Tutty & Klein, 2008; Patchan et al., 2013; Webb et al., 1995). Consequently, when considering teachers' knowledge of covariation as a reflection of their thoughts on behaviors, the peer interaction process might help them to develop a depth of understanding about how rate of change, slope, and variable concepts are connected.

Interestingly, but perhaps not surprisingly, prospective teachers generally preferred the expression of "increasingly/decreasingly increasing function" instead of "increasing at an increasing/decreasing rate function" for concave up/down increasing graphs. Such kind of incorrect language usage does not include a ratio-based conception of rate of change that requires consideration of the equal increments in the independent variable (Kertil, 2014; Thompson, 1994). This situation might be the reason for using a single term (i.e. oran) in Turkish language to explain both "ratio" and "rate" concepts. In this regard, we suggest that further studies are needed to investigate the possible effects of linguistic factors on covariational reasoning.

As a final point, we also want to share some possible methodological suggestions for future studies in the light of the results obtained from our in-depth analysis. In the current study, we investigated a high achiever's and a low achiever's reasoning abilities both in individual and peer interaction process, the latter providing a better understanding of their covariational reasoning abilities and mental actions of both members of each peer (Tabach et al., 2006). There might be more participants from different levels of achievement in a varied peer interaction process. Studying with more participants from different levels might give the opportunity to researchers to contrast and compare the contributions of the learners on the interactional process and the influence of peer learning on their cognitive development. In addition, since the dynamic event was limited to filling bottles in this study, prospective teachers' covariational reasoning abilities might be investigated in different dynamic events to reveal the mental actions involved in those events. Thus, analyzing the correspondence among mental actions in different dynamic events and even in different cultures might contribute to the development of a covariational reasoning framework.

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