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Oscillation Criteria for Nonlinear Higher-Order Forced Functional Difference Equations

Jehad Alzabut · Yasar Bolat

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Abstract Some oscillation criteria for solutions of nonlinear higher-order forced difference equations are established. The investigations are carried out without assuming that the coefficients of the equations are of a definite sign and by showing that the forcing term needs not be the mth difference of an oscillatory function. The proposed results are supported with some examples.

Keywords Forced oscillation · Higher order · Functional difference equations

Mathematics Subject Classification (2010) 39A10

1 Introduction and Preliminaries

It has been believed that the difference equations provide adequate natural descriptions for many observed evolution phenomena. This is justified because most measurements of time-evolving variables are discrete. Therefore, these equations are in their own right important mathematical models. Specifically, difference equations have extensive applications in computer, probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, electrical networks, and genetics in biology, economics, psychology, and sociology; see the remarkable monographs [1, 2]. The oscillation

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J. Alzabut (✉)
Department of Mathematics and Physical Sciences, Prince Sultan University, P.O. Box 66388, Riyadh 11586, Saudi Arabia
e-mail: jalzabut@psu.edu.sa

Y. Bolat
Department of Mathematics, Faculty of Science and Literatures, Kastamonu University, 37110 Kastamonu, Turkey
e-mail: ybolat@kastamonu.edu.tr

of delay difference equations, in particular, often appears in population growth with competitive species. The oscillation of first, second, and third-order forced delay difference equations has been studied by a variety of authors. For recent contributions to the literature, see for instance the papers [3–5, 10]. However, relatively few oscillation results are known for higher-order forced delay difference equations [6, 8, 9, 11–13].

In [11], Sun and Saker considered equations of the form

$$\Delta^m x(n) + q(n)g(x(n-\tau)) = f(n) \tag{1}$$

where $m \geq 1$ and $\tau \geq 0$ are integers, and $q(n)$ and $f(n)$ are real sequences defined for $n \geq 0$. No restrictions have been assumed on the forcing term f. They proved their results for $q(n) \geq 0$ and $q(n) < 0$, and $q(n)$ is oscillatory under the condition $xg(x) > 0$ for $x \neq 0$. In [9], Rath et al. established sufficient conditions for the oscillatory behavior of the following equation:

$$\Delta^m \left(x(n) - p(n)x(\tau(n))\right) + q(n)g(x(\sigma(n))) = f(n) \tag{2}$$

where $m \geq 1$ is an integer, and $p(n), q(n)$, and $f(n)$ are real sequences defined for $n \geq 0$ with $q(n) \geq 0$. The results are obtained by assuming that there exists a bounded sequence $F(n)$ such that $\Delta^m F(n) = f(n)$ and $\lim_{n\to\infty} F(n) = 0$. That is, the forcing term is mth difference of an oscillatory function. In [6], the authors considered the equation

$$\Delta^m \left(x(n) - p(n)x(n-\tau)\right) + q_1(n)\phi_\alpha(n-\sigma_1) + q_2(n)\phi_\beta(n-\sigma_2) = f(n) \tag{3}$$

where $m \geq 1, \tau, \sigma_1$, and σ_2 are integers, $\Phi_*(u) = |u|^* u$; $p(n), q_1(n), q_2(n)$, and $f(n)$ are real sequences defined for $n \geq 0$ with $p(n) > 0$, and $0 < \alpha < 1 < \beta$ are constants.

The principal purpose of this paper is to extend the above mentioned results and establish oscillation criteria for nonlinear higher-order forced difference equations of the forms

$$\sum_{i=1}^{m} a_i \Delta^i x(n) + \Phi\left(n, x(h(n)), x(g(n)), x(l(n))\right) = f(n) \tag{4}$$

and

$$\sum_{i=1}^{m} a_i \Delta^i x(n) - \Phi\left(n, x(h(n)), x(g(n)), x(l(n))\right) = f(n). \tag{5}$$

The investigations are carried out without assuming that the coefficients of the equations are of a definite sign and by showing that the forcing term f needs not be the mth difference of an oscillatory function. The proposed results are supported with some examples.

Let $\mathbb{N}_{n_0} = \{n_0, n_0+1, \ldots\}$. We shall consider (4) (or (5)) where a_i values are real numbers with $a_m \equiv 1$ and $h, g, l, f : \mathbb{N}_{n_0} \to \mathbb{Z}$ are real sequences satisfying $\lim_{n\to\infty} h(n) = \lim_{n\to\infty} g(n) = \lim_{n\to\infty} l(n) = \infty$. The function $\Phi : \mathbb{N}_{n_0} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is a sequence satisfying conditions (C1) and (C4) to be specified in the following section. By Δ^i, we mean $\Delta^i x(n) = \Delta(\Delta^{i-1} x(n))$ for $i = 2, 3, \ldots, m$ such that $\Delta x(n) = x(n+1) - x(n)$. Let $\nu = \min_{s\in\mathbb{Z}}\{h(s), g(s), l(s)\}$ and $n_0 \in \mathbb{Z}$ be fixed. By a solution of (4) (or (5)), we mean a real sequence $x(n)$ defined for all $n \geq \nu$ and satisfying (4) (or (5)) for $n \geq n_0$. The nontrivial solution of (4) (or (5)) is of our interest in this paper. Such a solution $x(n)$ is said to be *oscillatory* if for any $N \geq n_0$, there exists $n > N$ such that $x(n+1)x(n) \leq 0$. Otherwise, the solution is said to be *nonoscillatory*.

Let $H(n,s)$ be a nonnegative kernel defined on $\mathbb{D} := \{(n,s) : n \geq s \geq n_0\}$. We assume that $H(n,s)$ satisfies the following conditions:

(H.1) $H(n,n) = 0$ and $H(n,s) \geq 0$ for $n \geq s \geq n_0$,
(H.2) $H_i(n,s) = (-1)^i \Delta_s^i H(n,s),\ i = 0,1,\dots,m$ for $n > s \geq n_0$,
(H.3) $H_i(n,n) = 0,\ i = 0,1,\dots,m-1$,
(H.4) $\frac{H_i(n,n_0)}{H(n,n_0)} = O(1)$ as $n \to \infty$ for $i = 1,2,\dots,m-1$.

In what follows, we shall use the notation

$$\Phi_n := \Phi(n, x(h(n)), x(g(n)), x(l(n)))$$

unless stated otherwise. The following two facts will be also used in the sequel:

Fact 1. The function $F(x) := px^{\alpha-\gamma} - qx^{\beta-\gamma}$ for $x > 0$, $p,q > 0$, and $\alpha > \beta > \gamma > 0$ attains its minimum at

$$x = (\alpha-\gamma)^{1/(\beta-\alpha)}(\beta-\gamma)^{1/(\alpha-\beta)} p^{(\gamma-\beta)/((\alpha-\beta)(\alpha-\gamma))} q^{1/(\alpha-\beta)}$$

and thus

$$F_{\min} = -\sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)} q^{(\alpha-\gamma)/(\alpha-\beta)}$$

where $\sigma_1 := (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)}$.

Fact 2. The function $F(x) := px^{\alpha-\gamma} + qx^{\beta-\gamma}$ for $x > 0$, $p,q > 0$, and $\alpha > \gamma > \beta > 0$ attains its minimum at

$$x = (\alpha-\gamma)^{1/(\beta-\alpha)}(\gamma-\beta)^{1/(\alpha-\beta)} p^{(\gamma-\beta)/((\alpha-\beta)(\alpha-\gamma))} q^{1/(\alpha-\beta)}$$

and thus

$$F_{\min} = \sigma_2 p^{(\gamma-\beta)/(\alpha-\beta)} q^{(\alpha-\gamma)/(\alpha-\beta)}$$

where $\sigma_2 := (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\gamma-\beta)^{(\beta-\gamma)/(\alpha-\beta)}$.

2 Main Results

In this section, we shall provide sufficient conditions for the oscillation of (4) and (5) and their particular cases

$$\sum_{i=1}^{m} a_i \Delta^i x(n) + r(n)\psi_\gamma(x(h(n))) + p(n)\psi_\alpha(x(g(n))) + q(n)\psi_\beta(x(l(n))) = f(n) \quad (6)$$

and

$$\sum_{i=1}^{m} a_i \Delta^i x(n) - r(n)\psi_\gamma(x(h(n))) + p(n)\psi_\alpha(x(g(n))) + q(n)\psi_\beta(x(l(n))) = f(n) \quad (7)$$

where $\psi_\eta(u) := |u|^{\eta-1}u,\ \ \eta > 0$; conditions (C1) and (C4) are satisfied respectively, $r,q : [n_0,\infty) \to \mathbb{Z}$ and $p : [n_0,\infty) \to \mathbb{Z}^+$.

Theorem 1 *If there exists a nonnegative sequence* $G_1(n,s)$ *on* $[n_0,\infty)\times[n_0,\infty)$ *such that*

$$\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) + H(n,s)\Phi(s,y,z,w) \begin{cases} \geq G_1(n,s) & \text{if } x,y,z,w > 0, \\ \leq -G_1(n,s) & \text{if } x,y,z,w < 0 \end{cases} \quad \text{(C1)}$$

for $n > s \geq n_0$,

$$\limsup_{n\to\infty} \frac{1}{H(n,n_0)} \sum_{s=n_0}^{n-1} (H(n,s)f(s) + G_1(n,s)) = \infty \tag{C2}$$

and

$$\liminf_{n\to\infty} \frac{1}{H(n,n_0)} \sum_{s=n_0}^{n-1} (H(n,s)f(s) - G_1(n,s)) = -\infty, \tag{C3}$$

then every solution of (4) *is oscillatory.*

Proof For the sake of contradiction, we assume that (4) has a non-oscillatory solution on $[n_0, \infty)$. Without loss of generality, there is a solution $x(n)$ of (4) and $N \in [n_0, \infty)$ such that $x(n) > 0$ on $[N, \infty)$. Multiplying both sides of (4) by $H(n, s)$ and summing up with respect to s from N to $n-1$, we obtain

$$\begin{aligned}\sum_{s=N}^{n-1} H(n,s)f(s) &= \sum_{s=N}^{n-1} H(n,s)\sum_{i=1}^{m} a_i \Delta^i x(s) + \sum_{s=N}^{n-1} H(n,s)\Phi_s \\ &= \sum_{i=1}^{m} a_i \sum_{s=N}^{n-1} H(n,s)\Delta^i x(s) + \sum_{s=N}^{n-1} H(n,s)\Phi_s.\end{aligned} \tag{8}$$

By virtue of assumptions (H.1), (H.2), and (H.3), we observe that

$$\begin{aligned}\sum_{s=N}^{n-1} H(n,s)\Delta^i x(s) &= -H(n,N)\Delta^{i-1}x(N) - \sum_{j=1}^{i-1} H_j(n,N)\Delta^{i-j-1}x(N) \\ &\quad + \sum_{s=N}^{n-1} H_i(n,s)x(s+1),\end{aligned}$$

where $\sum_{j=1}^{0} = 0$. Thus, (8) becomes

$$\begin{aligned}\sum_{s=N}^{n-1} H(n,s)f(s) &= -\sum_{i=1}^{m} a_i \left[H(n,N)\Delta^{i-1}x(N) + \sum_{j=1}^{i-1} H_j(n,N)\Delta^{i-j-1}x(N) \right] \\ &\quad + \sum_{s=N}^{n-1} \left[\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) + H(n,s)\Phi_s \right].\end{aligned} \tag{9}$$

In view of (H.4), there exists a constant $M_1 > 0$ such that

$$-\sum_{i=1}^{m} a_i \left[H(n,N)\Delta^{i-1}x(N) + \sum_{j=1}^{i-1} H_j(n,N)\Delta^{i-j-1}x(N) \right] \geq M_1 H(n,N).$$

Thus, we may write (9) as follows:

$$\sum_{s=N}^{n-1} H(n,s)f(s) \geq M_1 H(n,N) + \sum_{s=N}^{n-1} \left[\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) + H(n,s)\Phi_s \right].$$

From condition (C1), it follows that

$$\frac{1}{H(n,N)}\sum_{s=N}^{n-1}(H(n,s)f(s)-G_1(n,s))\geq M_1,$$

which leads to a contradiction to (C3). The same arguments can be followed in case we assume that there is a solution of (4) and $N\in[n_0,\infty)$ such that $x(n)<0$ on $[N,\infty)$. We end up with a contradiction to condition (C2). The proof is complete. ☐

Theorem 2 *If there exists a nonnegative sequence* $G_2(n,s)$ *on* $[n_0,\infty)\times[n_0,\infty)$ *such that*

$$\sum_{i=1}^{m}a_iH_i(n,s)x(s+1)-H(n,s)\Phi(s,y,z,w)\begin{cases}\leq G_2(n,s) & \text{if } x,y,z,w>0,\\ \geq -G_2(n,s) & \text{if } x,y,z,w<0\end{cases}\tag{C4}$$

for $n>s\geq n_0$,

$$\limsup_{n\to\infty}\frac{1}{H(n,n_0)}\sum_{n_0}^{n-1}\big(H(n,s)f(s)-G_2(n,s)\big)=\infty\tag{C5}$$

and

$$\liminf_{n\to\infty}\frac{1}{H(n,n_0)}\sum_{n_0}^{n-1}\big(H(n,s)f(s)+G_2(n,s)\big)=-\infty,\tag{C6}$$

then every solution of (5) *is oscillatory.*

Proof For the sake of contradiction, we assume that (5) has a nonoscillatory solution on $[n_0,\infty)$. Without loss of generality, there is a solution x of (5) and $N\in[n_0,\infty)$ such that $x(n)>0$ on $[N,\infty)$. Proceeding as in the proof of Theorem 1, we arrive at

$$\begin{aligned}\sum_{s=N}^{n-1}H(n,s)f(s) &= -\sum_{i=1}^{m}a_i\left[H(n,N)\Delta^{i-1}x(N)+\sum_{j=1}^{i-1}H_j(n,N)\Delta^{i-j-1}x(N)\right]\\ &\quad+\sum_{s=N}^{n-1}\left[\sum_{i=1}^{m}a_iH_i(n,s)x(s+1)-H(n,s)\Phi_s\right].\end{aligned}$$

By virtue of (H.4), there exists a constant $M_2>0$ such that

$$-\sum_{i=1}^{m}a_i\left[H(n,N)\Delta^{i-1}x(N)+\sum_{j=1}^{i-1}H_j(n,N)\Delta^{i-j-1}x(N)\right]\leq M_2H(n,N),$$

which implies that

$$\sum_{s=N}^{n-1}H(n,s)f(s)\leq M_2H(n,N)+\sum_{s=N}^{n-1}\left[\sum_{i=1}^{m}a_iH_i(n,s)x(s+1)-H(n,s)\Phi_s\right].$$

The last inequality and (C4) imply that

$$\frac{1}{H(n,N)}\sum_{s=N}^{n-1}(H(n,s)f(s)-G_2(n,s))\leq M_2.$$

To this end, we reach a contradiction to (C5). In case we assume that there is a solution of (5) and $N\in[n_0,\infty)$ such that $x(n)<0$ on $[N,\infty)$, then by repeating the same arguments, we obtain a contradiction to condition (C6). The proof is complete. ☐

Throughout the remaining parts of the paper, we define

$$k_+ := \max\{0, k\}, \quad k_- := \max\{0, -k\},$$

and

$$h(n,s) := \sum_{i=1}^{m} a_i \frac{H_i(n,s)}{H^{1/\gamma}(n,s)}, \quad n \geq s \geq n_0.$$

Theorem 3 *Let $h(n) \equiv g(n) \equiv l(n) \equiv n$ on $[n_0, \infty)$. Assume that $0 < \gamma < 1$ and $\alpha > \beta > \gamma$ hold and $\sum_{i=1}^{m} a_i H_i(n,s) > 0$ for $n \geq s \geq n_0$. If (C2) and (C3) are satisfied where*

$$G_1(n,s) := (\gamma - 1)\gamma^{\gamma/(1-\gamma)} |Q_1(s)|^{1/(1-\gamma)} |h(n,s)|^{\gamma/(\gamma-1)},$$

and

$$Q_1(s) := r_-(s) + \sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)}(s) q_-^{(\alpha-\gamma)/(\alpha-\beta)}(s)$$

then every solution of (6) is oscillatory.

Proof For the sake of contradiction, we assume that (6) has a non-oscillatory solution on $[n_0, \infty)$. Then, without loss of generality, there is a solution $x(n)$ of (6) and $N \in [n_0, \infty)$ such that $x(n) > 0$ on $[N, \infty)$. We claim that (C1) is satisfied for $x = x(s), y = x(h(s)), z = x(g(s))$, and $w = x(l(s))$. Proceeding as in the proof of Theorem 1, we get

$$\begin{aligned}\sum_{i=1}^{m} a_i H_i(n,s) x(s+1) + H(n,s)\Phi_s &= \sum_{i=1}^{m} a_i H_i(n,s) x(s+1) \\ &\quad + H(n,s)\left[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\right] \\ &\geq \sum_{i=1}^{m} a_i H_i(n,s) x(s+1) - H(n,s) r_-(s) x^{\gamma}(s) \\ &\quad + H(n,s) x^{\gamma}(s)(p(s)x^{\alpha-\gamma}(s) - q_-(s)x^{\beta-\gamma}(s)).\end{aligned}$$

In view of Fact 1 , we deduce

$$\sum_{i=1}^{m} a_i H_i(n,s) x(s+1) + H(n,s)\Phi_s \geq \sum_{i=1}^{m} a_i H_i(n,s) x(s+1) - H(n,s) Q_1(s) x^{\gamma}(s). \quad (10)$$

Define $X \geq 0$ and $Y > 0$ by

$$X^{\gamma} := H Q_1 x^{\gamma} \quad \text{and} \quad Y^{\gamma-1} := \gamma^{-1} Q_1^{-1/\gamma} h.$$

By using the inequality in [7]

$$\gamma X Y^{\gamma-1} - X^{\gamma} \geq (\gamma - 1) Y^{\gamma}, \quad 0 < \gamma < 1,$$

we may write (10) in the form

$$\sum_{i=1}^{m} a_i H_i(n,s) x(s+1) + H(n,s)\Phi_s \geq (\gamma - 1)\gamma^{\gamma/(1-\gamma)} Q_1^{1/(1-\gamma)}(s) h^{\gamma/(\gamma-1)}(n,s).$$

However, the right hand side of the above inequality is nothing but $G_1(n,s)$. Therefore, by Theorem 1, we end up with the desired result. □

Theorem 4 *Let $h(n) \equiv n$ and $l(n) \equiv g(n)$ on $[n_0, \infty)$. Assume that $0 < \gamma < 1$ and $\alpha > \beta > 0$ hold and $\sum_{i=1}^{m} a_i H_i(n,s) > 0$ for $n \geq s \geq n_0$. If (C2) and (C3) are satisfied,* *where*

$$G_1(n,s) := (\gamma-1)\gamma^{\gamma/(1-\gamma)} r_-^{1/(1-\gamma)}(s) h^{\gamma/(\gamma-1)}(n,s) + \delta H(n,s) p^{\beta/(\beta-\alpha)}(s) q_-^{\alpha/(\alpha-\beta)}(s)$$

and

$$\delta := (\beta-\alpha)\alpha^{\alpha/(\beta-\alpha)}\beta^{\beta/(\alpha-\beta)},$$

then every solution of (6) *is oscillatory.*

Proof For the sake of contradiction, we assume (6) has a non-oscillatory solution on $[n_0,\infty)$. Then, without loss of generality, there is a solution $x(n)$ of (6) and $N \in [n_0,\infty)$ such that $x(n) > 0$ on $[N,\infty)$. We claim that (C1) holds for $x = x(s)$, $y = x(h(s))$, $z = x(g(s))$, and $w = x(l(s))$. Proceeding as in the proof of Theorem 1, we get

$$\begin{aligned}\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) + H(n,s)\Phi_s &= \sum_{i=1}^{m} a_i H_i(n,s)x(s+1)\\ &\quad + H(n,s)\left[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\right]\\ &\geq \sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)r_-(s)x^{\gamma}(s)\\ &\quad + H(n,s)(p(s)x^{\alpha}(g(s)) - q_-(s)x^{\beta}(g(s))). \end{aligned} \tag{11}$$

Similar to the steps followed in Theorem 3, we observe that

$$px^{\alpha} - q_-x^{\beta} \geq \delta p^{\beta/(\beta-\alpha)} q_-^{\alpha/(\alpha-\beta)}, \tag{12}$$

where $\delta = (\beta-\alpha)\alpha^{\alpha/(\beta-\alpha)}\beta^{\beta/(\alpha-\beta)}$. Therefore,

$$\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)r_-(s)x^{\gamma}(s) \geq (\gamma-1)\gamma^{\gamma/(1-\gamma)} r_-^{1/(1-\gamma)}(s) h^{\gamma/(\gamma-1)}(n,s). \tag{13}$$

Using (12) and (13) in the right hand side of (11), we have

$$\begin{aligned}\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) + H(n,s)\Phi_s &\geq (\gamma-1)\gamma^{\gamma/(1-\gamma)} r_-^{1/(1-\gamma)}(s) h^{\gamma/(\gamma-1)}(n,s)\\ &\quad + \delta H(n,s) p^{\beta/(\beta-\alpha)} q_-^{\alpha/(\alpha-\beta)} = G_1(n,s).\end{aligned}$$

From Theorem 1, we deduce that every solution of (6) is oscillatory. □

Corollary 1 *Let* $q(n)$ *be a nonnegative sequence and* $h(n) \equiv n$ *on* $[n_0,\infty)$. *Assume that* $0 < \gamma < 1$ *and* $\sum_{i=1}^{m} a_i H_i(n,s) > 0$ *for* $n \geq s \geq n_0$ *hold. If* (C2) *and* (C3) *are satisfied where*

$$G_1(n,s) := (\gamma-1)\gamma^{\gamma/(1-\gamma)} r_-^{1/(1-\gamma)}(s) h^{\gamma/(\gamma-1)}(n,s),$$

then every solution of (6) *is oscillatory.*

Corollary 2 *Let* $r(n)$ *and* $q(n)$ *be nonnegative sequences on* $[n_0,\infty)$. *Assume that* $\sum_{i=1}^{m} a_i H_i(n,s) \geq 0$ *for* $n \geq s \geq n_0$ *holds. If* (C2) *and* (C3) *are satisfied, then every solution of* (6) *is oscillatory.*

Theorem 5 *Let* $q(n)$ *be a nonnegative sequence and* $h(n) \equiv g(n) \equiv l(n) \equiv n$ *on* $[n_0,\infty)$. *Assume that* $\gamma > 1$, $\alpha > \gamma > \beta > 0$ *and* $Q_2(n) < 0$ *for* $n \geq n_0$. *If* (C5) *and* (C6) *are* *satisfied where*

$$G_2(n,s) := (\gamma-1)\gamma^{\gamma/(1-\gamma)}|Q_2(s)|^{1/(1-\gamma)}|h(n,s)|^{\gamma/(\gamma-1)},$$

and

$$Q_2(s) := -r(s) - \sigma_2 p^{(\gamma-\beta)/(\alpha-\beta)}(s) q^{(\alpha-\gamma)/(\alpha-\beta)}(s)$$

then every solution of (7) *is oscillatory.*

Proof For the sake of contradiction, we assume that (7) has a non-oscillatory solution on $[n_0,\infty)$. Then, without loss of generality, there is a solution $x(n)$ of (7) and a $n \in [n_0,\infty)$ such that $x(n) > 0$ on $[n,\infty)$. We claim that (C4) is satisfied for $x = x(s)$, $y = x(h(s))$, $z = x(g(s))$, and $w = x(l(s))$. Proceeding as in the proof of Theorem 2, we get

$$\begin{aligned}\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)\Phi_s &= \sum_{i=1}^{m} a_i H_i(n,s)x(s+1)\\ &- H(n,s)\left[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\right]\\ &\le \left|\sum_{i=1}^{m} a_i H_i(n,s)\right| x(s+1) - H(n,s)r(s)x^{\gamma}(s)\\ &- H(n,s)x^{\gamma}(s)(p(s)x^{\alpha-\gamma}(s) + q(s)x^{\beta-\gamma}(s)).\end{aligned} \tag{14}$$

In view of Fact 2, (14) becomes

$$\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)\Phi_s \le \left|\sum_{i=1}^{m} a_i H_i(n,s)\right| x(s+1) - H(n,s)|Q_2(s)|x^{\gamma}(s).$$

Define $X \ge 0$ and $Y \ge 0$ by

$$X^{\gamma} := H|Q_2|x^{\gamma}, \quad \text{and} \quad Y^{\gamma-1} := \gamma^{-1}|Q_2|^{-1/\gamma}|h|.$$

By employing the inequality in [7]

$$\gamma XY^{\gamma-1} - X^{\gamma} \le (\gamma-1)Y^{\gamma}, \quad \gamma > 1,$$

we obtain

$$\begin{aligned}&\left|\sum_{i=1}^{m} a_i H_i(n,s)\right| x(s+1) - H(n,s)|Q_2(s)|x^{\gamma}(s)\\ &\quad \le (\gamma-1)\gamma^{\gamma/(1-\gamma)}|Q_2(s)|^{1/(1-\gamma)}|h(n,s)|^{\gamma/(\gamma-1)}\\ &\quad = G_2(n,s).\end{aligned}$$

Then, from Theorem 2, we get the desired result. □

Theorem 6 *Let* $h(n) \equiv g(n) \equiv l(n) \equiv n$ *on* $[n_0,\infty)$. *Assume that* $\gamma > 1, \alpha > \beta > \gamma > 0$ *and* $Q_3(n) < 0$ *for* $n \ge n_0$. *If* (C5) *and* (C6) *are satisfied where*

$$\begin{aligned}G_2(n,s) &:= (\gamma-1)\gamma^{\gamma/(1-\gamma)}|Q_3(s)|^{1/(1-\gamma)}|h(n,s)|^{\gamma/(\gamma-1)},\\ Q_3(s) &:= -r(s) + \sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)}(s) q_{-}^{(\alpha-\gamma)/(\alpha-\beta)}(s)\end{aligned}$$

and

$$\sigma_1 := (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)}$$

then every solution of (7) *is oscillatory.*

Proof For the sake of contradiction, we assume (7) has a non-oscillatory solution on $[n_0, \infty)$. Then, without loss of generality, there is a solution $x(n)$ of (7) and $n \in [n_0, \infty)$ such that $x(n) > 0$ on $[n, \infty)$. We claim that (C4) holds for $x = x(s), y = x(h(s)), z = x(g(s)),$ *and* $w = x(l(s))$. Proceeding as in the proof of Theorem 2, we get

$$\begin{aligned}\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)\Phi_s &= \sum_{i=1}^{m} a_i H_i(n,s)x(s+1)\\ &\quad -H(n,s)\left[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\right]\\ &\le \left|\sum_{i=1}^{m} a_i H_i(n,s)\right| x(s+1) - H(n,s)r(s)x^{\gamma}(s)\\ &\quad -H(n,s)x^{\gamma}(s)(p(s)x^{\alpha-\gamma}(s) - q_-(s)x^{\beta-\gamma}(s)).\end{aligned}$$

However, we observe that

$$px^{\alpha-\gamma} - p_- x^{\beta-\gamma} \ge -\sigma_1 p^{(\gamma-\beta)/(\alpha-\beta)} q_-^{(\alpha-\gamma)/(\alpha-\beta)},$$

which implies

$$\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)\Phi_s \le \left|\sum_{i=1}^{m} a_i H_i(n,s)\right| x(s+1) - H(n,s)|Q_3(s)|x^{\gamma}(s).$$

The rest of the proof is the same as the proof of Theorem 5 with Q_2 replaced by Q_3. □

Theorem 7 *Let $r(n)$ be a positive sequence, and $h(n) \equiv n$ and $l(n) \equiv g(n)$ on $[n_0, \infty)$. Assume that $\gamma > 1$ and $\alpha > \beta > 0$. If (C5) and (C6) are satisfied, where*

$$\begin{aligned}G_2(n,s) := &(\gamma-1)\gamma^{\gamma/(1-\gamma)}|r(s)|^{1/(1-\gamma)}|h(n,s)|^{\gamma/(\gamma-1)}\\ &-\delta H(n,s)p^{\beta/(\beta-\alpha)}(s)q_-^{\alpha/(\alpha-\beta)}(s)\end{aligned}$$

and

$$\delta := (\beta-\alpha)\alpha^{\alpha/(\beta-\alpha)}\beta^{\beta/(\alpha-\beta)}$$

then every solution of (7) is oscillatory.

Proof For the sake of contradiction, we assume that (7) has a non-oscillatory solution on $[n_0, \infty)$. Then, without loss of generality, there is a solution $x(n)$ of (7) and $n \in [n_0, \infty)$ such that $x(n) > 0$ on $[n, \infty)$. As in the proof of Theorem 2, we get

$$\begin{aligned}\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)\Phi_s &= \sum_{i=1}^{m} a_i H_i(n,s)x(s+1)\\ &\quad -H(n,s)\left[r(s)x^{\gamma}(s) + (p(s)x^{\alpha}(s) + q(s)x^{\beta}(s))\right]\\ &\le \left|\sum_{i=1}^{m} a_i H_i(n,s)\right| x(s+1) - H(n,s)|r(s)|x^{\gamma}(s)\\ &\quad -H(n,s)(p(s)x^{\alpha}(g(s)) - q_-(s)x^{\beta}(g(s))).\end{aligned}$$

We observe that

$$px^{\alpha} - q_- x^{\beta} \ge \delta p^{\beta/(\beta-\alpha)} q_-^{\alpha/(\alpha-\beta)}.$$

As in the proof of Theorem 5, we get

$$\left|\sum_{i=1}^{m} a_i H_i(n,s)\right| x(s+1) - H(n,s)|r(s)|x^{\gamma}(s) \le (\gamma-1)\gamma^{\gamma/(1-\gamma)}|r(s)|^{1/(1-\gamma)}|h(n,s)|^{\gamma/\gamma-1}.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{m} a_i H_i(n,s)x(s+1) - H(n,s)\Phi_s &\le (\gamma-1)\gamma^{\gamma/(1-\gamma)}|r(s)|^{1/(1-\gamma)}|h(n,s)|^{\gamma/\gamma-1}\\ &\quad -H(n,s)\delta p^{\beta/(\beta-\alpha)} q_-^{\alpha/(\alpha-\beta)}\\ &= G_2(n,s),\end{aligned}$$

which implies that every solution of (7) is oscillatory. □

Corollary 3 *Let $r(n)$ be a nonnegative sequence and $h(n) \equiv n$ on $[n_0,\infty)$. Assume that $\gamma > 1$ holds. If (C5) and (C6) are satisfied, where*

$$G_2(n,s) := (\gamma-1)\gamma^{\gamma/(1-\gamma)} r^{1/(1-\gamma)}(s)|h(n,s)|^{\gamma/\gamma-1},$$

then every solution of (7) is oscillatory.

Corollary 4 *Let $r(n)$ and $q(n)$ be nonnegative sequences. We assume that $\sum_{i=1}^{m} a_i H_i(n,s) \le 0$ for $n \ge s \ge n_0$ holds. If (C5) and (C6) are satisfied, then every solution of (7) is oscillatory.*

3 Examples and Remarks

Example 1 Consider the equation

$$\begin{aligned}&\sum_{i=1}^{m} a_i \Delta^i x(n) + r(n)\psi_\gamma(x(n)) + n^{\frac{\mu}{3}}\psi_3(x(g(n)))\\ &\qquad + n^{\frac{8\mu}{27}}\cos^{\frac{1}{9}}(n)\psi_{\frac{8}{3}}(x(g(n))) = n^{\mu}\cos(n),\end{aligned} \tag{15}$$

where $0 < \gamma < 1, r(n) \ge 0$ for $n \ge n_0, a_i \ge 0,\ i = 1,2,\ldots,m-1, a_m \equiv 1, \alpha = 3,\ p(n) = n^{\frac{\mu}{3}}, \beta = \frac{8}{3},\ q(n) = n^{\frac{8\mu}{27}}\cos^{\frac{1}{9}}(n)$, and $f(n) = n^{\mu}\cos(n)$. By choosing $H(n,s) = (n-s)^m$, it is easy to see that H satisfies conditions (H.1)–(H.4) and $\sum_{i=1}^{m} a_i H_i(n,s) > 0$. By virtue of Theorem 4, every solution of (15) is oscillatory if $\mu > m$.

Example 2 Consider the equation

$$\begin{aligned}&\sum_{i=1}^{m} a_i \Delta^i x(n) + r(n)\psi_\gamma(x(h(n))) + p(n)\psi_\alpha(x(g(n)))\\ &\qquad + q(n)\psi_\beta(x(l(n))) = n^{\mu}\sin(n),\end{aligned} \tag{16}$$

where $m \ge 2,\ \mu,\ a_i \ge 0,\ i = 1,2,\ldots,m-1$, and $a_m \equiv 1$. Let $r(n)$ and $q(n)$ be nonnegative sequences on $[n_0,\infty)$. By choosing $H(n,s) = (n-s)^{\lambda},\ \lambda > m-1$, it is easy to see that H satisfies conditions (H.1)–(H.4) and $\sum_{i=1}^{m} a_i H_i(n,s) \ge 0$. By Corollary 2, every solution of (16) is oscillatory if $\mu > \lambda$.

Example 3 Consider the equation

$$\begin{aligned}&\Delta x(n) + r(n)\psi_\gamma(x(n)) - e^{\frac{5n}{3}}\psi_{\frac{7}{3}}(x(g(n)))\\ &\qquad - e^{\frac{10}{21}n}\sin^{\frac{5}{7}}(n)\psi_{\frac{2}{3}}(x(g(n))) = e^{n}\sin(n),\end{aligned} \tag{17}$$

where $\gamma > 1$ and $r(n) < 0$ for $n \geq n_0$, $a_1 \equiv 1$, $p(n) = -e^{\frac{5n}{3}}$, $\alpha = \frac{7}{3}$, $q(n) = -e^{\frac{10}{21}n} \sin^{\frac{5}{7}}(n)$, $\beta = \frac{2}{3}$, and $f(n) = e^n \sin(n)$. By choosing $H(n,s) = 1$, $n > s \geq n_0$, one can easily see that H satisfies (H.1)–(H.4). By the consequence of Theorem 7, every solution of (17) is oscillatory.

Example 4 Consider the equation

$$\sum_{i=1}^{m} a_i \Delta^i x(n) - r(n)\psi_\gamma(x(h(n))) + p(n)\psi_\alpha(x(g(n))) \\ + q(n)\psi_\beta(x(l(n))) = e^n \sin(n), \tag{18}$$

where $m \geq 2$, $a_i \leq 0$, $i = 1, 2, \dots, m-1$, and $a_m \equiv 1$. Let $r(n)$ and $q(n)$ be nonnegative sequences. By choosing $H(n,s) = (n-s)^\lambda$, $\lambda > m-1$, it is easy to see that H satisfies (H.1)–(H.4) and $\sum_{i=1}^m a_i H_i(n,s) \leq 0$. By Corollary 9, every solution of (18) is oscillatory.

Remark 1 The results of this paper are applicable to the following equation of neutral type

$$\sum_{i=1}^{m} a_i \Delta^i y(n) + r(n)\Phi_\gamma(x(h(n))) + \sum_{i=1}^{m} \left[p_i(n)\Phi_{\alpha_i}(x(g_i(n))) + q_i(n)\Phi_{\beta_i}(x(l_i(n)))\right] \\ + \overline{r}(n)\Phi_{\overline{\gamma}}(x(\overline{h}(n))) + \sum_{j=1}^{m} \left[\overline{p}_j(n)\Phi_{\overline{\alpha}_j}(x(\overline{g}_j(n))) + \overline{q}_j(n)\Phi_{\overline{\beta}_j}(x(\overline{l}_j(n)))\right] \\ = a(n)x(n) + b(n)x(n-\tau) + f(n),$$

where $y(n) := x(n) + \delta(n)x(\tau(n))$, $\Phi_\eta(u) := |u|^{\eta-1}u$, $\eta > 0$, a_i are real numbers with $a_n \equiv 1$ and r, $\overline{r}$, p_i, $\overline{p}_j$, q_i, $\overline{q}_j$, g_i, $\overline{g}_j$, l_i, $\overline{l}_j$, a, b, and f are real sequences such that a, b, p_i, and $\overline{p}_j$ are positive sequences, and $\lim_{n\to\infty} h(n) = \lim_{n\to\infty} \overline{h}(n) = \lim_{n\to\infty} g_i(n) = \lim_{n\to\infty} \overline{g}_j(n) = \lim_{n\to\infty} l_i(n) = \lim_{n\to\infty} \overline{l}_j(n) = \infty$. The details can be carried out using similar arguments.

Remark 2 The oscillation of solutions for (15), (16), (17), and (18) cannot be determined using the techniques considered in the literature. Indeed, one can easily figure out that when $\Phi = q(n)g(x(n-\tau))$, $\Phi = q(n)g(x(n-\tau))$ with $p(n) = 0$, and $\Phi = q_1(n)\phi_\alpha(n-\sigma_1) + q_2(n)\phi_\beta(n-\sigma_2)$, (4) (or (5)) reduces to (1), (2), and (3), respectively. This tells that our results in this paper extend as well as improve those existing in the literature.

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