



# Some Properties of Dual Fibonacci and Dual Lucas Octonions

Zafer Ünal\* , Ümit Tokeşer and Göksal Bilgici

**Abstract.** Halici (Adv Appl Clifford Algebr 25(4):905–914, 2015) defined dual Fibonacci and dual Lucas octonions by the relations  $\tilde{Q}_n = Q_n + \varepsilon Q_{n+1}$  and  $\tilde{P}_n = P_n + \varepsilon P_{n+1}$  for every integer  $n$  where  $Q_n$  and  $P_n$  are the Fibonacci and Lucas octonions respectively, and  $\varepsilon$  is the dual unit. The aim of this paper is to investigate properties of dual Fibonacci and dual Lucas octonions. After obtaining the Binet formulas for the sequences  $\{\tilde{Q}_n\}_{n=0}^{\infty}$  and  $\{\tilde{P}_n\}_{n=0}^{\infty}$ , we derive some identities for these sequences such as Catalan's, Cassini's and d'Ocagne's identities.

**Mathematics Subject Classification.** Primary 11B39; Secondary 11B37, 15A66.

**Keywords.** Fibonacci sequence, Lucas sequence, Dual Fibonacci octonion, Dual Lucas octonion.

## 1. Introduction

The quaternions were introduced by Sir William R. Hamilton in 1843. Any  $q$  quaternion number consists of four real coefficients and the set of all quaternions is denoted by

$$\mathbb{H} = \{q : q = a + bi + cj + dk, a, b, c, d \in \mathbb{R}\},$$

where  $i, j, k$  are the standard orthonormal basis in  $\mathbb{R}^3$ , which satisfy the following:

$$\begin{aligned}i^2 &= j^2 = k^2 = -1, \\ij &= -ji = k, \\jk &= -kj = i, \\ki &= -ik = j.\end{aligned}$$

\*Corresponding author.

In 1873, Clifford [3] extended the real numbers to the dual numbers. For  $a, a^*$  real numbers, a dual number is of the form

$$d = a + \varepsilon a^*,$$

where  $\varepsilon$  is the dual unit and  $\varepsilon^2 = 0, \varepsilon \neq 0$ . Also, the dual unit satisfies  $0\varepsilon = \varepsilon 0 = 0, 1\varepsilon = \varepsilon 1 = \varepsilon$ .

We denote the octonion algebra over the field  $\mathbb{R}$  by  $\mathbb{O}$ . Using the Cayley-Dickson method, an octonion  $p$  can be expressed by

$$p = q' + q''e,$$

where  $q', q'' \in \mathbb{H}$  and  $e$  is another square root of  $-1$  (a new imaginary unit). Let  $p_1 = q'_1 + q''_1e$  and  $p_2 = q'_2 + q''_2e$ . In the octonion algebra, the addition and the multiplication of any two octonions are defined by

$$\begin{aligned} p_1 + p_2 &= (q'_1 + q''_1e) + (q'_2 + q''_2e) = q'_1 + q'_2 + (q''_1 + q''_2)e \\ p_1p_2 &= (q'_1 + q''_1e)(q'_2 + q''_2e) = (q'_1q'_2 - \overline{q''_2}q''_1) + (q''_2q'_1 + q''_1\overline{q'_2}), \end{aligned}$$

where  $\overline{q'_2}, \overline{q''_2}$  are the conjugates of the quaternions  $q'_2, q''_2$ , respectively. According to these two binary operations, the octonions form an eight-dimensional non-commutative and non-associative division algebra over the real numbers  $\mathbb{R}$ . The natural basis for octonion algebra over  $\mathbb{R}$  consists of the elements

$$e_0 = 1, \quad e_1 = i, \quad e_2 = j, \quad e_3 = k, \quad e_4 = e, \quad e_5 = ie, \quad e_6 = je, \quad e_7 = ke$$

and any octonion  $p$  is in form of

$$p = \sum_{s=0}^7 a_s e_s, \quad a_s \in \mathbb{R}.$$

The following table is used for multiplication of two octonions.

Any  $p \in \mathbb{O}$  can be written as

$$\begin{aligned} p &= a_0 + \sum_{s=1}^7 a_s e_s \\ &= Re(p) + Im(p), \end{aligned}$$

where  $Re(p) = a_0$  and  $Im(p) = \sum_{s=1}^7 a_s e_s$ . The conjugate and the norm of an octonion  $p$  are defined by

$$\begin{aligned} \bar{p} &= Re(p) - Im(p), \\ N_p &= p\bar{p} = \bar{p}p = \sum_{s=0}^7 a_s^2. \end{aligned}$$

The inverse of any non-zero octonion  $p$  is given by

$$p^{-1} = \frac{\bar{p}}{N_p}.$$

The Fibonacci sequence  $\{F_n\}_{n=0}^\infty$  may be the most well-known one among the integer sequences. It satisfies the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions  $F_0 = 0$  and  $F_1 = 1$ . Another well-known integer sequence is Lucas sequence  $\{L_n\}_{n=0}^\infty$  which satisfies the same recurrence relation with Fibonacci sequence but, initial conditions  $L_0 = 2$  and  $L_1 = 1$ .

The Binet formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.1}$$

and

$$L_n = \alpha^n + \beta^n \tag{1.2}$$

where,  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$  are the solutions of the characteristic equation  $x^2 - x - 1 = 0$ . The positive root  $\alpha$  is known as golden ratio which plays very important role in theory of integer sequences.

Fibonacci and Lucas quaternions were defined by Horadam in [6]. Many authors dealt with Fibonacci or generalized Fibonacci quaternions in literature. Some of these studies can be found in [2, 4, 7-9, 12-14].

We define dual Fibonacci and dual Lucas numbers by

$$\tilde{F}_n = F_n + \varepsilon F_{n+1} \tag{1.3}$$

and

$$\tilde{L}_n = L_n + \varepsilon L_{n+1}, \tag{1.4}$$

respectively.

In [10], Keçilioğlu and Akkus introduced Fibonacci and Lucas octonions and gave their Binet formulas. They defined  $n$ -th Fibonacci and Lucas octonions by

$$Q_n = \sum_{s=0}^7 F_{n+s} e_s$$

and

$$P_n = \sum_{s=0}^7 L_{n+s} e_s$$

respectively, where  $F_n$  and  $L_n$  are  $n$ -th Fibonacci and Lucas numbers, and  $\{e_0, \dots, e_7\}$  is the standard basis of octonions. They also studied split Fibonacci and Lucas octonions in [1].

Halicı, studied dual Fibonacci octonions firstly in [5]. He/she gave generating function, Cassini's identity and some other identities. Dual Fibonacci octonions are defined by

$$\tilde{Q}_n = \sum_{s=0}^7 \tilde{F}_{n+s} e_s. \tag{1.5}$$

By using Eq. (1.3), we obtain

$$\begin{aligned} \tilde{Q}_n &= \sum_{s=0}^7 F_{n+s}e_s + \varepsilon \sum_{s=0}^7 F_{n+s+1}e_s \\ &= \sum_{s=0}^7 (F_{n+s} + \varepsilon F_{n+s+1})e_s \\ &= Q_n + \varepsilon Q_{n+1} \end{aligned} \tag{1.6}$$

Similarly, we define the dual Lucas octonion by the relation

$$\tilde{P}_n = \sum_{s=0}^7 \tilde{L}_{n+s}e_s \tag{1.7}$$

and we have

$$\tilde{P}_n = P_n + \varepsilon P_{n+1}.$$

For abbreviation, throughout this paper we use DFO and DLO for the dual Fibonacci and dual Lucas octonion.

Here, we express Binet formulas for DFOs and DLOs in a different forms from [5]. After having Binet formulas, we give many identities for DFOs and DLOs including Catalan’s, Cassini’s and d’Ocagne’s identities in §2 and §3.

Let  $\mathbb{O}_F$  and  $\mathbb{O}_L$  be the sets of all Fibonacci and Lucas octonions, respectively. The sets of all DFOs and DLOs are denoted by

$$\tilde{\mathbb{O}}_F = \{\tilde{Q}_n = Q_n + \varepsilon Q_{n+1} : Q_n, Q_{n+1} \in \mathbb{O}_F, \varepsilon^2 = 0, \varepsilon \neq 0\}$$

and

$$\tilde{\mathbb{O}}_L = \{\tilde{P}_n = P_n + \varepsilon P_{n+1} : P_n, P_{n+1} \in \mathbb{O}_L, \varepsilon^2 = 0, \varepsilon \neq 0\}$$

respectively.

The conjugate of a DFO and DLO are

$$\overline{\tilde{Q}_n} = \tilde{F}_n - \sum_{s=1}^7 \tilde{F}_{n+s}e_s, \quad \overline{\tilde{P}_n} = \tilde{L}_n - \sum_{s=1}^7 \tilde{L}_{n+s}e_s.$$

The norm of a DFO and DLO are defined by

$$N(\tilde{Q}_n) = \tilde{Q}_n \overline{\tilde{Q}_n} = \overline{\tilde{Q}_n} \tilde{Q}_n, \quad N(\tilde{P}_n) = \tilde{P}_n \overline{\tilde{P}_n} = \overline{\tilde{P}_n} \tilde{P}_n,$$

respectively. By [5], with a small correction, the norm  $N(\tilde{Q}_n)$  is

$$N(\tilde{Q}_n) = 21(\tilde{F}_{2n+7} + \varepsilon F_{2n+8}).$$

Similarly, we have

$$N(\tilde{P}_n) = \sum_{s=0}^7 (\tilde{L}_{n+s})^2$$

and, by using the identity  $L_n^2 = 5F_n^2 + 4(-1)^n$  (see [11, p. 75]) between Fibonacci and Lucas numbers, we get

$$N(\tilde{P}_n) = 105(\tilde{F}_{2n+7} + \varepsilon F_{2n+8}).$$

The DFO with negative indices was given in [5] as

$$\tilde{Q}_{-n} = \sum_{s=0}^7 (-1)^{n-s+1} F_{n-s} e_s + \varepsilon \sum_{s=1}^8 (-1)^{n-s+1} F_{n-s} e_{s-1}.$$

In the same way, we get

$$\begin{aligned} \tilde{P}_{-n} &= \sum_{s=0}^7 \tilde{L}_{-n+s} e_s \\ &= \sum_{s=0}^7 L_{-n+s} e_s + \varepsilon \sum_{s=1}^8 L_{-n+s} e_{s-1}. \end{aligned}$$

By the equation  $L_{-n} = (-1)^n L_n$  [11, p. 84], we obtain

$$\tilde{P}_{-n} = \sum_{s=0}^7 (-1)^{n-s} L_{n-s} e_s + \varepsilon \sum_{s=1}^8 (-1)^{n-s} L_{n-s} e_{s-1}.$$

By using the last equation, we have

$$\tilde{P}_n + \tilde{P}_{-n} = \sum_{s=0}^7 (\tilde{L}_{s+n} + \tilde{L}_{s-n}) e_s.$$

## 2. Binet Formulas for Dual Fibonacci and Lucas Octonions

In this section, we obtain Binet formulas for DFOs and DLOs. Using Binet formulas, we give Catalan’s, Cassini’s and d’Ocagne’s identities. The following theorem gives the Binet Formulas.

**Theorem 2.1.** *For  $n \geq 0$ , the  $n$ -th DFO and  $n$ -th DLO are given by*

$$\tilde{Q}_n = \frac{\alpha' \alpha^n - \beta' \beta^n}{\alpha - \beta}$$

and

$$\tilde{P}_n = \alpha' \alpha^n + \beta' \beta^n$$

where  $\alpha' = (1 + \varepsilon\alpha) \sum_{s=0}^7 \alpha^s e_s$  and  $\beta' = (1 + \varepsilon\beta) \sum_{s=0}^7 \beta^s e_s$ .

*Proof.* In [10], the Binet formulas for the Fibonacci and Lucas octonions are given as follows:

$$Q_n = \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta}$$

and

$$P_n = \alpha^* \alpha^n + \beta^* \beta^n$$

where  $\alpha^* = \sum_{s=0}^7 \alpha^s e_s$  and  $\beta^* = \sum_{s=0}^7 \beta^s e_s$ . By considering these two equations with Eqs. (1.5) and (1.7), the proof is completed.  $\square$

In [10], Catalan’s identity for Fibonacci octonions was given by

$$Q_n^2 - Q_{n+r} Q_{n-r} = (-1)^{n+r} [F_r^2 P_0 - F_{2r} (Q_0 - 14e_5 - 14e_6 - 7e_7)].$$

Catalan’s identities for DFOs and DLOs can be found in the following theorem.

**Theorem 2.2.** *For every integers  $n$  and  $r$ , we have*

$$\tilde{Q}_n^2 - \tilde{Q}_{n+r}\tilde{Q}_{n-r} = (-1)^{n-r} [F_r^2 P_0 - F_{2r}(Q_0 - 7\lambda)] (1 + \varepsilon) \tag{2.1}$$

and

$$\tilde{P}_n^2 - \tilde{P}_{n+r}\tilde{P}_{n-r} = 5(-1)^{n-r+1} [F_r^2 P_0 - F_{2r}(Q_0 - 7\lambda)] (1 + \varepsilon) \tag{2.2}$$

where  $\lambda = 2e_5 + 2e_6 + e_7$ .

*Proof.* For DFOs we have

$$\begin{aligned} \tilde{Q}_n^2 - \tilde{Q}_{n+r}\tilde{Q}_{n-r} &= (Q_n + \varepsilon Q_{n+1})^2 - (Q_{n+r} + \varepsilon Q_{n+r+1})(Q_{n-r} + \varepsilon Q_{n-r+1}) \\ &= Q_n^2 - Q_{n+r}Q_{n-r} \\ &\quad + \varepsilon(Q_n Q_{n+1} + Q_{n+1}Q_n - Q_{n+r}Q_{n-r+1} - Q_{n+r+1}Q_{n-r}). \end{aligned}$$

The dual part of the last equation is

$$\begin{aligned} &Q_n Q_{n+1} + Q_{n+1}Q_n - Q_{n+r}Q_{n-r+1} - Q_{n+r+1}Q_{n-r} \\ &= \frac{1}{5} [(\alpha^* \alpha^n - \beta^* \beta^n)(\alpha^* \alpha^{n+1} - \beta^* \beta^{n+1}) \\ &\quad + (\alpha^* \alpha^{n+1} - \beta^* \beta^{n+1})(\alpha^* \alpha^n - \beta^* \beta^n) \\ &\quad - (\alpha^* \alpha^{n+r} - \beta^* \beta^{n+r})(\alpha^* \alpha^{n-r+1} - \beta^* \beta^{n-r+1}) \\ &\quad - (\alpha^* \alpha^{n+r+1} - \beta^* \beta^{n+r+1})(\alpha^* \alpha^{n-r} - \beta^* \beta^{n-r})] \\ &= \frac{1}{5} [-\alpha^n \beta^n (\alpha + \beta)(\alpha^* \beta^* + \beta^* \alpha^*) \\ &\quad + \alpha^{n-r} \beta^{n-r} (\alpha + \beta)(\alpha^* \beta^* \alpha^{2r} + \beta^* \alpha^* \beta^{2r})]. \end{aligned}$$

By using Table 1, we obtain

$$\alpha^* \beta^* = P_0 - \sqrt{5}Q_0 + 7\sqrt{5}\lambda \tag{2.3}$$

and

$$\beta^* \alpha^* = P_0 + \sqrt{5}Q_0 - 7\sqrt{5}\lambda. \tag{2.4}$$

From Eqs. (2.3) and (2.4), we have

$$\alpha^* \beta^* + \beta^* \alpha^* = 2P_0 \tag{2.5}$$

TABLE 1. The multiplication table for the standard basis of octonions

|         |       |        |        |        |        |        |        |        |
|---------|-------|--------|--------|--------|--------|--------|--------|--------|
| $\cdot$ | 1     | $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
| 1       | 1     | $e_1$  | $e_2$  | $e_3$  | $e_4$  | $e_5$  | $e_6$  | $e_7$  |
| $e_1$   | $e_1$ | -1     | $e_3$  | $-e_2$ | $e_5$  | $-e_4$ | $-e_7$ | $e_6$  |
| $e_2$   | $e_2$ | $-e_3$ | -1     | $e_1$  | $e_6$  | $e_7$  | $-e_4$ | $-e_5$ |
| $e_3$   | $e_3$ | $e_2$  | $-e_1$ | -1     | $e_7$  | $-e_6$ | $e_5$  | $-e_4$ |
| $e_4$   | $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | -1     | $e_1$  | $e_2$  | $e_3$  |
| $e_5$   | $e_5$ | $e_4$  | $-e_7$ | $e_6$  | $-e_1$ | -1     | $-e_3$ | $e_2$  |
| $e_6$   | $e_6$ | $e_7$  | $e_4$  | $-e_5$ | $-e_2$ | $e_3$  | -1     | $-e_1$ |
| $e_7$   | $e_7$ | $-e_6$ | $e_5$  | $e_4$  | $-e_3$ | $-e_2$ | $e_1$  | -1     |

By using Eqs. (2.3)–(2.5), we get

$$\begin{aligned}
 &Q_n Q_{n+1} + Q_{n+1} Q_n - Q_{n+r} Q_{n-r+1} - Q_{n+r+1} Q_{n-r} \\
 &= \frac{1}{5} [2(-1)^{n+1} P_0 + (-1)^{n-r} (L_{2r} P_0 - 5F_{2r} (Q_0 - 7\lambda))].
 \end{aligned}$$

Now, we need the well-known identity between Fibonacci and Lucas numbers  $5F_r^2 = L_{2r} - 2(-1)^r$ . With the help of this identity, we obtain

$$\begin{aligned}
 &Q_n Q_{n+1} + Q_{n+1} Q_n - Q_{n+r} Q_{n-r+1} - Q_{n+r+1} Q_{n-r} \\
 &= (-1)^{n-r} [F_r^2 P_0 - F_{2r} (Q_0 - 7\lambda)] \tag{2.6}
 \end{aligned}$$

From Theorem 2.6 in [10], the real part is

$$Q_n^2 - Q_{n+r} Q_{n-r} = (-1)^{n-r} [F_r^2 P_0 - F_{2r} (Q_0 - 7\lambda)]. \tag{2.7}$$

Equations (2.6) and (2.7) give Eq. (2.1). Equation (2.2) can be obtained in a similar way.  $\square$

For  $r = 1$  this theorem gives Cassini’s identities for DFOs and DLOs.

**Corollary 2.3.** *For any integer  $n$ , we have*

$$\tilde{Q}_n^2 - \tilde{Q}_{n+1} \tilde{Q}_{n-1} = (-1)^{n-1} [P_0 + 7\lambda - Q_0] (1 + \varepsilon)$$

and

$$\tilde{P}_n^2 - \tilde{P}_{n+1} \tilde{P}_{n-1} = 5(-1)^n [P_0 + 7\lambda - Q_0] (1 + \varepsilon),$$

where  $\lambda = 2e_5 + 2e_6 + e_7$ .

In [10], d’Ocagne’s identity for Fibonacci octonions was given by

$$Q_{m+1} Q_n - Q_m Q_{n+1} = (-1)^m [F_{n-m} P_0 - L_{n-m} (Q_0 - 14e_5 - 14e_6 - 7e_7)].$$

D’Ocagne’s identities for DFOs and DLOs are given in the following theorem:

**Theorem 2.4.** *For every integers  $m, n$ , we have*

$$\tilde{Q}_{m+1} \tilde{Q}_n - \tilde{Q}_m \tilde{Q}_{n+1} = (-1)^m [F_{n-m} P_0 - L_{n-m} (Q_0 - 7\lambda)] (1 + \varepsilon) \tag{2.8}$$

and

$$\tilde{P}_{m+1} \tilde{P}_n - \tilde{P}_m \tilde{P}_{n+1} = 5(-1)^{m+1} [F_{n-m} P_0 - L_{n-m} (Q_0 - 7\lambda)] (1 + \varepsilon) \tag{2.9}$$

where  $\lambda = 2e_5 + 2e_6 + e_7$ .

*Proof.* From Theorem 2.8 in [10], we need the following identity.

$$Q_{m+1} Q_n - Q_m Q_{n+1} = (-1)^m [F_{n-m} P_0 - L_{n-m} (Q_0 - 7\lambda)]. \tag{2.10}$$

From the definition of DFO, we have

$$\tilde{Q}_{m+1} \tilde{Q}_n - \tilde{Q}_m \tilde{Q}_{n+1} = Q_{m+1} Q_n - Q_m Q_{n+1} + \varepsilon (-Q_m Q_{n+2} + Q_{m+2} Q_n). \tag{2.11}$$

If we take  $n \rightarrow n + 1$  and  $m \rightarrow m + 1$  in Eq. (2.10), respectively, we obtain

$$Q_{m+1} Q_{n+1} - Q_m Q_{n+2} = (-1)^m [F_{n-m+1} P_0 - L_{n-m+1} (Q_0 - 7\lambda)] \tag{2.12}$$

and

$$Q_{m+2} Q_n - Q_{m+1} Q_{n+1} = (-1)^{m+1} [F_{n-m-1} P_0 - L_{n-m-1} (Q_0 - 7\lambda)]. \tag{2.13}$$

Adding Eqs. (2.12) and (2.13) side by side, the dual part of Eq. (2.11) becomes

$$-Q_m Q_{n+2} + Q_{m+2} Q_n = (-1)^m [F_{n-m} P_0 - L_{n-m} (Q_0 - 7\lambda)]. \tag{2.14}$$

Considering Eqs. (2.10) and (2.14) together, we obtain Eq. (2.8).

Similarly, for the Eq.(2.9) it is sufficient to calculate the real part.

$$\begin{aligned} P_{m+1} P_n - P_m P_{n+1} &= (\alpha^* \alpha^{m+1} + \beta^* \beta^{m+1})(\alpha^* \alpha^n + \beta^* \beta^n) \\ &\quad - (\alpha^* \alpha^m + \beta^* \beta^m)(\alpha^* \alpha^{n+1} + \beta^* \beta^{n+1}) \\ &= \alpha^m \beta^m [\alpha^* \beta^* \beta^{n-m} (\alpha - \beta) - \beta^* \alpha^* \alpha^{n-m} (\alpha - \beta)] \\ &= (-1)^m \sqrt{5} (\alpha^* \beta^* \beta^{n-m} - \beta^* \alpha^* \alpha^{n-m}) \\ &= 5(-1)^{m+1} (F_{n-m} P_0 + L_{n-m} Q_0 - 7\lambda L_{n-m}). \end{aligned} \tag{2.15}$$

If we repeat same steps above, we reach to the dual part of Eq. (2.9). □

### 3. Some Identities

In this section, we give other identities for DFOs and DLOs. One can check these results by using Table 1 and definitions of DFO and DLO, easily.

We have the following identities by the help of definitions of DFO and DLO, directly.

$$\begin{aligned} \sum_{k=0}^n \tilde{Q}_k &= \tilde{Q}_{n+2} - \tilde{Q}_1, & \sum_{k=0}^n \tilde{P}_k &= \tilde{P}_{n+2} - \tilde{P}_1, \\ \sum_{k=1}^n \tilde{Q}_{2k-1} &= \tilde{Q}_{2n} - \tilde{Q}_0, & \sum_{k=1}^n \tilde{P}_{2k-1} &= \tilde{P}_{2n} - \tilde{P}_0, \\ \sum_{k=1}^n \tilde{Q}_{2k} &= \tilde{Q}_{2n+1} - \tilde{Q}_{-1}, & \sum_{k=1}^n \tilde{P}_{2k} &= \tilde{P}_{2n+1} - \tilde{P}_{-1}. \end{aligned}$$

The following theorem gives the relationship between DFO and DLO.

**Theorem 3.1.** *For every integer n we have*

$$\tilde{P}_n = \tilde{Q}_{n-1} + \tilde{Q}_{n+1}.$$

*Proof.* To prove the theorem we need the identity  $L_n = F_{n-1} + F_{n+1}$ . By using it, we obtain

$$\begin{aligned} \tilde{Q}_{n-1} + \tilde{Q}_{n+1} &= Q_{n-1} + \varepsilon Q_n + Q_{n+1} + \varepsilon Q_{n+2} \\ &= \sum_{s=0}^7 (F_{n+s-1} + F_{n+s+1}) e_s + \varepsilon \sum_{s=0}^7 (F_{n+s} + F_{n+s+2}) e_s \\ &= \sum_{s=0}^7 L_{n+s} e_s + \varepsilon \sum_{s=0}^7 L_{n+s+1} e_s \\ &= \tilde{P}_n. \end{aligned}$$

□

By using Binet formulas for DFOs and DLOs, the following identities can be shown easily.

$$\begin{aligned} \tilde{P}_n^2 + \tilde{Q}_n^2 &= -819L_{2n} - 1827F_{2n} + \frac{6}{5}L_{2n}P_0 + 6F_{2n}Q_0 + \frac{8}{5}(-1)^n P_0 \\ &\varepsilon \left( \frac{12}{5}L_{2n+1}P_0 + 12F_{2n+1}Q_0 - 1638L_{2n+1} - 3654F_{2n+1} + \frac{8}{5}(-1)^n P_0 \right), \end{aligned}$$

$$\begin{aligned} \tilde{P}_n^2 - \tilde{Q}_n^2 &= -546L_{2n} - 1218F_{2n} + \frac{4}{5}L_{2n}P_0 + 4F_{2n}Q_0 + \frac{12}{5}(-1)^n P_0 \\ &+\varepsilon \left( \frac{8}{5}L_{2n+1}P_0 + 8F_{2n+1}Q_0 - 1092L_{2n+1} - 2436F_{2n+1} + \frac{12}{5}(-1)^n P_0 \right), \end{aligned}$$

$$\begin{aligned} \tilde{Q}_m \tilde{P}_n - \tilde{P}_m \tilde{Q}_n &= \left[ \tilde{F}_{m+n}(2P_0 - 1365) + \tilde{L}_{m+n}(2Q_0 - 609) \right] \\ &+\varepsilon [F_{m+n+1}(2P_0 - 1365)L_{m+n+1}(2Q_0 - 609)], \end{aligned}$$

$$\begin{aligned} \tilde{Q}_n \tilde{P}_n &= P_0 F_{2n} + Q_0 L_{2n} - \frac{1365}{2} F_{2n} - \frac{609}{2} L_{2n} + (-1)^n (-2Q_0 + 14\lambda) \\ &+\varepsilon \left[ P_0 F_{2n+1} + Q_0 L_{2n+1} - \frac{1365}{2} F_{2n+1} - \frac{609}{2} L_{2n+1} \right], \end{aligned}$$

$$\begin{aligned} \tilde{Q}_{n+r} L_{n+r} + \tilde{Q}_{n-r} L_{n-r} &= L_{2r} \tilde{Q}_{2n} + 2(-1)^{n+r} \tilde{Q}_0, \\ \tilde{Q}_{n+r} F_{n+r} - \tilde{Q}_{n-r} F_{n-r} &= F_{2r} \tilde{Q}_{2n}, \\ \tilde{P}_{n+r} \tilde{Q}_{n+s} - \tilde{P}_{n+s} \tilde{Q}_{n+r} &= 2(-1)^{n+r} P_0 F_{s-r} (1 + \varepsilon), \\ \tilde{Q}_{m+n} + (-1)^n \tilde{Q}_{m-n} &= Q_m L_n (1 + \varepsilon), \\ \tilde{P}_{n+r} L_{n+r} + \tilde{P}_{n-r} L_{n-r} &= L_{2r} \tilde{P}_{2n} + 2(-1)^{n+r} \tilde{P}_0, \\ \tilde{P}_{n+r} L_{n+r} - \tilde{P}_{n-r} L_{n-r} &= 5F_{2r} \tilde{Q}_{2n}, \\ \tilde{P}_{n+r} F_{n+t} + \tilde{P}_{n+t} F_{n+r} &= 2\tilde{Q}_{2n+r+t} - (-1)^{n+t} L_{r-t} \tilde{Q}_0. \end{aligned}$$

**References**

- [1] Akkus, I., Keçilioğlu, O.: Split Fibonacci and Lucas octonions. *Adv. Appl. Clifford Algebras* **25**(3), 517–525 (2015)
- [2] Akyiğit, M., Köksal, H.H., Tosun, M.: Fibonacci generalized quaternions. *Adv. Appl. Clifford Algebras* **24**(3), 631–641 (2014)
- [3] Clifford, W.K.: Preliminary sketch of bi-quaternions. *Proc. Lond. Math. Soc.* **4**(1), 381–395 (1871)
- [4] Halici, S.: On Fibonacci quaternions. *Adv. Appl. Clifford Algebras* **22**(2), 321–327 (2012)
- [5] Halici, S.: On dual Fibonacci octonions. *Adv. Appl. Clifford Algebras* **25**(4), 905–914 (2015)
- [6] Horadam, A.F.: Complex Fibonacci numbers and Fibonacci quaternions. *Am. Math. Mon.* **70**(3), 289–291 (1963)

- [7] Horadam, A.F.: Quaternion recurrence relations. *Ulam Quart.* **2**(2), 23–33 (1993)
- [8] Iakin, A.L.: Generalized quaternions of higher order. *Fibonacci Quart.* **15**(4), 343–346 (1977)
- [9] Iyer, M.R.: A note on Fibonacci quaternions. *Fibonacci Quart.* **7**(3), 225–229 (1969)
- [10] Keçilioğlu, O., Akkus, I.: The Fibonacci octonions. *Adv. Appl. Clifford Algebras* **25**(1), 151–158 (2015)
- [11] Koshy, T.: *Fibonacci and Lucas Numbers with Applications*. Wiley-Interscience Publication, Canada (2001)
- [12] Ramirez, J.L.: Some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions. *An. Stiint. Univ. “Ovidius” Constanta Ser. Mat.* **23**(2), 201–212 (2015)
- [13] Swamy, M.N.S.: On generalized Fibonacci quaternions. *Fibonacci Quart.* **11**(5), 547–549 (1973)
- [14] Tan, E., Yilmaz, S., Sahin, M.: On a new generalization of Fibonacci quaternions. *Chaos Solitons Fract.* **82**, 1–4 (2016)

Zafer Ünal and Ümit Tokeşer  
Department of Mathematics  
Faculty of Arts and Sciences  
Kastamonu University  
Kastamonu  
Turkey  
e-mail: [zunal@kastamonu.edu.tr](mailto:zunal@kastamonu.edu.tr)

Ümit Tokeşer  
e-mail: [utokeser@kastamonu.edu.tr](mailto:utokeser@kastamonu.edu.tr)

Göksal Bilgici  
Department of Computer Education and Instructional Technology  
Education Faculty  
Kastamonu University  
Kastamonu  
Turkey  
e-mail: [gbilgici@kastamonu.edu.tr](mailto:gbilgici@kastamonu.edu.tr)

Received: March 3, 2016.

Accepted: August 22, 2016.