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ORIGINAL RESEARCH

Stability criteria for volterra type linear nabla fractional difference equations

Murat Gevgeşoğlu[1] · **Yaşar Bolat**[1]

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Abstract
In this study, we give some necessary and sufficient conditions on the stability for Volterra type linear nabla fractional difference equations of the form $\nabla_{-1}^{v} x(t) = \lambda x(t)$, $t \in \mathbb{N}_1$, with initial condition $\nabla_{-1}^{v-1} x(t) \mid_{t=0} = x_0$. For this, first of all we show that the above equation is a convolution-type Volterra equation, then give the stability conditions by using the stability analysis methods of the convolution type Volterra equations. Also we give some examples to illustrate our theoretic results.

Keywords Volterra difference equations · Stability · Nabla fractional difference equations

Mathematics Subject Classification 39A10 · 39A30

1 Introduction

The fractional calculus is the generalization of ordinary differentiation and integration to the arbitrary non-integer order and the history of fractional calculus begins with the classical calculus. Fractional calculus have valuable tools to model many phenomena in physics, engineering, economics and mechanics, see the monographs [1–6] and the references therein. Fractional difference and differential equations have various applications in viscoelasticity, chemistry, tumor growth model, modeling of pandemic diseases such as corona and flu, electromagnetics and control theory, see [7–15].

Murat Gevgeşoğlu and Yaşar Bolat contributed equally to this work.

✉ Murat Gevgeşoğlu
mgevgesoglu@kastamonu.edu.tr

Yaşar Bolat
ybolat@kastamonu.edu.tr

[1] Department of Mathematics, Kastamonu University, 37150 Kastamonu, Turkey

In [16] Qian et al. studied qualitative properties of the fractional differential equation

$$_{o}D^{\alpha}x(t) = \lambda x(t), \quad t > 0 \tag{1.1}$$

where $_{o}D^{\alpha}$ is a Riemann–Liouville derivative operator and the authors focused on establishing stability theorems for fractional differential system with Riemann–Liouville derivative, in particular their analysis covered the linear system, the perturbed system and the time-delayed system.

In [17] Cermak et al. studied qualitative properties of two term linear fractional difference equation

$$_{0}\nabla^{\alpha}y(n) = \lambda y(n) \tag{1.2}$$

where $\alpha, \lambda \in \mathbb{R}$, $0 < \alpha < 1$, $\lambda \neq 1$ and $_{0}\nabla^{\alpha}$ is the αth order Riemann Liouville difference operator. Authors showed that this fractional equation is the Volterra equation of convolution type and derived a sharp condition for the asymptotic stability of the studied equation and, moreover, gave a precise asymptotic description of its solutions.

In [18] Saris and Al-Mdall investigated the stability of the equilibrium solution of the vth order linear system of difference equations

$$\left(\Delta^{v}_{a+v-1}\, y\right)(t) = \Lambda y(t+v-1); \quad t \in \mathbb{N}_a,\ a \in \mathbb{R},\ \Lambda \in \mathbb{R}^{p\times p,} \tag{1.3}$$

subject to the initial condition

$$y(a+v-1) = y_{-1}$$

where $0 < v \leq 1$ and $y_{-1} \in \mathbb{R}^p$. Authors first focused on the scalar case with $\Lambda = \lambda$ a scalar, then focused on the systems of fractional order difference equations following the same line of reasoning.

In this manuscript we consider the stability of Volterra type linear nabla fractional difference equation

$$\nabla^{v}_{-1}x(t) = \lambda x(t), \quad t \in \mathbb{N}_1, \tag{1.4}$$

with initial condition

$$\nabla^{v-1}_{-1}x(t)\mid_{t=0} = x_0. \tag{1.5}$$

Here $0 < v < 1$, $\lambda \in \mathbb{R} - \{1\}$, ∇^{v}_{-1} is vth order Riemann–Liouville nabla fractional difference operator and $x : \mathbb{N}_0 \to \mathbb{R}$.

Equation (1.4) can be considered as a discrete analogue of Eq. (1.1). In this paper we give some necessary and sufficient conditions on the stability of zero solution of Volterra type linear nabla fractional difference equation (1.4). Firstly we show that the above equation is a convolution-type Volterra equation, then we give stability conditions by using some binomial relations and the stability analysis methods of the convolution type Volterra equations.

2 Preliminaries

Throughout this article $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ where a is a fixed real number.

Definition 1 [19] For an arbitrary function $f : \mathbb{N}_a \to \mathbb{R}$ the nabla operator (backwards difference operator), ∇, defined by

$$(\nabla f)(t) := f(t) - f(t-1), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2 [20]

(i) For a natural number m, the m rising (ascending) factorial of t is defined by

$$t^{\overline{m}} = \prod_{k=0}^{m-1} (t+k), \quad t^{\overline{0}} = 1$$

(ii) For any real number α the rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{\ldots, -2, -1, 0\},\ 0^{\overline{\alpha}} = 0. \tag{2.1}$$

Definition 3 The extended binomial coefficient $\binom{t}{n}$, $(t \in \mathbb{R}, n \in \mathbb{Z})$ is defined by

$$\binom{t}{n} = \begin{cases} \frac{\Gamma(t+1)}{\Gamma(t-n+1)\Gamma(n+1)}, & n > 0 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases} \tag{2.2}$$

Lemma 1 [13] *The quotient expansion of two gamma functions at infinity:*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + O\left(\frac{1}{z}\right)\right], (|\arg(z+a)| < \pi, |z| \to \infty).$$

Definition 4 [20] Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$ be given. Then the αth-order nabla fractional sum of f is given by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s), \quad \text{for } t \in \mathbb{N}_a. \tag{2.3}$$

where $\rho(s) = s - 1$ and $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{\ldots, -2, -1, 0\},\ 0^{\overline{\alpha}} = 0.$

Definition 5 [21] Let $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ be given and let $N \in \mathbb{N}$ be chosen such that $N - 1 < \alpha \leq N$. Then the αth-order Riemann–Liouuville nabla fractional difference of f is given by

$$\nabla_a^{\alpha} f(t) = \nabla^N \nabla_a^{-(N-\alpha)} f(t) \quad \text{for } t \in \mathbb{N}_{a+N}. \tag{2.4}$$

For $\alpha = 0$ we set $\nabla_a^0 f(t) = f(t), t \in \mathbb{N}_a$.

Definition 6 Let $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ be given and let $N \in \mathbb{N}$ be chosen such that $N - 1 < \alpha \leq N$. Then the αth-order fractional difference of f is given by

$$\nabla_a^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \sum\limits_{s=a+1}^{t} (t - \rho(s))^{\overline{-\alpha-1}} f(s), & \alpha \notin \mathbb{N} \\ \nabla^N f(t), & \alpha = N \in \mathbb{N} \end{cases} \quad \text{for } t \in \mathbb{N}_{a+N}. \tag{2.5}$$

Definition 7 [22] Volterra difference equations of convolution type are of the form

$$x(t+1) = Ax(t) + \sum_{j=0}^{t} B(t-j)x(j), \tag{2.6}$$

where $A \in \mathbb{R}$ and $B : \mathbb{Z}^+ \to \mathbb{R}$ is a discrete function.

Equation (2.6) represents systems in which the future state $x(t+1)$ does not depend only on the present value $x(t)$ but also on all past values $x(t-1), x(t-2), \ldots, x(0)$. If the initial condition x_0 is given then by iteration $x(1)$, $x(2), \ldots, x(t+1)$ can be generated. In the literature this equation is called as *hereditary.*

Definition 8 [17] Consider (2.6) along with the initial condition $x_0 = \phi_0$. Then (2.6) is said to be

(i) Stable if for any real ϕ_0 there exists $\varepsilon > 0$ such that the corresponding solution $x(t)$ of (2.6) satisfies $|x(t)| < \varepsilon$ for all $t \in \mathbb{N}_1$.
(ii) Asymptotically stable if $x(t) \to 0$ as $t \to \infty$ for any real ϕ_0.

Definition 9 [17] Consider (2.6) with the initial conditions $x(k) = \phi_k$, k =0,1,…,m, m being an arbitrary nonnegative integer. Then (2.6) is said to be

(i) Uniformly stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if ϕ_k are reals with $|\phi_k| < \delta$, $k = 0, \ldots, m$, then the corresponding solution $x(t)$ of (2.6) satisfies $|x(t)| < \varepsilon$ for all $t \in \mathbb{N}_1, t > m$;
(ii) Uniformly asymptotically stable if it is uniformly stable and if there exists $\eta > 0$ such that, for any $\varepsilon > 0$ there is $N = N(\varepsilon) \in \mathbb{N}_1$ such that if $|\phi_k|$ are reals with $|\phi_k| < \eta$, $k = 0, \ldots, m$, then $|x(t)| < \varepsilon$ for all $t \in \mathbb{N}_1, t \geq m + N$.

To study the qualitative behaviour of (2.6) we use Z-transform method below.

Definition 10 [22] The Z-transform of a sequence $x(t)$, which is identically zero for negative integers t(i.e $x(t) = 0$ for $t = -1, -2, \ldots$), is defined by

$$\tilde{x}(z) = Z(x(t)) = \sum_{j=0}^{\infty} x(j) z^{-j},$$

where z is a complex number.

Definition 11 [22] The convolution $*$ of two sequences $x(t)$ and $y(t)$ is defined by

$$(x * y)(t) = \sum_{j=0}^{n} x(t-j)y(j) = \sum_{j=0}^{n} x(t)y(t-j).$$

Lemma 2 [22] *Let* $Z(x(t)) = \tilde{x}(z)$ *be the* Z*-transform of the* $x(t)$*.Then*

$$Z(x(t+1)) = z\tilde{x}(z) - zx(0).$$

If we write (2.6) Using Definition 11, we can write Eq. (2.6) as follows.

$$x(t+1) = Ax(t) + (B * x)(t). \tag{2.7}$$

By using Lemma 2, if we take Z-transforms of both sides of (2.7) ,we obtain

$$z\tilde{x}(z) - zx(0) = A\tilde{x}(z) + \tilde{B}(z)\tilde{x}(z). \tag{2.8}$$

From (2.8) we have

$$\tilde{x}(z) = \frac{zx(0)}{z - A - \tilde{B}(z)}. \tag{2.9}$$

In (2.9) let

$$g(z) = z - A - \tilde{B}(z). \tag{2.10}$$

The characteristic function $g(z)$ plays very important role for the stability of zero solution of (2.6) Necessary and sufficient conditions are given for the uniformly asymptotical stability of zero solution of (2.6)with the following theorem

Theorem 1 [22] *The zero solution of* (2.6) *is uniformly asymptotically stable if and only if*

$$z - A - \tilde{B}(z) \neq 0 \ \textit{for all} \ \ |z| \geq 1.$$

In the followcondition of zero solutions of (2.6) is given.

Theorem 2 [22] *Suppose that* $B(t)$ *does not change sign for* $t \in \mathbb{N}_1$*. Then the zero solution of* (2.6) *is asymptotically stable if*

$$|A| + \left| \sum_{t=0}^{\infty} B(t) \right| < 1.$$

3 Main Results

In this section firstly we will show that Eq. (1.4) is a Volterra equation of convolution type. Using the definition of Riemann–Liouville nabla fractional difference operator for Eq. (1.4) we obtain

$$\begin{aligned}
\nabla_{-1}^{v}x(t) &= \frac{1}{\Gamma(-v)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{-v-1}}x(s)\\
&= \sum_{s=0}^{t}\frac{1}{\Gamma(-v)}\frac{\Gamma(t-s+1-v-1)}{\Gamma(t-s+1)}x(s)\\
&= \sum_{s=0}^{t}\frac{\Gamma(t-s-v)}{\Gamma(-v)\Gamma(t-s+1)}x(s)\\
&= \sum_{s=0}^{t}\binom{t-s-v-1}{-v-1}x(s)\\
&= x(t)+\sum_{s=0}^{t-1}\binom{t-s-v-1}{-v-1}x(s)=\lambda x(t).
\end{aligned}$$

From this we have

$$x(t)=\frac{1}{\lambda-1}\sum_{s=0}^{t-1}\binom{t-s-v-1}{-v-1}x(s). \tag{3.1}$$

In Eq. (3.1) by replacing t with $t+1$, we reach

$$x(t+1)=\frac{1}{\lambda-1}\sum_{s=0}^{t}\binom{t-s-v}{-v-1}x(s), \tag{3.2}$$

which is in the form Eq. (2.6), where $A=0$ and

$$B(t)=\frac{1}{\lambda-1}\binom{t-v}{-v-1}. \tag{3.3}$$

Now we give our results in the following theorems.

Theorem 3 *The zero solution of* (1.4) *is uniformly asymptotically stable if and only if*

$$\lambda<0 \ \ or \ \ \lambda>2^{v}. \tag{3.4}$$

Proof Suppose that the zero solution of Eq. (1.4) is uniformly asymptotically stable. Consedering (2.8) and (2.9),we obtain the characteristic equation of (3.2)

$$g(z)=z-\widetilde{B}(z). \tag{3.5}$$

Now we will use Theorem 1 to find the interval of zeros of $g(z)$. Taking Z-transform of both sides of (3.3) and using Lemma 2 and binomial theorem respectively we obtain

$$\begin{aligned}
\widetilde{Z}(B(t)) = B(z) &= \frac{1}{\lambda-1}\sum_{s=0}^{\infty}\binom{s-v}{-v-1}z^{-s}\\
&= \frac{1}{\lambda-1}\left[\binom{-v}{-v-1}+\binom{1-v}{-v-1}z^{-1}+\binom{2-v}{-v-1}z^{-2}+\cdots\right]\\
&= \frac{1}{\lambda-1}\left[\binom{-v}{1}+\binom{1-v}{2}z^{-1}+\binom{2-v}{3}z^{-2}+\cdots\right]\\
&= \frac{z}{\lambda-1}\left[-vz^{-1}+\frac{(1-v)(-v)}{2!}z^{-2}+\frac{(2-v)(1-v)(-v)}{3!}z^{-3}+\cdots\right]\\
&= \frac{z}{\lambda-1}\left[-vz^{-1}-\frac{(v)(v-1)}{2!}z^{-2}-\frac{(v)(v-1)(v-2)}{3!}z^{-3}+\cdots\right]\\
&= \frac{z}{\lambda-1}\left[-1+1-vz^{-1}-\frac{(v)(v-1)}{2!}z^{-2}-\frac{(v)(v-1)(v-2)}{3!}z^{-3}+\cdots\right]\\
&= \frac{z}{\lambda-1}\left[\left(1-\frac{1}{z}\right)^{v}-1\right]. \qquad (3.6)
\end{aligned}$$

If we write Eq. (3.6) in Eq. (3.5) then we reach the characteristic equation of Eq. (3.2)

$$z-\frac{z}{\lambda-1}\left[\left(1-\frac{1}{z}\right)^{v}-1\right]=0. \qquad (3.7)$$

By Theorem 1, the zero solution of Eq. (3.2) is uniformly asymptoticaly stable if and only if

$$1-\frac{1}{\lambda-1}\left[\left(1-\frac{1}{z}\right)^{v}-1\right]\neq 0 \ \text{ for all } \ |z|\geq 1. \qquad (3.8)$$

Let z_r be a nonzero root of characteristic equation (3.7). Then z_r satisfies the equation

$$\left(1-\frac{1}{z_r}\right)^{v}=\lambda. \qquad (3.9)$$

We will study the Eq. (3.9) in two cases. First we assume that $\lambda<0$. It is clear that Eq. (3.9) has no root. Hence the condition (3.8) is satisfied. Secondly we assume that $\lambda>0$ $(\lambda\neq 1)$. From (3.9) we obtain

$$z_r=\frac{1}{1-\lambda^{\frac{1}{v}}}$$

which is the real nonzero root of Eq. (3.7). From this, we have $|z|=\frac{1}{\left|1-\lambda^{\frac{1}{v}}\right|}<1$. If $\left|z\right|=\frac{1}{\left|1-\lambda^{\frac{1}{v}}\right|}<1$, that is $\left|1-\lambda^{\frac{1}{v}}\right|>1$,then (3.8) is satisfied. From this we obtain $\lambda>2^{v}$. Therefore conditions (3.4) are satisfied.

Conversely suppose that $\lambda < 0$ or $\lambda > 2^v$. Then (3.8) is obviously satisfied. Therefore by Theorem 1 the zero solution of Eq. (1.4) is uniformly asymptotically stable. Hence the proof is completed. □

Theorem 4 *The zero solution of Eq.* (1.4) *is asymptotically stable if*

$$\lambda < 0 \ \ or \ \ \lambda > 2. \tag{3.10}$$

Proof We will make the proof using Theorem 2. For this, first we will show that $B(t)$ in (3.3) does not change sign for $t \in \mathbb{N}_1$. Since $\Gamma(1-v) = (-v)\Gamma(-v)$ we have

$$\begin{aligned} B(t) &= \frac{1}{\lambda - 1}\binom{t-v}{-v-1} \\ &= \frac{\Gamma(t-v+1)}{(\lambda-1)\,\Gamma(t+2)\Gamma(-v)} \\ &= \frac{-v\Gamma(t-v+1)}{(\lambda-1)\,\Gamma(t+2)\Gamma(1-v)}. \end{aligned} \tag{3.11}$$

Since $0 < v < 1$ and $\lambda \neq 1$ for $\forall t \in \mathbb{N}_0$ we obtain $\Gamma(t-v+1) > 0$, $\Gamma(t+2) > 0$ and $\Gamma(1-v) > 0$. Hence the sign of $\frac{-v\Gamma(t-v+1)}{(\lambda-1)\Gamma(t+2)\Gamma(1-v)}$ depends on only λ, not on t.

Consedering Lemma 1, from (3.11) we have

$$\begin{aligned} \lim_{k\to\infty}\binom{k-v+1}{-v} &= \lim_{k\to\infty}\frac{\Gamma(k-v+2)}{\Gamma(k+2)\Gamma(-v+1)} = \frac{1}{\Gamma(-v+1)}\lim_{k\to\infty}\frac{\Gamma(k-v+2)}{\Gamma(k+2)} \\ &= \frac{1}{\Gamma(-v+1)}\lim_{k\to\infty}k^{-v}\left[1+O\left(\frac{1}{k}\right)\right] = 0. \end{aligned} \tag{3.12}$$

Using binomial relation $\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+}$, from (3.12) we obtain

$$\begin{aligned} |A| + \left|\sum_{t=0}^{\infty}B(t)\right| &= \left|\sum_{t=0}^{\infty}\frac{1}{\lambda-1}\binom{t-v}{-v-1}\right| = \frac{1}{|\lambda-1|}\left|\lim_{k\to\infty}\sum_{t=0}^{k}\binom{t-v}{-v-1}\right| \\ &= \frac{1}{|\lambda-1|}\left|\lim_{k\to\infty}\sum_{t=0}^{k}\left[\binom{t-v+1}{-v}-\binom{t-v}{-v}\right]\right| \\ &= \frac{1}{|\lambda-1|}\left|\lim_{k\to\infty}\left[\binom{-v+1}{-v}-\binom{-v}{-v}\right.\right. \\ &\quad +\binom{-v+2}{-v}-\binom{1-v}{-v} \\ &\quad +\binom{-v+3}{-v}-\binom{2-v}{-v} \\ &\quad \vdots \end{aligned}$$

$$
\begin{aligned}
&+\binom{k-v+1}{-v}-\binom{k-v}{-v}\Bigg]\Bigg| \\
&= \frac{1}{|\lambda-1|}\left|\lim_{k\to\infty}\left[\binom{k-v+1}{-v}-1\right]\right| \\
&= \frac{1}{|\lambda-1|}.
\end{aligned}
$$

By Theorem 2, if $\frac{1}{|\lambda-1|} < 1$, then the zero solution of Eq. (1.4) is asymptotically stable. $\frac{1}{|\lambda-1|} < 1$ implies that $\lambda < 0$ or $\lambda > 2$. Hence the proof is completed. □

4 Examples

In the following examples, numerical datas of solutions are computed in Mathematica and the graphs are drawn

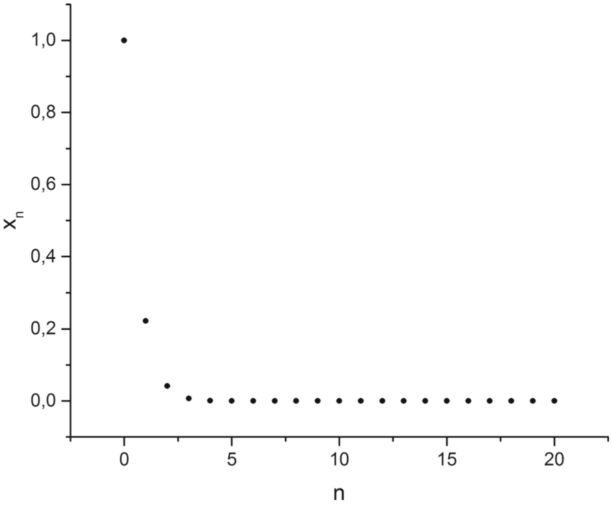
Example 1 Consider the fractional difference equation

$$\nabla_{-1}^{\frac{1}{3}} x(t) = 3x(t), \quad t \in \mathbb{N}_1, \tag{4.1}$$

with initial condition

$$\nabla_{-1}^{-\frac{2}{3}} x(t)\,|_{t=0} = 1, \tag{4.2}$$

where $v = \frac{1}{3}$ and $\lambda = 3$. Since $3 > 2^{\frac{1}{3}}$, by Theorem 3 the zero solution of Eq. (4.1) is uniformly asymptotically stable. The graph of solution is below.



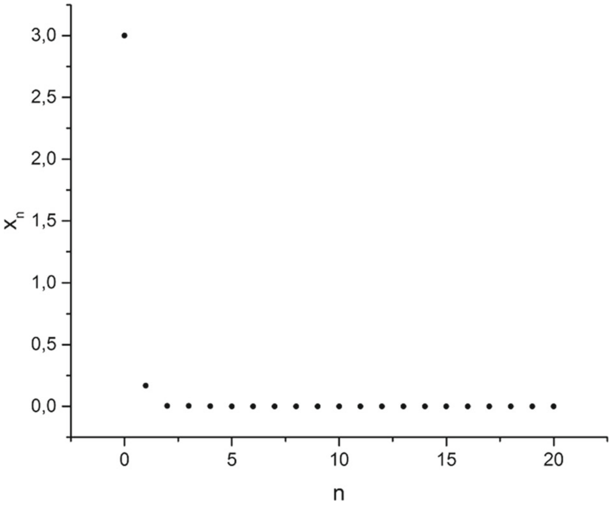
Example 2 Consider the fractional difference equation

$$\nabla_{-1}^{\frac{2}{3}} x(t) = 4x(t), \quad t \in \mathbb{N}_1, \tag{4.3}$$

with initial condition

$$\nabla_{-1}^{-\frac{1}{3}} x(t) \mid_{t=0} = 3, \tag{4.4}$$

where $v = \frac{2}{3}$and $\lambda = 4$. Since $\lambda = 4 > 2$, by Theorem 4 the zero solution of Eq. (4.3) is asymptotically stable. The graph of solution is below.



5 Conclusion

In this paper we investigated the stability of linear nabla fractional difference equations of order v where $0 < v < 1$. In this investigation, we first showed that Eq. (1.4) is a convolution-type Volterra equation. Then using some binomial relations and stability analysis methods of the convolution type Volterra equations we have obtained some stability results. We hope that using the results established above, system of linear and nonlinear fractional order difference equations can be handled.

References

1. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V., Amsterdam (2006)
3. Kiryakova, V.: Generalized Fractional Calculus and Applications. Longman Scientific & Technical, Harlow. Copublished in the United States with John Wiley & Sons Inc, New York (1994)
4. Lakshmikantham, V., Leela, S., Vasundhara Devi, J.: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
5. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
6. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
7. Abdeljawad, T., Al-Mdallalb, Q.M.: Discrete Mittag-Leffler kernel type fractional difference initial value problems and Gronwall's inequality. J. Comput. Appl. Math. **339**, 218–230 (2018)

8. Abdeljawad, T., Al-Mdallal, Q.M., Hajji, M.A.: Arbitrary order fractional difference operators with discrete exponential kernels and applications. Theor. Numeri. Results Fract. Differ. Differ. Equ. **2017**, Article ID 4149320
9. Atici, F.M., Sengül, S.: Modeling with fractional difference equations. J. Math. Anal. Appl. **369**, 1–9 (2010)
10. Coussot, C.: Fractional derivative models and their use in the characterization of hydropolymer and in-vivo breast tissue viscoelasticity. Master thesis, University of Illinois at Urbana-Champain (2008)
11. Diethelm, K., Freed, A.D.: On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity. In: Keil, F., Mackens, W., Voss, H., Werther, J. (eds.) Scientific Computing in Chemical Engineering II—Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, pp. 217–224. Springer, Heidelberg (1999)
12. Jumarie, G.: Stock exchange fractional dynamics defined as fractional exponential growth driven by (usual) Gaussian white noise. Application to fractional Black–Scholes equations. Insur. Math. Econ. **42**, 271–287 (2008)
13. Khan, A., Alshehri, H.M., Abdeljawad, T., Al-Mdallale, Q.M., Khan, H.: Stability analysis of fractional nabla difference COVID-19 model. Results Phys. **22**, 103888 (2021)
14. Mainardi, F.: Fractional calculus, some basic problems in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 291–348. Springer, Wien (1997)
15. Norton, L.A.: Gompertzian model of human breast cancer growth. Cancer Res. **48**, 7067–7071 (1998)
16. Qian, D.L.C., Agarwal, R.P., Wong, P.J.Y.: Stability analysis of fractional differential system with Riemann–Liouville derivative. Math. Comput. Model. **52**(5–6), 862–874 (2010)
17. Cermak, J., Kisela T., Nechvatal, L.: Stability and asymptotic properties of linear fractional difference equation. Adv. Differ. Equ. **2012**:122 (2012)
18. Saris, R.A., Al-Mdall, Q.: On the asymptotic stability of linear system of fractional-order difference equations. Fract. Calculus Appl. Anal. **16**(3) (2013)
19. Goodrich, C., Peterson, A.: Discrete Fractional Calculus. Springer, Berlin (2015)
20. Abdeljavad, T., Atıcı, F.: On the definitions of nabla fractional operators. Abstr. Appl. Anal. **2012**, Article ID 406757, 13 pages (2012)
21. Thabet, A.: On Riemann and Caputo fractional differences. Comput. Math. Appl. **62**, 1602–1611 (2011)
22. Elaydi, S.: An Introduction to Difference Equations, 3rd edn. Springer, Berlin (2000)