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Oscillation criteria for nonlinear second-order difference equations with a nonlinear damped term

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Abstract

Sufficient conditions for the oscillation of solutions of the nonlinear second-order difference equation of the form

$$\begin{aligned}&\Delta[p(k)\psi(y(k))\Delta y(k)] + q(k)h(y(k))g(\Delta y(k-r(k)))\Delta y(k)\\&\quad + f(k, y(k), y(k-s_1(k)), y(k-s_2(k)), \ldots, y(k-s_n(k))) = 0\end{aligned}$$

are established. We obtain a series of results for oscillatory behaviour.

MSC: 39A10

Keywords: Oscillation; Second-order nonlinear difference equation; Damped term

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1. Introduction

Studies on oscillatory and nonoscillatory solutions of second-order linear and nonlinear difference equations with a nonlinear damped term are scarce; and many of them have been done only in the last decade. This shows that there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions for different classes of linear and nonlinear second-order difference equations with a damped term. Most of the known results are for special cases of Eq. (1) and related equations; see, for example, [5,7–11]. We refer the reader to the recent papers [5–11] where further references can be found.

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For the general theory of difference equations, the reader could refer to [1–4]. Many references to applications of the difference equations can be found in [4].

Our main objective in this work is to study the oscillatory behaviour of the general nonlinear second-order difference equation

$$\begin{aligned}&\Delta[p(k)\psi(y(k))\Delta y(k)] + q(k)h(y(k))g(\Delta y(k-r(k)))\Delta y(k)\\&\quad + f(k, y(k), y(k-s_1(k)), y(k-s_2(k)), \ldots, y(k-s_n(k))) = 0 \qquad (1)\end{aligned}$$

where $N = \{0, 1, 2, \ldots\}$, $N(k_0) = \{k_0, k_0+1, \ldots\}$, and $k \in N$ and the following conditions are always assumed to hold:

(i) $p(k)$ is defined on $N(k_0)$ and $p(k) > 0$,
(ii) $q(k) \geq 0$ for every $k \in N(k_0)$,
(iii) $\psi(y(k))$ is defined on R and there exist positive constants a and b such that $a \leq \psi(y(k)) \leq b$ for every $y(k)$,
(iv) $h(y(k))$ is defined on R and $h(y(k)) > 0$,
(v) there exists a positive constant λ such that $\psi(y(k)) \geq \lambda h(y(k))$ for every $y(k)$,
(vi) $g(v)$ is defined on R and $0 < g(v) \leq m$ where m is a positive constant,
(vii) $f(k, u_0, u_1, u_2, \ldots, u_n) : N(k_0) \times R^{n+1} \to R$ and f is continuous with respect to $u_0, u_1, u_2, \ldots, u_n$; further, $f(k, u_0, u_1, u_2, \ldots, u_n) > 0$ if $u_j > 0$ for all $j = 0, 1, 2, \ldots, n$ and $f(k, u_0, u_1, u_2, \ldots, u_n) < 0$ if $u_j < 0$ for all $j = 0, 1, 2, \ldots, n$,
(viii) $r(k) \in N(k_0)$ and $\lim_{k\to\infty}(k - r(k)) = \infty$; $s_i(k) \in N(k_0)$ and $\lim_{k\to\infty}(k - s_i(k)) = \infty$ for $i = 1, 2, \ldots, n$.

The operator Δ is a forward difference operator which is defined by $\Delta y(k) = y(k+1) - y(k)$.

Let $\sigma = \max\{r(k), s_i(k)\}$, $i = 1, 2, \ldots, n$, and let N_0 be a fixed nonnegative integer. By a solution of (1), we mean a real sequence $\{y(k)\}$ which is defined for all $k \geq N_0 - \sigma$ and satisfies (1) for $k \geq N_0$. A solution $\{y(k)\}$ of (1) is said to be nonoscillatory if all the terms $y(k)$ are eventually of fixed sign. Otherwise, the solution $\{y(k)\}$ is called oscillatory. In this work, we shall be concerned only with the nontrivial solutions of (1).

2. Main results

To obtain our results we need the following lemma.

Lemma 2.1. *Assume that*

$$\lim_{k\to\infty} \sum_{j=k_0}^{k-1} \frac{1}{p(j)} \prod_{l=k_0}^{k-1} \left[1 - \frac{m}{\lambda}\frac{q(l)}{p(l)}\right] = +\infty \qquad (C_0)$$

is satisfied. If $y(k)$ is a nonoscillatory solution of Eq. (1), *then we must have*

$y(k)\Delta y(k) > 0$ *for all large* k.

Proof. Let $y(k)$ be a nonoscillatory solution of Eq. (1). Assume, without loss of generality, that $y(k) > 0$ for $k \geq k_1$ such that $k_1 \geq k_0 \geq 0$. Then, we can find a $k_2 \geq k_1$ such that $y(k - r(k)) > 0$ and $y(k - s_i(k)) > 0$ for $i = 1, 2, \ldots, n$ and $k \geq k_2$. Since $y(k) > 0$ and $y(k)\Delta y(k) > 0$, $\Delta y(k)$ is positive.

Let us suppose the contrary. That is, let us have $\Delta y(k) < 0$ for $k \geq k_3$ such that $k_3 \geq k_2$. Then, we can write from (1)

$$\Delta w(k) + \frac{m}{\lambda}\frac{q(k)}{p(k)}w(k) > 0 \qquad \text{for } k \geq k_3, \tag{2}$$

where $w(k) = -p(k)\psi(y(k))\Delta y(k)$. The difference inequality (2) has the solution

$$w(k) > \prod_{l=k_3}^{k-1}\left[1 - \frac{m}{\lambda}\frac{q(l)}{p(l)}\right]w(k_3). \tag{3}$$

If we divide both sides of the inequality (3) by $-p(k)$, we obtain

$$\psi(y(k))\Delta y(k) < -\frac{1}{p(k)}\prod_{l=k_3}^{k-1}\left[1 - \frac{m}{\lambda}\frac{q(l)}{p(l)}\right]w(k_3). \tag{4}$$

If we sum (4) from k_3 to $k-1$ and then take $k \to \infty$, the right side of (4) tends to $-\infty$ by (C_0). However, since the left side of (4) is finite, this is a contradiction. Hence, the proof is complete.

Note: If $q(k) = 0$, then condition (C_0) takes the form

$$\lim_{k\to\infty}\sum_{j=k_0}^{k-1}\frac{1}{p(j)} = \infty. \quad \square$$

Corollary 2.2. *If y_k is a nonoscillatory solution of Eq.* (1), *then Δy_k is also nonoscillatory.*

Proof. Let $y(k)$ be a nonoscillatory solution of Eq. (1). We can assume, without loss of generality, that $y(k) > 0$ for $k \geq k_1$ such that $k_1 \geq k_0 \geq 0$. Then, we can find a $k_2 \geq k_1$ such that $y(k - r(k)) > 0$ and $y(k - s_i(k)) > 0$ for $i = 1, 2, \ldots, n$ and $k \geq k_2$. Suppose to the contrary that Δy_k is oscillatory. Then from (1) we have

$$\Delta[p(k)\psi(y(k))\Delta y(k)] + q(k)h(y(k))g(\Delta y(k - r(k)))\Delta y(k) < 0. \tag{5}$$

If we consider (ii), (iv) and (vi), $q(k)h(y(k))g(\Delta y(k - r(k)))\Delta y(k)$ is not negative and not positive. Therefore $\Delta[p(k)\psi(y(k))\Delta y(k)]$ has to be negative and $|\Delta[p(k)\psi(y(k))\Delta y(k)]| > |q(k)h(y(k))g(\Delta y(k - r(k)))\Delta y(k)|$. Since

$$\Delta[p(k)\psi(y(k))\Delta y(k)] < 0, \tag{6}$$

if we sum (6) from k_2 to $(k-1)$, we obtain

$$p(k)\psi(y(k))\Delta y(k) - C < 0 \qquad \text{for all } k \geq k_2 \tag{7}$$

where $C = p(k_2)\psi(y(k_2))\Delta y(k_2)$ is any constant (may be positive or negative). Inequality (7) contradicts $p(k) > 0$, $\psi(y(k)) > 0$ and $\Delta y(k)$ is oscillatory. Hence the proof is complete. $\square$

Theorem 2.3. *Assume that* (C_0) *is satisfied. Further, if the conditions*

$$f(k, u_0, u_1, u_2, \ldots, u_n) \text{ is nondecreasing for } u_0, u_1, u_2, \ldots, u_n \tag{C_1}$$

and

$$\lim_{k\to\infty}\sum_{l=k_0}^{k-1}|f(l, c, c, \ldots, c)| = +\infty \text{ for any } c \neq 0 \tag{C_2}$$

are satisfied, then every solution of Eq. (1) *is oscillatory.*

Proof. Suppose that Eq. (1) has a nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive for $k \geq k_1 \geq k_0$ (the proof is similar when $y(k)$ is eventually negative). Then, we can find a $k_2 \geq k_1$ such that $y(k - r(k)) > 0$ and $y(k - s_i(k)) > 0$ for $i = 1, 2, \ldots, n$ and $k \geq k_2$. There exists a $k_3 \geq k_2$ such that $\Delta y(k) > 0$ for $k \geq k_3$ by Lemma 2.1. Therefore $y(k)$ is increasing. Then there exist a $k_4 \geq k_3$ and a constant $c > 0$ such that $y(k) > c$ for $k \geq k_4$. Thus, we can find a $k_5 \geq k_4$ such that $y(k - s_i(k)) > c$ for $k \geq k_5$ and $i = 1, 2, \ldots, n$. Hence, we obtain from (1) by Lemma 2.1 and condition (C_1)

$$\Delta[p(k)\psi(y(k))\Delta y(k)] + f(k, c, c, \ldots, c) \leq 0 \tag{8}$$

for $k \geq k_5$. Summing (8) from k_5 to $k - 1$, we obtain

$$p(k)\psi(y(k))\Delta y(k) \leq S_1 - \sum_{l=k_5}^{k-1} f(l, c, c, \ldots, c) \tag{9}$$

where $S_1 = p(k_5)\psi(y(k_5))\Delta y(k_5)$ is positive. Since $f(k, u_0, u_1, u_2, \ldots, u_n) > 0$, we obtain from (9)

$$p(k)\psi(y(k))\Delta y(k) \to -\infty$$

as $k \to \infty$ by using condition (C_2). Therefore, $\Delta y(k)$ must be negative for all sufficiently large k. This contradicts Lemma 2.1. This completes the proof. □

Theorem 2.4. *Assume that conditions* (C_0), *and* (C_1) *hold. Moreover, suppose that the following conditions are satisfied:*

$$\lim_{k\to\infty} \sum_{l=k_0}^{k-1} |\Delta\psi(y(l))| < +\infty. \tag{C_3}$$

There exist a positive function $\beta(k)$, *with* $\Delta\beta(k) \geq 0$, *and a positive number* A *such that*

$$\Delta\beta(k)p(k) \leq A \tag{C_4}$$

and

$$\lim_{k\to\infty} \sum_{l=k_0}^{k-1} \beta(l)|f(l, c, c, \ldots, c)| = +\infty \text{ for any } c \neq 0. \tag{C_5}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Proof. Assume that Eq. (1) has a nonoscillatory solution $y(k)$. Without loss of generality, suppose that $y(k)$ is eventually positive for $k \geq k_1 \geq k_0$ (the proof is similar when $y(k)$ is eventually negative). Then, we can find a $k_2 \geq k_1$ such that $y(k - r(k)) > 0$ and $y(k - s_i(k)) > 0$ for $i = 1, 2, \ldots, n$ and $k \geq k_2$. There exists a $k_3 \geq k_2$ such that $\Delta y(k) > 0$ for $k \geq k_3$ by Lemma 2.1. Therefore, $y(k)$ is increasing. Then there exist a $k_4 \geq k_3$ and a constant $c > 0$ such that $y(k) > c$ for $k \geq k_4$. Thus, we can find a $k_5 \geq k_4$ such that $y(k - s_i(k)) > c$ for $k \geq k_5$ and $i = 1, 2, \ldots, n$. Hence, we obtain from (1) by Lemma 2.1 and condition (C_1)

$$\Delta[p(k)\psi(y(k))\Delta y(k)] + f(k, c, c, \ldots, c) \leq 0 \tag{10}$$

for $k \geq k_5$. Let $\beta(k)$ be a positive function which satisfies conditions (C_4) and (C_5). If we multiply the inequality (10) by $\beta(k)$ and later take its sum from k_5 to $k-1$, we obtain

$$\begin{aligned}\beta(k-1)p(k)\psi(y(k))\Delta y(k) &+ \sum_{l=k_5}^{k-1}\beta(l)f(l,c,c,\dots,c)\\ &\leq S_2 + \sum_{l=k_5}^{k-1}\Delta\beta(l)p(l)\psi(y(l))\Delta y(l) \qquad (11)\end{aligned}$$

where $S_2 = \beta(k_5)p(k_5)\psi(y(k_5))\Delta y(k_5)$ is positive. Therefore applying the condition (C_4) to (11), we have

$$\begin{aligned}\beta(k-1)p(k)\psi(y(k))\Delta y(k) \leq S_2 &+ A\sum_{l=k_0}^{k-1}\psi(y(l))\Delta y(l)\\ &- \sum_{l=k_5}^{k-1}\beta(l)f(l,c,c,\dots,c). \qquad (12)\end{aligned}$$

If we take $k \to \infty$ in the inequality (12), since $y(k)$ is increasing and bounded and $\psi(y(k))$ is bounded from above by some positive constant b, in the inequality (12) the term

$$\begin{aligned}\sum_{l=k_0}^{k-1}\psi(y(l))\Delta y(l) &= \psi(y(k))y(k) - \psi(y(k_0))y(k_0) - \sum_{l=k_0}^{k-1}\Delta\psi(y(l))y(l)\\ &\leq \psi(y(k))y(k) < +\infty\end{aligned}$$

as $k \to \infty$ by (C_3). Thus, we obtain from (12) by condition (C_5)

$$\beta(k-1)p(k)\psi(y(k))\Delta y(k) \to -\infty$$

as $k \to \infty$ where $0 < a \leq \psi(y(k)) \leq b$ for all $k \geq k_0$, a and b given by statement (iii). Therefore $\Delta y(k)$ must be negative for all sufficiently large k. This contradicts Lemma 2.1. Hence the proof is complete. □

Corollary 2.5. *Assume that the conditions (C_0), and (C_1) hold. In addition, if*

$$\lim_{k\to\infty}\sum_{l=k_0}^{k-1}\sum_{j=k_0}^{l}\frac{1}{p(j)}|f(l,c,c,\dots,c)| = +\infty \text{ for any constant } c \neq 0 \qquad (C_6)$$

is satisfied, then every bounded solution of Eq. (1) is oscillatory.

Proof. The proof follows from Theorem 2.4 if we take

$$\beta(k-1) = \sum_{l=k_0}^{k-1}\frac{1}{p(l)}. \quad \square$$

Theorem 2.6. *Suppose that the conditions (C_0) and (C_1) hold. Furthermore, if*

$$\lim_{k\to\infty}\left(b\sum_{l=k_0}^{k-1}\frac{1}{p(l)} - \sum_{l=k_0}^{k-1}\left[\frac{1}{p(l)}\sum_{j=k_0}^{k-1}|f(j,c,c,\dots,c)|\right]\right) = -\infty \qquad (C_7)$$

is satisfied for any constants b and c with the property $bc > 0$, *then every bounded solution of Eq.* (1) *is oscillatory.*

Proof. Assume that Eq. (1) has a nonoscillatory solution $y(k)$. Suppose, without loss of generality, that $y(k)$ is eventually positive for $k \geq k_1 \geq k_0$ (the proof is similar when $y(k)$ is eventually negative). Then, we can find a $k_2 \geq k_1$ such that $y(k - r(k)) > 0$ and $y(k - s_i(k)) > 0$ for $i = 1, 2, \ldots, n$ and $k \geq k_2$. There exists a $k_3 \geq k_2$ such that $\Delta y(k) > 0$ for $k \geq k_3$ by Lemma 2.1. Therefore $y(k)$ is increasing. Then there exists a $k_4 \geq k_3$ and a constant $c > 0$ such that $y(k) > c$ for $k \geq k_4$. Thus, we can find a $k_5 \geq k_4$ such that $y(k - s_i(k)) > c$ for $k \geq k_5$ and $i = 1, 2, \ldots, n$. Hence, we obtain from (1) by Lemma 2.1 and condition (C_1)

$$\Delta[p(k)\psi(y(k))\Delta y(k)] + f(k, c, c, \ldots, c) \leq 0 \tag{13}$$

for $k \geq k_5$. Summing (13) from k_5 to $k - 1$, we obtain

$$p(k)\psi(y(k))\Delta y(k) \leq b - \sum_{l=k_5}^{k-1} f(l, c, c, \ldots, c) \tag{14}$$

where $b = p(k_5)\psi(y(k_5))\Delta y(k_5)$ is a positive constant. If we divide both sides of the inequality (14) by $p(k)$ and later take its sum from k_5 to $k - 1$, we obtain

$$\sum_{l=k_5}^{k-1} \psi(y(l))\Delta y(l) \leq \sum_{l=k_5}^{k-1} \frac{b}{p(l)} - \sum_{l=k_5}^{k-1} \left(\frac{1}{p(l)} \sum_{j=k_5}^{k-1} f(j, c, c, \ldots, c) \right). \tag{15}$$

Therefore the left side of the inequality (15) tends to $-\infty$ as $k \to \infty$ by condition (C_7) and since $y(k)$ is bounded and $\psi(y(k))$ is bounded from above by some positive constant b, the right side of the inequality (15) is positive valued. This is a contradiction. Hence the proof is complete. □

Theorem 2.7. *Suppose that the conditions* (C_0), (C_3) *and* (C_4) *hold. In addition, if conditions*

$$\frac{f(k, u_0, u_1, u_2, \ldots, u_n)}{f_1(u_0, u_1, u_2, \ldots, u_n)} \geq \eta(k) > 0 \tag{C_8}$$

where the function $f_1(u_0, u_1, u_2, \ldots, u_n)$ *is a continuous function and* $u_j f_1(u_0, u_1, u_2, \ldots, u_n) > 0$, *if every* $u_j > 0$ *or every* $u_j < 0$, *and*

$$\lim_{k\to\infty} \sum_{l=k_0}^{k-1} \beta(l)\eta(l) = +\infty \tag{C_9}$$

are satisfied, then every bounded solution of Eq. (1) *is oscillatory.*

Proof. Assume that Eq. (1) has a nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive for $k \geq k_1 \geq k_0$ (the proof is similar when $y(k)$ is eventually negative). Then, we can find a $k_2 \geq k_1$ such that $y(k - r(k)) > 0$ and $y(k - s_i(k)) > 0$ for $i = 1, 2, \ldots, n$ and $k \geq k_2$. Hence, we have for $k \geq k_2$,

$$\begin{aligned} &f(k, y(k), y(k - s_1(k)), y(k - s_2(k)), \ldots, y(k - s_n(k))) \geq \\ &\quad \eta(k).\, f_1(y(k), y(k - s_1(k)), y(k - s_2(k)), \ldots, y(k - s_n(k))) > 0. \end{aligned} \tag{16}$$

Since $\Delta y(k) > 0$ by Lemma 2.1, $y(k)$ is increasing. Because $y(k)$ is bounded and $f_1(u_0, u_1, u_2, \dots, u_n)$ is continuous, we have $\lim_{k\to\infty} y(k) = \delta$ $(0 < \delta < +\infty)$ and

$$\lim_{k\to\infty} f_1(y(k), y(k - s_1(k)), y(k - s_2(k)), \dots, y(k - s_n(k))) = f_1(\delta, \delta, \dots, \delta).$$

Thus, we obtain $0 < f_1(\delta, \delta, \dots, \delta) < +\infty$. Choose γ such that $0 < \gamma < f_1(\delta, \delta, \dots, \delta) < +\infty$. Then there exists $k_3 \geq k_2$ such that

$$f_1(y(k), y(k - s_1(k)), y(k - s_2(k)), \dots, y(k - s_n(k))) > \gamma. \tag{17}$$

Therefore, from (1), (16) and (17) we obtain by condition (C_8)

$$\Delta[p(k)\psi(y(k))\Delta y(k)] + \gamma\eta(k) \leq 0 \tag{18}$$

for $k \geq k_3$. If we treat (18) as we treat (10) in the proof of Theorem 2.4, we obtain by condition (C_4)

$$\beta(k-1)p(k)\psi(y(k))\Delta y(k) - A\sum_{l=k_4}^{k-1}\psi(y(l))\Delta y(l) + \gamma\sum_{l=k_4}^{k-1}\beta(l)\eta(l) \leq S_4 \tag{19}$$

where $S_4 = \beta(k_3)p(k_3)\psi(y(k_3))\Delta y(k_3)$ is a positive constant. Because $y(k)$ is increasing and bounded and $\psi(y(k))$ is bounded from above by some positive constant b bounded, the second term of

$$\sum_{l=k_4}^{k-1}\psi(y(l))\Delta y(l)$$

in the inequality (19) is finite as $k \to \infty$ by (C_3). Hence, we have

$$\beta(k-1)p(k)\psi(y(k))\Delta y(k) \leq S_5 - \gamma\sum_{l=k_4}^{k-1}\beta(l)\eta(l) \tag{20}$$

where $S_5 = S_2 + A\sum_{s=k_4}^{k-1}\psi(y(l))\Delta y(l)$. Thus if we take $k \to \infty$ in the inequality (20), we obtain

$$\beta(k-1)p(k)\psi(y(k))\Delta y(k) \to -\infty.$$

Therefore $\Delta y(k)$ must be negative for all sufficiently large k. Hence, we reach a contradiction by Lemma 2.1. This completes the proof. □

Corollary 2.8. *Suppose that Eq.* (1) *satisfies conditions* (C_0) *and* (C_8). *Moreover if condition*

$$\lim_{k\to\infty}\sum_{l=k_0}^{k-1}\sum_{j=k_0}^{k-1}\frac{1}{p(j)}\eta(l) = +\infty \tag{C_{10}}$$

is satisfied, then every bounded solution of Eq. (1) *is oscillatory.*

Proof. The proof follows from Theorem 2.7 if we take $\beta(k-1) = \sum_{l=k_0}^{k-1}\frac{1}{p(l)}$. □

Example 2.1. Consider a difference equation of the form

$$\begin{aligned}&\Delta\left[k\left(1 - \frac{1}{y_k^2}\right)\Delta y_k\right] + \frac{2}{4^k}[(3k+1)(2^{2k+1} - 1) + 2^{2k+1}]\\ &\quad\times\frac{1}{y_k^2}\left(1 - \frac{1}{\Delta y_{k-1}}\right)\Delta y_k - \frac{3k+1}{2^{15}}\frac{y_k y_{k-1} y_{k-2}}{y_{k-3}^2 y_{k-4}^3} + 6k + 4 = 0\end{aligned} \tag{21}$$

where $p_k = k$; $\psi(y_k) = (1-\frac{1}{y_k^2})$; $q_k = \frac{2}{4^k}[(3k+1)(24^k-1)+24^k]$; $h(y_k) = \frac{1}{y_k^2}$; $g(\Delta y_{k-r(k)}) = (1-\frac{1}{\Delta y_{k-1}})$ where $r(k) = 1$;

$f(k, y_k, y_{k-s_1(k)}, y_{k-s_2(k)}, y_{k-s_3(k)}, y_{k-s_4(k)}) = \frac{3k+1}{2^{15}} \frac{y_k y_{k-1} y_{k-2}}{y_{k-3}^2 y_{k-4}^3} + 6k + 4$ where $s_1(k) = 1$, $s_2(k) = 2$, $s_3(k) = 3$, $s_4(k) = 4$ where $n = 4$.

(i)–(viii) are satisfied. Let us show that (C_0), (C_1), (C_2) of Theorem 2.3 are satisfied. Since

$$\lim_{k\to\infty} \sum_{j=k_0}^{k-1} \frac{1}{p(j)} \prod_{l=k_0}^{k-1} \left[1 - \frac{m}{\lambda}\frac{q(l)}{p(l)}\right] = \lim_{k\to\infty} \sum_{j=k_0}^{k-1} \frac{1}{j} \prod_{l=k_0}^{k-1} \left[1 - \frac{1}{20}\frac{12}{l}\right] = +\infty$$

where we can choose $m = \frac{4}{3}$ and $\lambda = 12$ by (v) for large k, condition (C_0) is satisfied. In addition, since $f(k, y_k, y_{k-s_1(k)}, y_{k-s_2(k)}, y_{k-s_3(k)}, y_{k-s_4(k)}) = \frac{3k+1}{2^{15}} \frac{y_k y_{k-1} y_{k-2}}{y_{k-3}^2 y_{k-4}^3} + 6k + 4$ is a nondecreasing function for $(y_0, y_1, y_2, y_3, y_4)$ and

$$\lim_{k\to\infty} \sum_{l=k_0}^{k-1} |f(l, c, c, c, c, c)| = \lim_{k\to\infty} \sum_{l=k_0}^{k-1} \left|\frac{3l+1}{2^{15}c^2} + 6l + 4\right| = +\infty \text{ for any } c \neq 0,$$

(C_1), (C_2) of Theorem 2.3 are also satisfied. Hence, every solutions of Eq. (21) is oscillatory eventually. One of the solutions is

$$y_k = (-2)^k.$$

Example 2.2. We consider a difference equation of the form

$$\Delta\left[k\left(3 + \frac{2}{y_k}\right)\Delta y_k\right] + 12\frac{1}{y_k^2}\frac{1}{(\Delta y_{k-1})^2}\Delta y_k + 4y_k y_{k-1} y_{k-2}^2 y_{k-3}^4 y_{k-5} = 0 \tag{22}$$

where $y(k) = y_k$; $p_k = k$; $\psi(y_k) = 3 + \frac{2}{y_k}$; $q_k = 12$; $h(y_k) = \frac{1}{y_k^2}$; $g(\Delta y_{k-r(k)}) = \frac{1}{(\Delta y_{k-1})^2}$ where $r(k) = 1$;

$f(k, y_k, y_{k-s_1(k)}, y_{k-s_2(k)}, y_{k-s_3(k)}, y_{k-s_4(k)}) = 4y_k y_{k-1} y_{k-2}^2 y_{k-3}^4 y_{k-5}$ where $n = 4$ and $s_1(k) = 1$, $s_2(k) = 2$, $s_3(k) = 3$, $s_4(k) = 5$;

(i)–(viii) are satisfied. Let us show that all conditions of Theorems 2.4, 2.6 and 2.7 are satisfied. Since

$$\lim_{k\to\infty} \sum_{j=k_0}^{k-1} \frac{1}{p(j)} \prod_{l=k_0}^{k-1} \left[1 - \frac{m}{\lambda}\frac{q(l)}{p(l)}\right] = \lim_{k\to\infty} \sum_{j=k_0}^{k-1} \frac{1}{j} \prod_{l=k_0}^{k-1} \left[1 - \frac{1}{20}\frac{12}{l}\right] = +\infty$$

where we can choose $m = \frac{1}{4}$, $\lambda = 5$ by (v) for large k, (C_0) is satisfied. Since y_k is bounded and increasing,

$$\begin{aligned}\lim_{k\to\infty} \sum_{l=k_0}^{k-1} |\Delta\psi(y(l))| &= \lim_{k\to\infty} \sum_{l=k_0}^{k-1} \left|-\frac{2\Delta y_k}{y_k E y_k}\right| \\ &= \lim_{k\to\infty} \sum_{l=k_0}^{k-1} \left|\frac{2}{y_{k+1}} - \frac{2}{y_k}\right| < +\infty.\end{aligned}$$

Then condition (C_3) is satisfied. If we take a positive function $\beta(k) = (1 - \frac{1}{k^2})$, with $\Delta\beta(k) = \frac{2k+1}{k^2(k+1)^2} > 0$, and a positive number A, we have

$$\Delta\beta(k)p(k) = \frac{2k^2 + k}{k^2(k+1)^2} \leq A.$$

Hence condition (C_4) is satisfied. Since

$$\lim_{k\to\infty}\sum_{l=k_0}^{k-1}\beta(l)|f(l,c,c,c,c,c)| = \lim_{k\to\infty}\sum_{l=k_0}^{k-1}\left(1-\frac{1}{l^2}\right)|4c^9| = +\infty \text{ for any } c \neq 0,$$

condition (C_5) is satisfied. Hence since all conditions of Theorem 2.4 are satisfied, every bounded solution of Eq. (22) is oscillatory eventually.

Furthermore, condition (C_7) of Theorem 2.6

$$\lim_{k\to\infty}\left(b\sum_{l=k_0}^{k-1}\frac{1}{p(l)} - \sum_{l=k_0}^{k-1}\left[\frac{1}{p(l)}\sum_{j=k_0}^{k-1}|f(j,c,c,\ldots,c)|\right]\right)$$
$$= \lim_{k\to\infty}\left[b\sum_{l=k_0}^{k-1}\frac{1}{l} - \sum_{l=k_0}^{k-1}\frac{1}{l}\left(\sum_{j=k_0}^{k-1}|4c^9|\right)\right] = -\infty$$

is satisfied. Hence all conditions of Theorem 2.6 are satisfied. Since

$$\frac{f(k,u_0,u_1,u_2,\ldots,u_n)}{f_1(u_0,u_1,u_2,\ldots,u_n)} = \frac{4y_k y_{k-1} y_{k-2}^2 y_{k-3}^4 y_{k-5}}{y_k y_{k-1} y_{k-2}^2 y_{k-3}^4 y_{k-5}}$$
$$= 4 \geq \eta(k) > 0,$$

condition (C_8) of Theorem 2.7 is satisfied. In addition, since condition (C_9)

$$\lim_{k\to\infty}\sum_{l=k_0}^{k-1}\beta(l)\eta(l) = \lim_{k\to\infty}\sum_{l=k_0}^{k-1}\left(1-\frac{1}{l^2}\right)4 = +\infty$$

is also satisfied, all conditions of Theorem 2.7 are satisfied. Hence, every solution of Eq. (22) is oscillatory eventually by Theorems 2.4, 2.6 and 2.7. One of the solutions is

$$y_k = (-1)^k.$$

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