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Applied Mathematics Letters 17 (2004) 1073–1078
Applied Mathematics Letters
www.elsevier.com/locate/aml

Oscillatory Behaviour of a Higher-Order Nonlinear Neutral Type Functional Difference Equation with Oscillating Coefficients

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(Received January 2003; revised and accepted October 2003)

Abstract—In this paper, we are concerned with the oscillation of solutions of a certain higher-order nonlinear neutral type difference equation with oscillating coefficients. We obtain two sufficient criteria for oscillatory behaviour.

Keywords—Difference equation, Higher-order difference equation, Higher-order nonlinear neutral difference equation, Oscillating coefficients, Oscillatory.

1. INTRODUCTION

We consider the higher-order nonlinear difference equation of the form

$$\Delta^n \left[y(k) + p(k)\, y(k-\tau)\right] + q(k)\, f(y(\sigma(k))) = 0, \qquad n \geq 2 \in N, \quad k \in N, \tag{1.1}$$

where $N = \{0, 1, 2, \dots\}$, $p(k) : N \to R = (-\infty, \infty)$, and it is an oscillating function; $q(k) : N \to [0, \infty)$; τ is a positive integer; $\sigma(k) : N \to Z$ (Z denotes the set of integers) with $\sigma(k) \leq k$, and $\sigma(k) \to +\infty$ as $k \to \infty$; $f(u) \in C(R, R)$ is a nondecreasing function, $uf(u) > 0$ for $u \neq 0$.

By a solution of equation (1.1), we mean any function $y(k) : Z \to R$, which is defined for all $k \geq \min_{i\geq 0}\{i - \tau, \sigma(i)\}$, and satisfies equation (1.1) for sufficiently large k. We consider only such solutions which are nontrivial, for all large k. As it is customary, a solution $\{y(k)\}$ is said to be oscillatory if the terms $y(k)$ of the sequence are not eventually positive or not eventually negative. Otherwise, the solution is called nonoscillatory. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory. In this paper, we restrict our attention to real valued solutions $y(k)$.

Recently, much research has been done on the oscillatory and asymptotic behaviour of solutions of higher-order delay and neutral delay type difference equations. The results obtained here are

0893-9659/04/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.aml.2004.07.011

an extension of work in [1]. Most of the known results are for special cases of equation (1.1) and related equations; see, for example, [1,2; 3, Chapter 7; 4, Section 22; 5–9].

The purpose of this paper is to study oscillatory behaviour of solutions of equation (1.1). For the general theory of difference equations, one can refer to [2–7]. Many references to applications of the difference equations can be found in [5–7].

For the sake of convenience, the function $z(k)$ is defined by

$$z(k) = y(k) + p(k)\, y(k-\tau) \tag{2.1}$$

and $N(a) = \{a, a+1, \dots\}$, $N(a,b) = \{a, a+1, \dots, b\}$.

2. SOME AUXILIARY LEMMAS

LEMMA 2.1. *(See [2].) Let $y(k)$ be defined for $k \ge k_0 \in N$, and $y(k) > 0$ with $\Delta^n y(k)$ of constant sign for $k \ge k_0$, $n \in N(1)$, and not identically zero. Then, there exists an integer m, $0 \le m \le n$ with $(n+m)$ even for $\Delta^n y(k) \ge 0$ or $(n+m)$ odd for $\Delta^n y(k) \le 0$, such that*

i) *$m \le n-1$ implies $(-1)^{m+i}\Delta^i y(k) > 0$, for all $k \ge k_0$, $m \le i \le n-1$*
ii) *$m \ge 1$ implies $\Delta^i y(k) > 0$, for all large $k \ge k_0$, $1 \le i \le m-1$.*

LEMMA 2.2. *(See [2].) Let $y(k)$ be defined for $k \ge k_0$, and $y(k) > 0$ with $\Delta^n y(k) \le 0$ for $k \ge k_0$ and not identically zero. Then, there exists a large $k_1 \ge k_0$, such that*

$$y(k) \ge \frac{1}{(n-1)!}(k-k_1)^{n-1}\Delta^{n-1} y\left(2^{n-m-1}k\right), \qquad k \ge k_1,$$

where m is defined as in Lemma 2.1. Further, if $y(k)$ is increasing, then,

$$y(k) \ge \frac{1}{(n-1)!}\left(\frac{k}{2^{n-1}}\right)^{n-1}\Delta^{n-1} y(k), \qquad k \ge 2^{n-1}k_1.$$

3. MAIN RESULTS

THEOREM 3.1. *Assume that n is odd and*

(C_1) $\lim_{k\to\infty} p(k) = 0$,
(C_2) $\sum_{s=k_0}^{+\infty} s^{n-1}q(s) = +\infty$.

Then, every bounded solution of equation (1.1) is either oscillatory or tends to zero as $k \to +\infty$.

PROOF. Assume that equation (1.1) has a bounded nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive (the proof is similar when $y(k)$ is eventually negative). That is, $y(k) > 0$, $y(k-\tau) > 0$, and $y(\sigma(k)) > 0$ for $k \ge k_1 \ge k_0$. Further, we assume that $y(k)$ does not to zero as $k \to \infty$. By (1.1),(1.2), we have for $k \ge k_1$

$$\Delta^n z(k) = -q(k)\, f(y(\sigma(k))) \le 0. \tag{3.1}$$

That is, $\Delta^n z(k) \le 0$. It follows that $\Delta^a z(k)$ $(a = 0,1,2,\dots,n-1)$ is strictly monotone and eventually of constant sign. Since $p(k)$ is an oscillating sequence and $\lim_{k\to\infty} p(k) = 0$, there exists a $k_2 \ge k_1$, such that for $k \ge k_2$, we have $z(k) > 0$. Since $y(k)$ is bounded, by virtue of (C_1) and (1.2), there is a $k_3 \ge k_2$ such that $z(k)$ is also bounded, for $k \ge k_3$. Because n is odd and $z(k)$ is bounded, by Lemma 2.1, since $m = 0$ (otherwise, $z(k)$ is not bounded), there exists $k_4 \ge k_3$, such that for $k \ge k_4$, we have $(-1)^i\Delta^i z(k) > 0$ $(i = 0,1,2,\dots,n-1)$. In particular, since $\Delta z(k) < 0$ for $k \ge k_4$, $z(k)$ is decreasing. Since $z(k)$ is bounded, we may write $\lim_{k\to\infty} z(k) = L(-\infty < L < +\infty)$. Assume that $0 \le L < +\infty$. Let $L > 0$. Then, there exists a constant $c > 0$ and a k_5 with $k_5 \ge k_4$, such that $z(k) > c > 0$ for $k \ge k_5$. Since $y(k)$ is bounded, $\lim_{k\to\infty} p(k)y(k-\tau) = 0$ by (C_1). Therefore, there exists a constant $c_1 > 0$ and a k_6 with $k_6 \geq k_5$, such that $y(k) = z(k) - p(k)y(k-\tau) > c_1 > 0$ for $k \geq k_6$. So, we may find k_7 with $k_7 \geq k_6$, such that $y(\sigma(k)) > c_1 > 0$ for $k \geq k_7$. From (3.1), we have

$$\Delta^n z(k) \leq -q(k) f(c_1) \qquad (k \geq k_7). \tag{3.2}$$

If we multiply (3.2) by k^{n-1}, and summing it from k_7 to $k-1$, we obtain

$$F(k) - F(k_7) \leq -f(c_1) \sum_{s=k_7}^{k-1} q(s) s^{n-1}, \tag{3.3}$$

where

$$F(k) = \sum_{\gamma=2}^{n-1} (-1)^\gamma \Delta^\gamma k^{n-1} \Delta^{n-\gamma-1} z(k+\gamma).$$

Since $(-1)^i \Delta^i z(k) > 0$, for $i = 0, 1, 2, \ldots, n-1$ and $k \geq k_4$, we have $F(k) > 0$ for $k \geq k_7$. From (3.3), we have

$$-F(k_7) \leq -f(c_1) \sum_{s=k_7}^{k-1} q(s) s^{n-1}.$$

By (C_2), we obtain

$$-F(k_7) \leq -f(c_1) \sum_{s=k_7}^{\infty} q(s) s^{n-1} = -\infty,$$

as $k \to \infty$. This is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{k\to\infty} z(k) = 0$. Since $y(k)$ is bounded, by virtue of (C_1) and (1.2), we obtain

$$\lim_{k\to\infty} y(k) = \lim_{k\to\infty} z(k) - \lim_{k\to\infty} p(k) y(k-\tau) = 0.$$

Now, let us consider the case of $y(k) < 0$ for $k \geq k_1$. By (1.1) and (1.2),

$$\Delta^n z(k) = -q(k) f(y(\sigma(k))) \geq 0 \qquad (k \geq k_1).$$

That is, $\Delta^n z(k) \geq 0$. It follows that $\Delta^a z(k)$ ($a = 0, 1, 2, \ldots, n-1$) is strictly monotone and eventually of constant sign. Since $p(k)$ is an oscillating sequence and $\lim_{k\to\infty} p(k) = 0$, there exists a $k_2 \geq k_1$, such that for $k \geq k_2$, we have $z(k) < 0$. Since $y(k)$ is bounded, by virtue of (C_1) and (1.2), there is a $k_3 \geq k_2$, such that $z(k)$ is also bounded for $k \geq k_3$. Assume that $x(k) = -z(k)$. Then, $\Delta^n x(k) = -\Delta^n z(k)$. Therefore, $x(k) > 0$ and $\Delta^n x(k) \leq 0$ for $k \geq k_3$. From this, we observe that $x(k)$ is bounded. Because n is odd and $x(k)$ is bounded, by Lemma 2.1, since $m = 0$ (otherwise, $x(k)$ is not bounded) there exists a $k_4 \geq k_3$, such that $(-1)^i \Delta^i x(k) > 0$ for $i = 0, 1, 2, \ldots, n-1$ and $k \geq k_4$. That is, $(-1)^i \Delta^i z(t) < 0$ for $i = 0, 1, 2, \ldots, n-1$ and $k \geq k_4$. In particular, for $k \geq k_4$, we have $\Delta z(k) > 0$. Therefore, $z(k)$ is increasing. So, we can assume that $\lim_{k\to\infty} z(k) = L$ ($-\infty < L \leq 0$). As in the proof of $y(k) > 0$, we may prove that $L = 0$. As for the rest, it is similar to the case of $y(k) > 0$. That is, $\lim_{k\to\infty} y(k) = 0$. This contradicts our assumption. Hence, the proof is completed. ∎

THEOREM 3.2. *Assume that n is even and (C_1) holds. Further, we suppose that the following condition holds.*

(C_3) *There is a sequence $\varphi(k) > 0$ which is defined on $N(k_0)$, such that*

$$\lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} \varphi(s) q(s) = +\infty,$$

and

$$\lim_{k\to\infty} \sup \sum_{s=k_9}^{k-1} \frac{(\Delta\varphi(s))^2}{\varphi(s)\, s^{n-2}} < +\infty.$$

Then, every bounded solution of equation (1.1) is oscillatory.

PROOF. Assume that equation (1.1) has a bounded nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive (the proof is similar when $y(k)$ is eventually negative). That is, $y(k) > 0$, $y(k-\tau) > 0$, and $y(\sigma(k)) > 0$ for $k \geq k_1 \geq k_0$. By (1.1),(1.2), we have for $k \geq k_1$

$$\Delta^n z(k) = -q(k) f(y(\sigma(k))) \leq 0. \tag{3.4}$$

That is, $\Delta^n z(k) \leq 0$. It follows that $\Delta^a z(k)$ $(a = 0,1,2,\dots,n-1)$ is strictly monotone and eventually of constant sign. Since $p(k)$ is an oscillating sequence and $\lim_{k\to\infty} p(k) = 0$, there exists a $k_2 \geq k_1$, such that for $k \geq k_2$, we have $z(k) > 0$ eventually. Since $y(k)$ is bounded, by virtue of (C_1) and (1.2), there is a $k_3 \geq k_2$, such that $z(k)$ is also bounded for sufficiently large $k \geq k_3$. Because n is even, by Lemma 2.1, since $m = 1$ (otherwise, $z(k)$ is not bounded) there exists $k_4 \geq k_3$, such that for $k \geq k_4$,

$$(-1)^{i+1} \Delta^j z(k) > 0 \qquad (i = 0,1,2,\dots,n-1). \tag{3.5}$$

In particular, since $\Delta z(k) > 0$ for $k \geq k_4$, $z(k)$ is increasing. Since $y(k)$ is bounded, $\lim_{k\to\infty} p(k)y(k-\tau) = 0$ by (C_1). Then, there exists a $k_5 \geq k_4$ by (1.2),

$$y(k) = z(k) - p(k)\, y(k-\tau) \geq \frac{1}{2} z(k) > 0,$$

for $k \geq k_5$. We may find a $k_6 \geq k_5$, such that for $k \geq k_6$, we have

$$y(\sigma(k)) \geq \frac{1}{2} z(\sigma(k)) > 0. \tag{3.6}$$

From (3.4),(3.6) and the properties of f, we have

$$\begin{aligned} \Delta^n z(k) &\leq -q(k) f\left(\frac{1}{2} z(\sigma(k))\right) \\ &= -q(k) \frac{f((1/2)\, z(\sigma(k)))}{z(\sigma(k))} z(\sigma(k)), \end{aligned} \tag{3.7}$$

for $k \geq k_6$. Since $z(k)$ is bounded and increasing, $\lim_{k\to\infty} z(k) = L$ $(0 < L < +\infty)$. By the continuity of f, we have

$$\lim_{k\to\infty} \frac{f((1/2)\, z(\sigma(k)))}{z(\sigma(k))} = \frac{f((1/2)\, L)}{L} > 0.$$

Then, there is a $k_7 \geq k_6$, such that for $k \geq k_7$, we have

$$\lim_{k\to\infty} \frac{f(z(\sigma(k)))}{z(\sigma(k))} \geq \frac{f((1/2)\, L)}{2L} = \alpha > 0. \tag{3.8}$$

By (3.7) and (3.8),

$$\Delta^n z(t) \leq -\alpha q(k)\, z(\sigma(k)), \qquad \text{for } k \geq k_7. \tag{3.9}$$

Let us set

$$w(k) = \frac{\Delta^{n-1} z(k)}{z((1/2)\, \sigma(k))}. \tag{3.10}$$

Since $\Delta^{n-1}z(k) > 0$ and $z(\sigma(k)) > 0$ by (3.5), we can find a k_8 with $k_8 \geq k_7$, such that for sufficiently large $k \geq k_8$, we have $w(k) > 0$. If we apply the forward difference operator Δ to (3.10), since $\Delta^{n-1}z(k)$ and $\Delta z(k)$ are decreasing and $z(k)$ is increasing by (3.5), we obtain

$$\begin{aligned}\Delta w(k) &= \frac{z((1/2)\sigma(k))\Delta^n z(k) - \Delta z((1/2)\sigma(k))\Delta^{n-1}z(k)}{z((1/2)\sigma(k))Ez((1/2)\sigma(k))} \\ &\leq \frac{z((1/2)\sigma(k))\Delta^n z(k)}{(z((1/2)\sigma(k)))^2} - \frac{\Delta z((1/2)\sigma(k))\Delta^{n-1}z(k)}{(z((1/2)\sigma(k)))^2} \\ &\leq \frac{\Delta^n z(k)}{z((1/2)\sigma(k))} - w(k)\frac{\Delta z(k)}{z((1/2)\sigma(k))},\end{aligned} \tag{3.11}$$

where E is shift operator as defined $Ex(k) = x(k+1)$. Since $\Delta^n z(k) \leq 0$, it is obvious that $\Delta^{n-1}z(k)$ is decreasing. Since $\Delta^{n-1}z(k) > 0$ for $k \geq k_9$, by Lemma 2.2 there exists a large $k_9 \geq k_8$, such that for all $k \geq k_9$, we can write

$$\begin{aligned}z(k) &\geq z(\sigma(k)) \geq z\left(\frac{1}{2}\sigma(k)\right) \\ &\geq \frac{1}{(n-1)!}\left|\Delta^{n-1}z\left(2^{n-m-1}k\right)\right|(k-k_9)^{n-1}.\end{aligned} \tag{3.12}$$

If we apply forward difference operator Δ to (3.12), since $\sigma(k) \leq k$, by Lemma 2.2, we have

$$\begin{aligned}\Delta z(k) &\geq \frac{1}{(n-2)!}\Delta^{n-1}z\left(2^{n-m-1}k\right)k^{n-2} \\ &\geq \frac{1}{(n-2)!}\frac{1}{(2^{n-1})^{n-1}}k^{n-2}\Delta^{n-1}z(k) \qquad \left(k \geq 2^{n-2}k_9\right).\end{aligned} \tag{3.13}$$

Therefore, from (3.11) and (3.13), we obtain

$$\Delta w(k) \leq -\alpha q(k) - w^2(k)\frac{1}{(n-2)!}\frac{1}{(2^{n-1})^{n-1}}k^{n-2} \tag{3.14}$$

We may again write (3.14),

$$\alpha q(k) \leq -\Delta w(k) - w^2(k)\frac{1}{(n-2)!}\frac{1}{(2^{n-1})^{n-1}}k^{n-2} \qquad \left(k \geq 2^{n-2}k_9\right). \tag{3.15}$$

If we multiply (3.15) by $\varphi(k)$, and summing it from k_9 to $k-1$, we obtain

$$\begin{aligned}\alpha\sum_{s=k_9}^{k-1}\varphi(s)q(s) &\leq -\sum_{s=k_9}^{k-1}\varphi(s)\Delta w(s) - C\sum_{s=k_9}^{k-1}\varphi(s)w^2(s)s^{n-2} \\ &= -\varphi(k)w(k) + \varphi(k_9)w(k_9) + \sum_{s=k_9}^{k-1}\Delta\varphi(s)w(s) - C\sum_{s=k_9}^{k-1}\varphi(s)w^2(s)s^{n-2} \\ &\leq \varphi(k_9)w(k_9) - C\sum_{s=k_9}^{k-1}\varphi(s)s^{n-2}\left[w(s) - \frac{\Delta\varphi(s)}{2C\varphi(s)s^{n-2}}\right]^2 + \sum_{s=k_9}^{k-1}\frac{[\Delta\varphi(s)]^2}{4C\varphi(s)s^{n-2}} \\ &\leq \varphi(k_9)w(k_9) + \sum_{s=k_9}^{k-1}\frac{[\Delta\varphi(s)]^2}{4C\varphi(s)s^{n-2}},\end{aligned}$$

where $C = (1/(n-2)!)(1/2^{n-1})^{n-1}$. Therefore, by (C_3)

$$+\infty = \alpha\limsup_{k\to\infty}\sum_{s=k_9}^{k-1}\varphi(s)q(s) \leq \varphi(k_9)w(k_9) + \frac{1}{4C}\limsup_{k\to\infty}\sum_{s=k_9}^{k-1}\frac{[\Delta\varphi(s)]^2}{\varphi(s)s^{n-2}} < +\infty.$$

This is a contradiction.

Now, let us consider the case of $y(k) < 0$. We do the proof similar to Theorem 1, as in the case of $y(k) < 0$. Therefore, there is a $k \geq k_3$, such that $\Delta^n z(k) \geq 0$, $z(k) < 0$, and $z(k)$ is bounded and, at the same time, there exists an integer $m = 1$ and a $k_4 \geq k_3$, such that $(-1)^{i+1}\Delta^i z(k) < 0$, for $k \geq k_4$ and $i = 1, 2, \dots, n-1$. In particular, $\Delta z(k) < 0$, for $k \geq k_4$. Let us set $x(k) = -z(k)$. The rest of proof is similar to the case of $y(k) > 0$. Hence, the proof is completed. ∎

EXAMPLE 3.1. We consider difference equation of the form

$$\Delta^3 \left[y(k) + e^{-k}\sin(\ln k)\, y(k-1)\right] + k y^3(k-2) = 0, \qquad k \geq 2, \tag{3.16}$$

where $n = 3$, $q(k) = k$, $\sigma(k) = k-2$, $\tau = 1$, $p(k) = e^{-k}\sin(\ln k)$. Hence, we have

$$\lim_{k\to\infty} p(k) = \lim_{k\to\infty} e^{-k}\sin(\ln k) = 0$$

and

$$\sum_{s=k_0}^{+\infty} s^{n-1} q(s) = \sum_{s=k_0}^{+\infty} s^3 = +\infty.$$

Since Conditions (C_1) and (C_2) of the Theorem 1 are satisfied, every bounded and does not to zero as $k \to \infty$ solution of equation (3.16) is oscillatory.

EXAMPLE 3.2. We consider difference equation of the form

$$\Delta^n \left[y(k) + \frac{(-1)^k}{k} y(k-2)\right] + 9y^3(k-1) = 0, \tag{3.17}$$

where $n = 2$, $\tau = 2$, $p(k) = (-1)^k/k$, $q(k) = 9$, $\sigma(k) = k-1$. If we get $\varphi(k) = 1/k$, then, $\Delta\varphi(k) = -1/k(k+1)$. Hence, we have

$$\lim_{k\to\infty} p(k) = \lim_{k\to\infty} \frac{(-1)^k}{k} = 0,$$

$$\lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} \varphi(s)\, q(s) = \lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} \frac{1}{s} 9 = +\infty,$$

and

$$\lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} \frac{[\Delta\varphi(s)]^2}{\varphi(s)\, s^{n-2}} = \lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} \frac{[\Delta(1/s)]^2}{(1/s).1} = \lim_{k\to\infty} \sup \sum_{s=k_0}^{k-1} \frac{1}{s(s+1)^2} < +\infty.$$

Since Conditions (C_1) and (C_3) of the Theorem 2 are satisfied, every bounded solution of equation (3.17) is oscillatory.

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