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ORIGINAL PAPER

Dynamics of Lump, Breather, Two-Waves and Other Interaction Solutions of (2+1)-Dimensional KdV Equation

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Abstract

In this investigation, we address a particular variant of the Korteweg–de Vries (KdV) equation, specifically focusing on the (2+1)-dimensional KdV equation. The equation can model various physical phenomena in different fields, including fluid dynamics, plasma physics, nonlinear optics, and other areas where coupled wave interactions are important. To commence, we establish the Auto-Bäcklund and Cole–Hopf transformations for the given model, resulting in the derivation of numerous soliton-like solutions characterized by hyperbolic, trigonometric, and exponential function waves. Furthermore, we effectively elucidate the behavior of lump, lump–kink, breather, two-wave, and three-wave solutions using the Hirota bilinear technique. Extensive numerical simulations employing 3-D profiles are conducted with meticulous consideration of pertinent parameter values, providing additional insights into the distinctive traits of the obtained solutions. Moreover, employing the extended transformed rational function method grounded in the bilinear form of the underlying equation, we uncover complexiton solutions. These solutions are depicted using 3-D and 2-D visualizations to portray their dynamics. Our findings reveal that the approach adopted to derive analytical solutions for nonlinear partial differential equations proves to be both efficient and potent. The combination of numerical simulations and visual representations enhances our understanding of these solutions, ultimately affirming the effectiveness and robustness of the employed methodology in tackling nonlinear partial differential equations.

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Introduction

Nonlinear differential equations have an important role in mathematical physics, fluid dynamics, finance, control theory, chemistry, optics and so on. One of these equations is the classical (1 + 1)-dimensional Korteweg–de Vries (KdV) equation that is given in the following form [1]:

$$u_t + 6uu_x + u_{xxx} = 0.$$

This equation is utilized to characterize the waves in shallow water surfaces. Some scholars have suggested novel implications for the classical KdV equation, including the formation of acoustic waves on crystal lattices and ionized plasma. The classical KdV equation comes in a variety of forms, some of them are generalized KdV, modified KdV and the (2+1)-dimensional KdV equations [2]. The (2+1)-dimensional KdV equation is presented in the following way in this essay

$$u_t + 3vu_x + 3uv_x + u_{xxx} = 0, \qquad u_x - v_y = 0, \tag{1}$$

The bilinear form of the underlying equation is given as

$$(\Delta_y\Delta_t + \Delta_x^3\Delta_y)\mu\cdot\mu = \mu_{yt}\mu - \mu_y\mu_t + \mu\mu_{xxxy} + 3\mu_{xy}\mu_{xx} - 3\mu_x\mu_{xxy} - \mu_y\mu_{xxx} = 0, \tag{2}$$

using logarithmic transformations

$$u = 2(\ln\mu)_{xy}, \quad v = 2(\ln\mu)_{xx}.$$

The weak Lax pair concept was used to generate the equation [3]. The (1+1)-dimensional KdV equation is obtained by taking $v = u$ and $y = x$ in Eq. (3). The Boiti–Leon–Manna–Pempinelli (BLMP) expression is established by considering $u = w_y, v = w_x$ in Eq. (3). Many researchers were interested in the (2+1) dimensional KdV equation for example Wazwaz derived multiple-soliton solutions by the Hirota's bilinear method in [4], Raza et al. derived variety of soliton solutions by different analytic methods [5]. Cao et al. generated deformed multi-solitons and deformed breathers in [6], Ozkan et al. applied the improved $\tan(\phi/2)$- expansion and Jacobi elliptic function expansion methods for finding the exact solutions in [7], Peng obtained periodic waves and periodic solitons of the given model in [8], Wang et al. introduce two subequations with different independent variables for constructing exact solutions in [9], Zhang and Chen derived the deformation rogue wave by the bilinear method in [10], Zhai and Zhao applied the Pfaffian Technique to the given model in [11], Lou obtained the generalized dromion solutions of the given equation in [12].

In the literature, there are lots of methods for finding the exact solutions of nonlinear partial differential equations, for example the Legendre reproducing kernel method [13], the Nucci's reduction method [14], Heir-equations method [15], the enhanced Kudryashov's and improved extended tanh-function techniques [16], Kudryashov and exponential methods [17], the generalized Kudryashov, modified extended tanh and exponential rational function methods [18], the Jacobi elliptic function method [19], the modified simple equation [20], the Homotopy analysis method [21], trial equation technique [22], the extended simplest equation method [23], the extended trial function approach [24], the extended simple equation method

[25], Hamiltonian's system [26], generalized exponential rational function method [27–30], and so on.

In this study, we will apply one of the important methods for finding the complexiton solutions of nonlinear evolution equations which is the extended transformed rational function method to the given model. This method based on the Hirota bilinear forms of the equations. Ma and Lee have offered direct method with the help of the rational function transformations. Then Zhang and Ma have developed the extended transformed rational function method as an extension of this method in [31]. Many researchers applied this method to the nonlinear partial differential equations for example Yaşar et al. applied the used method to the two different nonlinear differential equations which are (2+1) and (3+1) dimensional in [32], Ünsal et al. used for obtaining the the complexiton solutions of Sawada–Kotera and ninth-order KdV equations in [33], Mirzazadeh applied the discussed method to the (3+1)-dimensional generalized KP equation with variable coefficients in [34] and so on.

The aim of the paper is to obtain the soliton and complexiton solutions of the (2+1)-dimensional KdV equation. For this purpose, some necessary informations are given in SECTION, then we find the Auto-Bäcklund transformation, Cole–Hopf transformation of the given model in Sect. 2. In Sect. 3, solitary waves solutions to (2+1)-dimensional KdV are extracted. In Sect. 4, various interaction aspects have been discussed. In the next section, we applied the extended transformed rational function method to the given model also, 3D and 2D figures of the obtained solutions are represented. Finally, conclusion is given.

Auto-Bäcklund Transformation for (2+1)-Dimensional KdV Equation

We can use the solution of Eq. (1) based on the extended homogeneous balancing approach as follows [35]

$$u(x,y,t) = p''(\mu)\mu_x^2 + p'(\mu)\mu_{xx} + \breve{u}, \quad v(x,y,t) = q''(\mu)\mu_x^2 + q'(\mu)\mu_{xx} + \breve{v}, \tag{3}$$

p, q, and μ are the three functions that will be determined, and $\breve{u}(x,y,t)$ is a seed solution of Eq. (1). This is a simple seed solution that can be used to build a variety of additional solutions. When Eq. (3) is inserted for Eq. (1), the result is

$$\begin{aligned}
&[p^{(5)} + 3p''q''' + 3p'''q'']\mu_x^5 + [10p^{(4)} + 3p'''q' + 18p''q'' + 3p'q''']\mu_x^3\mu_{xx} \\
&\quad + [(10\mu_x^2\mu_{xxx} + 15\mu_x\mu_{xx}^2 + 3\breve{v}\mu_x^3 + \mu_t\mu_x^2)p''' + 3\breve{u}\mu_x^3 q''' + (9\mu_x\mu_{xx}^2 + 3\mu_x^2\mu_{xxx})p''q' \\
&\quad + (9\mu_x\mu_{xx}^2 + 3\mu_x^2\mu_{xxx})p'q''] + [(10\mu_{xx}\mu_{xxx} + 5\mu_x\mu_{xxxx} + 2\mu_x\mu_{xt} + \mu_t\mu_{xx} + 3\breve{v}_x\mu_x^2 \\
&\quad + 9\breve{v}\mu_x\mu_{xx})p'' + (3\breve{u}_x\mu_x^2 + 9\breve{u}\mu_x\mu_{xx})q'' + 6p'q'\mu_{xx}\mu_{xxx}] + [(3\breve{v}\mu_{xxx} + 3\breve{v}_x\mu_{xx} \\
&\quad + \mu_{xxt} + \mu_{xxxxx})p' + (3\breve{u}_x\mu_{xx} + 3\breve{u}\mu_{xxx})q'] + [\breve{u}_t + \breve{u}_{xxx} + 3\breve{v}\breve{v}_x + 3\breve{u}\breve{v}_x] = 0,
\end{aligned} \tag{4}$$

$$\begin{aligned}
&(p'''\mu_x^3 - q'''\mu_y\mu_x^2) + [3p''\mu_x\mu_{xx} - (\mu_x\mu_{xy} + \mu_y\mu_{xx})q''] + (p'\mu_{xxx} - q'\mu_{xxy}) \\
&\quad + (\breve{u}_x - \breve{v}_y) = 0.
\end{aligned} \tag{5}$$

In order to simplify the expression [36], we suppose that

$$p = q = s\ln(\mu), \tag{6}$$

where s is an arbitrary constant. Then we get

$$\begin{aligned}
&p^{''}p^{'''}=-\frac{s}{12}p^{(5)}, \qquad p^{'}p^{'''}=p^{'}q^{'''}=p^{'''}q^{'}=-\frac{s}{3}p^{(4)},\\
&p^{''}q^{''}=-\frac{s}{6}p^{(4)}, \qquad p^{'}q^{''}=p^{''}q^{'}=-\frac{s}{2}p^{'''},\\
&p^{'}q^{'}=-sp^{''}.
\end{aligned} \tag{7}$$

Eqs. (4) and (5) can be recast as the sum of certain terms of $p^{(0)}$, $p^{'}$, $p^{''}$, $p^{'''}$, $p^{(4)}$, and $p^{(5)}$ by utilizing the equalities

$$\begin{aligned}
p^{(5)} &: 1-\frac{s}{2}=0,\\
p^{(4)} &: 10\mu_x^3\mu_{xx}+3\left(-\frac{s}{3}\right)\mu_x^3\mu_{xx}+18\left(-\frac{s}{6}\right)\mu_x^3\mu_{xx}+3\left(-\frac{s}{3}\right)\mu_x^3\mu_{xx}=0,\\
p^{'''} &: 10\mu_x^2\mu_{xxx}+15\mu_x\mu_{xx}^2+3\breve{v}\mu_x^3+\mu_t\mu_x^2+3\breve{u}\mu_x^3+(9\mu_x\mu_{xx}^2+3\mu_x^2\mu_{xxx}\\
&\quad +9\mu_x\mu_{xx}^2+3\mu_x^2\mu_{xxx})\left(-\frac{s}{2}\right)=0,\\
p^{''} &: 10\mu_{xx}\mu_{xxx}+5\mu_x\mu_{xxxx}+2\mu_x\mu_{xt}+\mu_t\mu_{xx}+3\breve{v}_x\mu_x^2+9\breve{v}\mu_x\mu_{xx}-6s\mu_{xx}\mu_{xxx}=0,\\
p^{'} &: 3\breve{v}\mu_{xxx}+3\breve{v}_x\mu_{xx}+\mu_{xxt}+\mu_{xxxxx}+3\breve{u}_x\mu_{xx}+3\breve{u}\mu_{xxx}=0,\\
p^{(0)} &: \breve{u}_t+\breve{u}_{xxx}+3\breve{v}\breve{v}_x+3\breve{u}\breve{v}_x=0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
p^{'''} &: \mu_x^3-\mu_y\mu_x^2,\\
p^{''} &: 3\mu_x\mu_{xx}-\mu_x\mu_{xy}+\mu_y\mu_{xx}=0,\\
p^{'} &: \mu_{xxx}-\mu_{xxy},\\
p^{(0)} &: \breve{u}_x-\breve{v}_y=0.
\end{aligned} \tag{9}$$

We deduce the following from the system (8) initial equation:

$$s=2,$$

The systems (8) and (9) can be expressed as follows utilizing the equalities (7)

$$\begin{aligned}
&[\mu_x(4\mu_x\mu_{xxx}-3\mu_{xx}^2+\mu_t\mu_x+3\breve{v}\mu_x^2+3\breve{u}\mu_x^2)]p^{'''}+[10\mu_{xx}\mu_{xxx}+5\mu_x\mu_{xxxx}\\
&\quad +2\mu_x\mu_{xt}+\mu_t\mu_{xx}+3\breve{v}_x\mu_x^2+9\breve{v}\mu_x\mu_{xx}-12\mu_{xx}\mu_{xxx}]p^{''}+[3\breve{v}\mu_{xxx}+3\breve{v}_x\mu_{xx}\\
&\quad +\mu_{xxt}+\mu_{xxxxx}+3\breve{u}_x\mu_{xx}+3\breve{u}\mu_{xxx}]p^{'}+[\breve{u}_t+\breve{u}_{xxx}+3\breve{v}\breve{v}_x+3\breve{u}\breve{v}_x]=0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&[\mu_x(\mu_x^2-\mu_y\mu_x)]p^{'''}+[3\mu_x\mu_{xx}-\mu_x\mu_{xy}+\mu_y\mu_{xx}]p^{''}\\
&\quad +[\mu_{xxx}-\mu_{xxy}]p^{'}+[\breve{u}_x-\breve{v}_y]=0.
\end{aligned} \tag{11}$$

We obtain the following differential equations by substituting zero for the coefficients of $p^{'''}$, $p^{''}$, $p^{'}$, and $p^{(0)}$ in (10):

$$4\mu_x\mu_{xxx}-3\mu_{xx}^2+\mu_t\mu_x+3\breve{v}\mu_x^2+3\breve{u}\mu_x^2=0, \tag{12}$$

$$10\mu_{xx}\mu_{xxx}+5\mu_x\mu_{xxxx}+2\mu_x\mu_{xt}+\mu_t\mu_{xx}+3\breve{v}_x\mu_x^2+9\breve{v}\mu_x\mu_{xx}-12\mu_{xx}\mu_{xxx}=0, \tag{13}$$

$$3\breve{v}\mu_{xxx} + 3\breve{v}_x\mu_{xx} + \mu_{xxt} + \mu_{xxxxx} + 3\breve{u}_x\mu_{xx} + 3\breve{u}\mu_{xxx} = 0, \tag{14}$$
$$\breve{u}_t + \breve{u}_{xxx} + 3\breve{v}\breve{v}_x + 3\breve{u}\breve{v}_x = 0. \tag{15}$$

Similarly, we obtain the following differential equations by substituting zero for the coefficients of $p^{'''}$, $p^{''}$, $p^{'}$, and $p^{(0)}$ in (11):

$$\mu_x^2 - \mu_y\mu_x = 0, \tag{16}$$
$$3\mu_x\mu_{xx} - \mu_x\mu_{xy} + \mu_y\mu_{xx} = 0, \tag{17}$$
$$\mu_{xxx} - \mu_{xxy} = 0, \tag{18}$$
$$\breve{u}_x - \breve{v}_y = 0. \tag{19}$$

We can get the relationship between Eqs. (12)-(14) by differentiating Eq. (12) and integrating Eq. (14) with respect to x.

$$\begin{aligned}\frac{\partial}{\partial x}(4\mu_x\mu_{xxx} - 3\mu_{xx}^2 + \mu_t\mu_x + 3\breve{v}\mu_x^2 + 3\breve{u}\mu_x^2) + \mu_x\int(3\breve{v}\mu_{xxx}\\ + 3\mu_{xx}\breve{v}_x + 3\breve{u}\mu_{xxx} + 3\mu_{xx}\breve{u}_x + \mu_{xxt} + \mu_{xxxxx})dx = 10\mu_{xx}\mu_{xxx}\\ + 5\mu_x\mu_{xxxx} + 2\mu_x\mu_{xt} + \mu_t\mu_{xx} + 3\breve{v}_x\mu_x^2 + 9\breve{v}\mu_x\mu_{xx} - 12\mu_{xx}\mu_{xxx}.\end{aligned} \tag{20}$$

Then the following linearly independent equations satisfies Eq. (20),

$$4\mu_x\mu_{xxx} - 3\mu_{xx}^2 + \mu_t\mu_x + 3\breve{v}\mu_x^2 + 3\breve{u}\mu_x^2 = 0, \tag{21}$$
$$3\breve{v}\mu_{xx} + 3\breve{u}\mu_{xx} + \mu_{xt} + \mu_{xxxx} = 0, \tag{22}$$
$$\breve{u}_t + \breve{u}_{xxx} + 3\breve{v}\breve{v}_x + 3\breve{u}\breve{v}_x = 0. \tag{23}$$

Similarly, we can obtain the relationship between Eqs. (16)–(18).

$$\begin{aligned}\frac{\partial}{\partial x}(\mu_x^2 - \mu_x\mu_y) + \mu_x\int(\mu_{xxx} - \mu_{xxy})dx\\ = 3\mu_x\mu_{xx} - \mu_x\mu_{xy} + \mu_y\mu_{xx}.\end{aligned} \tag{24}$$

Then the following linearly independent equations satisfies Eq. (24),

$$\mu_x^2 - \mu_y\mu_x = 0, \tag{25}$$
$$\mu_{xx} - \mu_x\mu_y = 0, \tag{26}$$
$$\breve{u}_x - \breve{v}_y = 0. \tag{27}$$

Hence, the auto-Bäcklund transformation to the underlying problem is expressed as

$$u(x, y, t) = -2\frac{\mu_x}{\mu^2} + 2\frac{\mu_{xx}}{\mu} + \breve{u}, \qquad v(x, y, t) = -2\frac{\mu_x}{\mu^2} + 2\frac{\mu_{xx}}{\mu} + \breve{v}, \tag{28}$$

where $\mu(x, y, t)$, $\breve{u}(x, y, t)$ and $\breve{v}(x, y, t)$ satisfy Eqs. (21)–(23) and Eqs. (25)–(27). Thus, the traveling wave solutions to Eq. (1) from Eq. (28) can be generated by solving Eqs. (21), (22), (25) and (26) for the given sets of solution $\breve{u}$, $\breve{v}$ to Eq. (1).
1. Assuming $\breve{u} = k$ and $\breve{v} = l$ in Eq. (28), we obtain

$$u(x, y, t) = -2\frac{\mu_x}{\mu^2} + 2\frac{\mu_{xx}}{\mu} + k, \qquad v(x, y, t) = -2\frac{\mu_x}{\mu^2} + 2\frac{\mu_{xx}}{\mu} + l, \tag{29}$$

where $\mu(x, y, t)$ verifies the equations below

$$4\mu_x\mu_{xxx} - 3\mu_{xx}^2 + \mu_t\mu_x + 3\breve{v}\mu_x^2 + 3\breve{u}\mu_x^2 = 0, \tag{30}$$

$$3l\mu_{xx} + 3k\mu_{xx} + \mu_{xt} + \mu_{xxxx} = 0, \tag{31}$$

$$\mu_x^2 - \mu_y\mu_x = 0, \tag{32}$$

$$\mu_{xx} - \mu_x\mu_y = 0. \tag{33}$$

2. Setting $\breve{u} = 0$ and $\breve{v} = 0$ in Eq. (28), yields Cole–Hopf transformation

$$u(x, y, t) = -2\frac{\mu_x}{\mu^2} + 2\frac{\mu_{xx}}{\mu}, \qquad v(x, y, t) = -2\frac{\mu_x}{\mu^2} + 2\frac{\mu_{xx}}{\mu}, \tag{34}$$

where $\mu(x, y, t)$ satisfies Eqs. (30)–(33).

Solitary Wave Solutions

This section comprises the auto-Bäcklund transformation introduced in the previous part to derive several soliton-like solutions of Eq. (1). Commencing with the mentioned transformation (29) and the seed solutions $\breve{u}$, $\breve{v}$ of Eq. (1) to obtain our aim.
A. Take $\mu(x, y, t)$ be of the following type

$$\mu(x, y, t) = d\cosh(fx + gy + ct + \nu) + e\sinh(fx + gy + ct + \nu) + m, \tag{35}$$

$d, e, f, g, c,$ and$m$ are constants that will be determined later, while ν is arbitrary. An algebraic equation system is confirmed by substituting (35) into Eqs. (30)–(33). We get the following two cases from the system's solutions.
Case-1

$$d = e, \qquad g = f, \qquad c = -f^3 - 3kf - 3lf, \tag{36}$$

where e, f, m and ν are arbitrary parameters. Thus, we obtain

$$\begin{aligned}\mu(x, y, t) = e[&\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)\\ &+ \sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)] + m,\end{aligned} \tag{37}$$

Then by entering Eq. (37) into Eq. (29), solitary wave solution for (2 + 1)-dimensional KdV is given by (Figs. 1 and 2):

$$\begin{aligned}u_1 = &-\frac{2ef[\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + \cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{(b\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + b\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m)^2}\\ &+\frac{2ef^2[\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + \sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{e\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + e\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m} + k,\end{aligned} \tag{38}$$

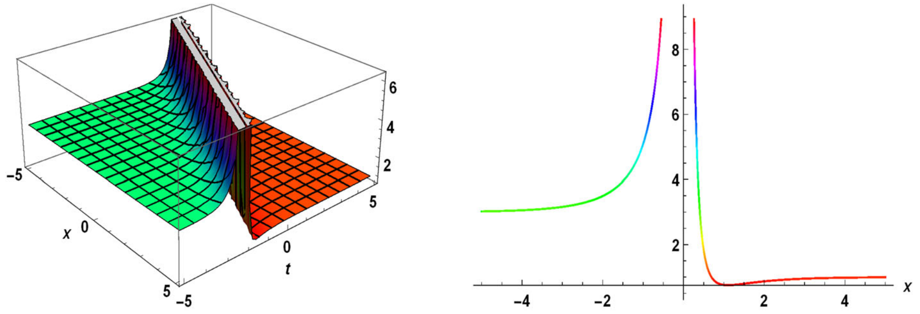
$$\begin{aligned}v_1 = &-\frac{2ef[\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + \cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{(b\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + b\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m)^2}\\ &+\frac{2ef^2[\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + \sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{e\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + e\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m} + l.\end{aligned} \tag{39}$$

Case-2

$$d = -e, \qquad g = f, \qquad c = -f^3 - 3kf - 3lf, \tag{40}$$

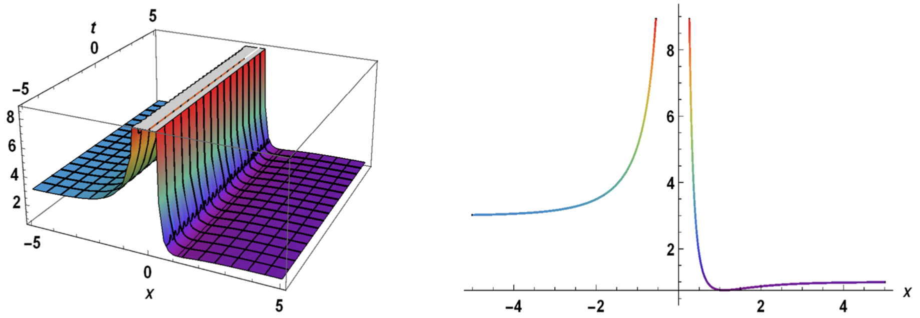
where e, f, m and ν are arbitrary constants. Thus, we obtain

$$\mu(x, y, t) = e[\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)$$



(a) **(b)**

Fig. 1 Solitary wave solution for $u_1(x, y, t)$ acquired in Case-1 for $e = 2,\ f = -1, k = 1, \nu = 1, l = -2, m = -2, y = 0$



(a) **(b)**

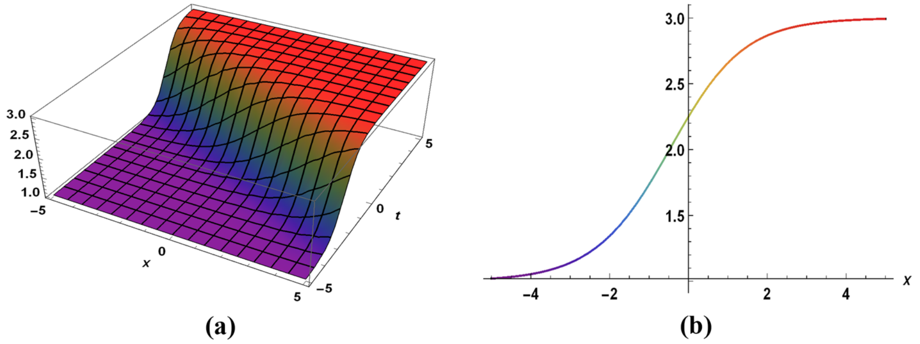
Fig. 2 Solitary wave solution for $v_1(x, y, t)$ acquired in Case-1 for $e = 2,\ f = -1, k = 1, \nu = 1, l = -2, m = -2, y = 0$

$$-\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)] + m, \tag{41}$$

Then by entering Eq. (41) into Eq. (29), solitary wave solution for (2 + 1)-dimensional KdV is given by (Figs. 3 and 4):

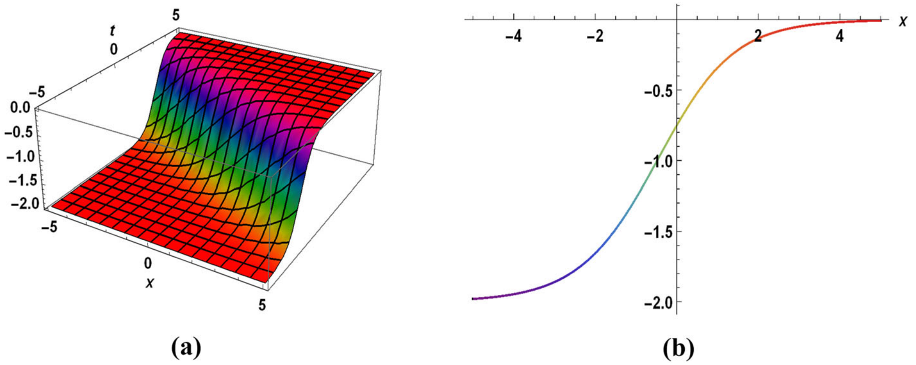
$$\begin{aligned} u_2 = &-\frac{2ef[\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - \sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{(b\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - b\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m)^2} \\ &+ \frac{2ef^2[\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - \cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{e\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - e\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m} + k, \end{aligned} \tag{42}$$

$$\begin{aligned} v_2 = &-\frac{2ef[\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - \sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{(b\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - b\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m)^2} \\ &+ \frac{2ef^2[\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - \cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu)]}{e\sinh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) - e\cosh(fx + fy + (-f^3 - 3kf - 3lf)t + \nu) + m} + l, \end{aligned} \tag{43}$$



(a) (b)

Fig. 3 Traveling wave solution for $u_2(x, y, t)$ acquired in Case-2 for $e = 2$, $f = -1$, $k = 1$, $\nu = 1$, $l = -2$, $m = -2$, $y = 0$



(a) (b)

Fig. 4 Traveling wave solution for $v_2(x, y, t)$ acquired in Case-2 for $e = 2$, $f = -1$, $k = 1$, $\nu = 1$, $l = -2$, $m = -2$, $y = 0$

B. Assume $\mu(x, y, t)$ of the following form

$$\mu(x, y, t) = d\cos(fx + gy + ct + \nu) + e\sin(fx + gy + ct + \nu) + m, \tag{44}$$

where d, e, f, g, c and m are unknown constants and ν represents an arbitrary parameter. Entering (44) into Eqs. (30)–(33) an equation system is satisfied.

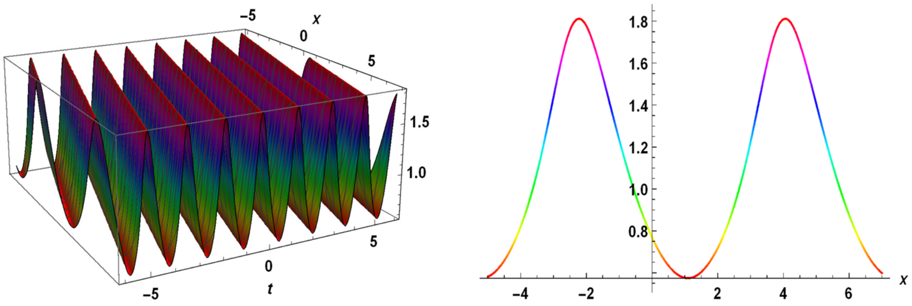
$$d = \iota e, \qquad g = f, \qquad c = f^3 - 3kf - 3lf, \tag{45}$$

where e, f, m and ν are arbitrary constants.

$$\begin{aligned} \mu(x, y, t) = e[&\iota\sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu) \\ &+ \cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu)] + m, \end{aligned} \tag{46}$$

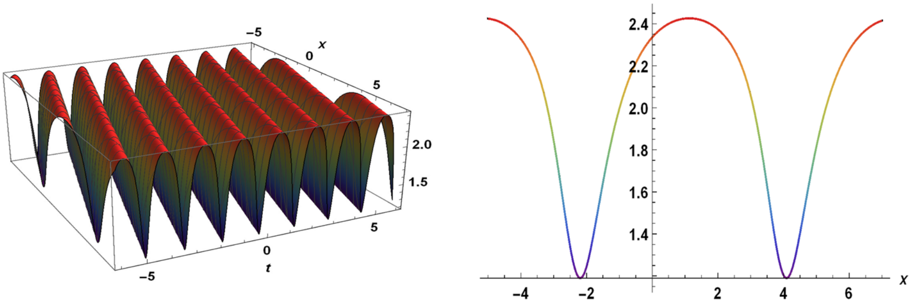
Then by inserting Eq. (46) into Eq. (29), travelling wave solution for (2 + 1)- dimensional KdV is given by (Figs. 5 and 6):

$$\begin{aligned} u_3 = &-\frac{2ef[-\iota\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + \sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu)]}{(\iota e\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + e\sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + m)^2} \\ &+ \frac{2ef^2[-\iota e\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) - e\sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu)]}{\iota e\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + e\sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + m} + k, \end{aligned} \tag{47}$$



(a) **(b)**

Fig. 5 Periodic wave solution for $u_3(x, y, t)$ acquired in (B) for $e = \frac{1}{2}$, $f = -1$, $k = 1$, $\nu = 1$, $l = -2$, $m = -2$, $y = 0$



(a) **(b)**

Fig. 6 Periodic wave solution for $v_3(x, y, t)$ acquired in (B) for $e = \frac{1}{2}$, $f = -1$, $k = 1$, $\nu = 1$, $l = -2$, $m = -2$, $y = 0$

$$v_3 = -\frac{2ef[-\iota\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + \sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu)]}{(\iota e\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + e\sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + m)^2} + \frac{2ef^2[-\iota e\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) - e\sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu)]}{\iota e\cos(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + e\sin(fx + fy + (f^3 - 3kf - 3lf)t + \nu) + m} + l, \tag{48}$$

C. Now we assume the solution of Eq. (1) of the following form

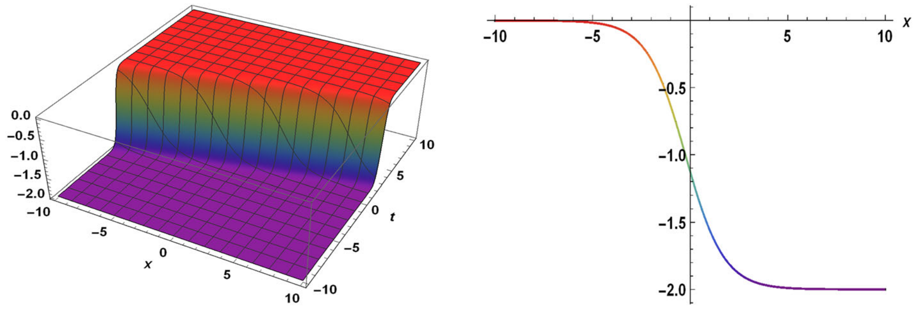
$$\mu(x, y, t) = d\, e^{f_1x+g_1y+c_1t+\nu_1} + e\, e^{f_2x+g_2y+c_2t+\nu_2}, \tag{49}$$

where $d, e, f_i, g_i, c_i, \nu_i$ $(i = 1, 2)$ are arbitrary parameters. Inserting (49) into Eqs. (30)–(33) an equation system is fulfilled.

$$f_1 = 0, \qquad g_1 = 0, \qquad g_2 = f_2, \qquad c_1 = 0, \qquad c_2 = -f_2^3 - 3kf_2 - 3lf_2, \tag{50}$$

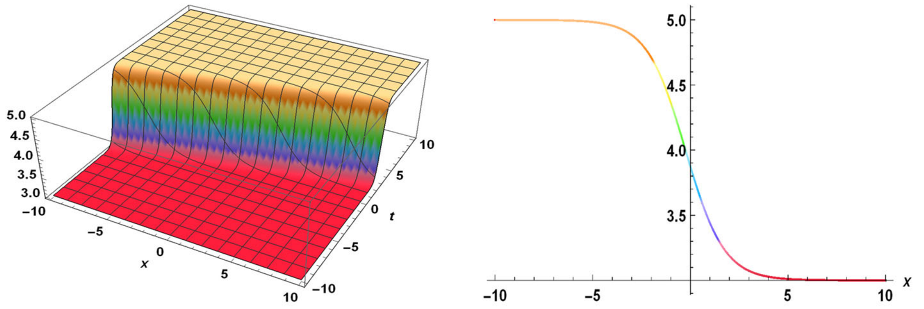
where f_2, m, ν_1 and ν_2 are arbitrary constants

$$\mu(x, y, t) = de^{\nu_1} + ee^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2} + m, \tag{51}$$



(a) **(b)**

Fig. 7 Travelling wave solution for $u_4(x, y, t)$ acquired in C for $d = 1$, $e = 1$, $f_2 = -1$, $\nu_1 = -1$, $k = -2$, $\nu_2 = -1$, $l = 3$, $m = 4$, $y = 0$



(a) **(b)**

Fig. 8 Traveling wave solution for $v_4(x, y, t)$ acquired in C for $d = 1$, $e = 1$, $f_2 = -1$, $\nu_1 = -1$, $k = -2$, $\nu_2 = -1$, $l = 3$, $m = 4$, $y = 0$

Then by entering Eq. (51) into Eq. (29), periodic wave solution for (2 + 1)-dimensional KdV is given by (Fig. 7 and 8):

$$\begin{aligned} u_4 = {} & \frac{2df_2 e^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}}{\left(de^{\nu_1}+ee^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}+m\right)^2} \\ & +\frac{2df_2^2 e^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}}{de^{\nu_1}+ee^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}+m}+k, \end{aligned} \tag{52}$$

$$\begin{aligned} v_4 = {} & \frac{2df_2 e^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}}{\left(de^{\nu_1}+ee^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}+m\right)^2} \\ & +\frac{2df_2^2 e^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}}{de^{\nu_1}+ee^{-tf-2^3-3ktf_2-3ltf_2+f_2x+f_2y+\nu_2}+m}+l, \end{aligned} \tag{53}$$

Note that by substituting zero for the arbitrary constant m in the given solutions, the precise explicit solutions to Eq. (1) for Cole-Hopf transformation Eq. (34) can be easily verified.

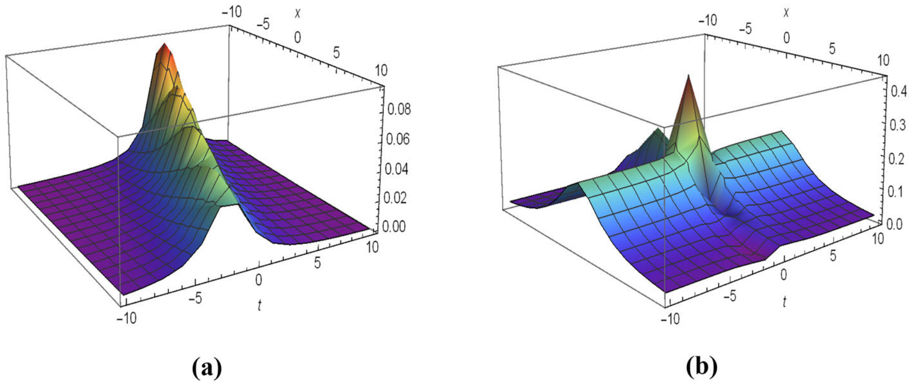


Fig. 9 **a** Lump solution to Set-1 for $b_4 = -1.05, b_1 = 0.45, b_5 = 0.05, b_8 = 2.5, b_6 = 9.9, b_3 = 0.8, y = 1$. **b** Lump solution to Set-2 for $b_4 = -7.95, b_1 = 1.95, b_5 = 2.5, b_8 = 5.5, b_6 = 0.59, b_3 = 8.8, y = 1$

Interaction Aspects

A lump solution is a real-analytic, rational function solution that decays in all spatial variable directions. In this section we have discussed lump, lump–kink, breather wave, two-wave and three wave solutions along with their graphical visualization [37, 38].

Lump Solutions

To construct the lump type solutions of Eq. (1) we consider the following ansatz function,

$$\mu(x, y, t) = g^2 + h^2 + b_9, \tag{54}$$

where $g = b_1x + b_2y + b_3t + b_4$ and $h = b_5x + b_6y + b_7t + b_8$. By inserting Eq. (54) for Eq. (2) and equating like coefficients to zero, we get:
Set-1:

$$b_2 = -\iota b_6, \qquad b_7 = -\iota b_3, \qquad b_9 = \frac{-3(b_1 - \iota b_5)(b_1 + \iota b_5)^2}{b_3}, b_3 \neq 0. \tag{55}$$

Putting Eq. (55) in Eq. (54), yields

$$\begin{aligned}\mu(x, y, t) = &(b_1x - \iota b_6y + b_3t + b_4)^2 + (b_5x + b_6y - \iota b_3t + b_8)^2 \\ &+ \frac{-3(b_1 - \iota b_5)(b_1 + \iota b_5)^2}{b_3}.\end{aligned} \tag{56}$$

Set-2:

$$b_2 = \iota b_6, \qquad b_7 = \iota b_3, \qquad b_9 = \frac{-3(b_1 - \iota b_5)(b_1 + \iota b_5)^2}{b_3}, b_3 \neq 0. \tag{57}$$

Putting Eq. (57) in Eq. (54), yields

$$\begin{aligned}\mu(x, y, t) = &(b_1x + \iota b_6y + b_3t + b_4)^2 + (b_5x + b_6y + \iota b_3t + b_8)^2 \\ &- \frac{3(b_1 - \iota b_5)(b_1 + \iota b_5)^2}{b_3}.\end{aligned} \tag{58}$$

Lump–Kink Solutions

To construct the lump–kink solutions of Eq. (1) we consider the following ansatz function,

$$\mu(x, y, t) = f^2 + g^2 + e^{\zeta} + b_{13}, \tag{59}$$

where $f = b_1x + b_2y + b_3t + b_4$, $g = b_5x + b_6y + b_7t + b_8$ and $\zeta = b_9x + b_{10}y + b_{11}t + b_{12}$. By inserting Eq. (59) for Eq. (2) and equating like coefficients to zero, we get:
Set-1:

$$\begin{aligned} &b_2 = \iota b_6, \quad b_3 = -\frac{3(b_1^2b_9^2 - \iota b_5b_9^2)}{2}, \quad b_4 = \frac{b_1b_8}{b_5}, \quad b_7 = -\frac{3\iota(b_1^2b_9^2 - \iota b_5b_9^2)}{2},\\ &b_{10} = 0, \quad b_{11} = -b_9^3, \quad b_{13} = \frac{2(b_1^2 + b_5^2)}{b_9^2}, \end{aligned} \tag{60}$$

putting Eq. (60) in Eq. (59), yields

$$\begin{aligned} \mu(x, y, t) = {} & (b_1x + \iota b_6y - \frac{3(b_1b_9^2 - \iota b_5b_9^2)}{2}t + \frac{b_1b_8}{b_5})^2 + (b_5x + b_6y\\ & - \frac{3\iota(b_1b_9^2 - \iota b_5b_9^2)}{2}t + b_8)^2\\ & + e^{b_9x - b_9^3t + b_{12}} + \frac{2(b_1^2 + b_5^2)}{b_9^2}. \end{aligned} \tag{61}$$

Set-2:

$$\begin{aligned} &b_2 = -\iota b_6, \quad b_3 = -\frac{3b_1b_9^2}{2}, \quad b_5 = 0, \quad b_7 = \frac{3\iota b_1b_9^2}{2}, \quad b_8 = 0,\\ &b_{10} = 0, \quad b_{11} = -b_9^3, \quad b_{13} = \frac{2b_1^2}{b_9^2}, \end{aligned} \tag{62}$$

putting Eq. (62) in Eq. (59), yields

$$\begin{aligned} \mu(x, y, t) = {} & (b_1x - \iota b_6y - \frac{3\iota b_1b_9^2}{2} + b_4)^2\\ & + \left(b_6y + \frac{3b_1b_9^2}{2}\right)^2 + e^{b_9x - b_9^3t + b_{12}} + \frac{2b_1^2}{b_9^2}. \end{aligned} \tag{63}$$

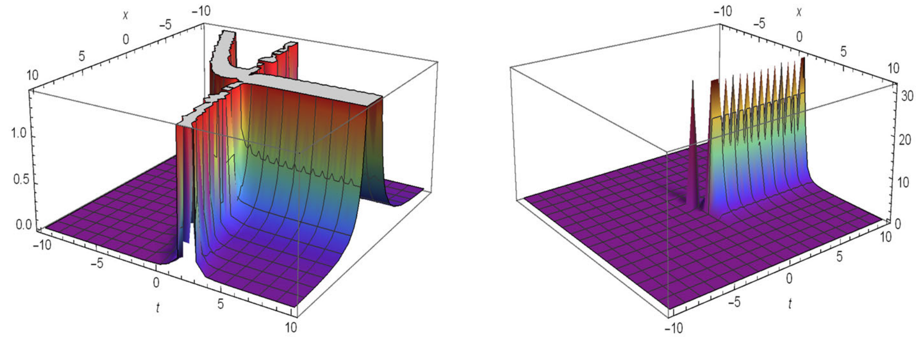
Breather Wave Solutions

To construct the breather wave solutions of Eq. (1) we consider the following ansatz function,

$$\mu(x, y, t) = l_2e^{p_1(b_1y + c_1t + x)} + e^{-p_1(b_1y + c_1t + x)} + l_1\cos(p_0(b_2y + c_2t + x)). \tag{64}$$

By inserting Eq. (64) for Eq. (2) and equating like coefficients to zero, we get:
Set-1:

$$l_2 = -\frac{b_2l1^2p_0^2}{4b_1}, \quad c_2 = p_0^2 - 3p_1^2, \quad c_1 = 3p_0^2 - p_1^2, \tag{65}$$



(a) **(b)**

Fig. 10 **a** Lump–Kink solution to Set-1 for $b_1 = 1.9, b_5 = 9.5, b_6 = 2.5, b_8 = 0.2, b_9 = 1.5, b_{12} = 0.01, y = 1$. **b** Lump–Kink solution to Set-2 for $b_1 = -7.45, b_4 = 8.5, b_5 = -9.05, b_6 = -0.9, b_8 = 5.5, b_9 = 7.25, b_{12} = 9.8, y = 1$

putting Eq. (65) in Eq. (64), yields

$$\begin{aligned}\mu(x,y,t) = &-\frac{b_2 l_1^2 p_0^2}{4b_1} e^{p_1(x+b_1y+(3p_0^2-p_1^2)t)} \\ &+ e^{-p_1(x+b_1y+(3p_0^2-p_1^2)t)} + \cos(p_0(x+b_1y+(3p_0^2-p_1^2)t).\end{aligned} \tag{66}$$

Set-2:

$$l_1 = 0, \qquad c_2 = \frac{p_0^3 + 3\iota p_0^2 p_1 - 3p_0 p_1^2 + 3\iota p_1^3}{p_0}, \qquad b_2 = \frac{\iota b_1 p_1}{p_0}, \qquad c_1 = -4p_1^2, \tag{67}$$

putting Eq. (67) in Eq. (64), yields

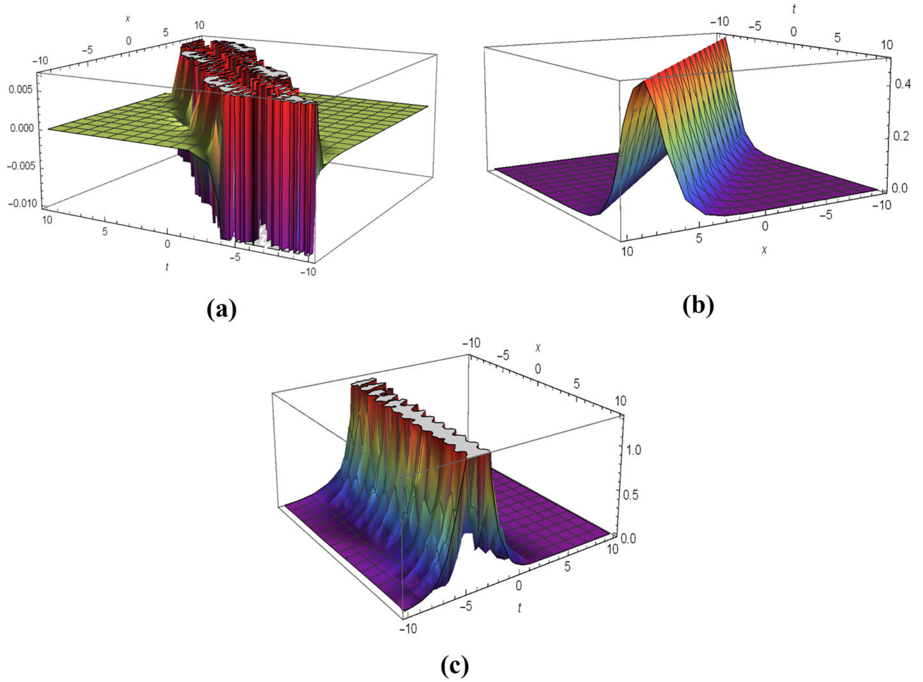
$$\mu(x,y,t) = e^{-p_1(x+b_1y-4p_1^2t)} + l_2 e^{p_1(x+b_1y-4p_1^2t)}. \tag{68}$$

Set-3:

$$\begin{aligned} &l_2 = -\frac{l_1^2(c_2-4p_0^2)}{-4c_2+4p_0^2-12\iota p_0 p_1 - 12p_1^2 - \frac{12\iota p_1^3}{p_0}}, \qquad b_2 = -\frac{\iota b_1 p_1}{p_0}, \\ &c_1 = \frac{\iota p_0^3 - \iota c_2 p_0 + 3p_0^2 p_1 - 3\iota p_0 p_1^2 - p_1^3}{p_1},\end{aligned} \tag{69}$$

putting Eq. (69) in Eq. (64), yields

$$\begin{aligned}\mu(x,y,t) = &-\frac{l_1^2(c_2-4p_0^2)}{-4c_2+4p_0^2-12\iota p_0 p_1 - 12p_1^2 - \frac{12\iota p_1^3}{p_0}} e^{(p_1(x+b_1+\frac{\iota p_0^3 - \iota c_2 p_0 + 3p_0^2 p_1 - 3\iota p_0 p_1^2 - p_1^3}{p_1}t))} \\ &+ e^{(-p_1(x+b_1+\frac{\iota p_0^3 - \iota c_2 p_0 + 3p_0^2 p_1 - 3\iota p_0 p_1^2 - p_1^3}{p_1}t))} + l_1\cos(p_0(x - \frac{\iota b_1 p_1}{p_0}y + c_2 t)).\end{aligned} \tag{70}$$



(a) **(b)** **(c)**

Fig. 11 **a** Breather solution to Set-1 for $p_0 = 0.9$, $p_1 = 1.9$, $l_1 = 1.5$, $b_2 = 5.5$, $b_1 = 1.9$, $y = 1$. **b** Breather solution to Set-2 for $l_2 = -9.9$, $p_1 = 1.9$, $l_1 = 1.5$, $b_2 = 5.5$, $b_1 = 1.9$, $y = 1$. **c** Breather solution to Set-3 for $l_2 = -7.7$, $p_0 = -2.0$, $b_2 = 4.5$, $l_1 = -5.5$, $p_1 = 0.07$, $b_1 = -1.1$, $y = 1$

Two-Wave Solutions

To construct the two-wave solutions of Eq. (1) we consider the following ansatz function,

$$\mu(x, y, t) = d_1 e^{\xi} + d_2 e^{-\xi} + d_3 \sin(\alpha) + d_4 \sinh(\eta), \tag{71}$$

where $\xi = p(a_1 x + a_2 y + a_3 t)$, $\alpha = a_4 x + a_5 y + a_6 t$ and $\eta = a_7 x + a_8 y + a_9 t$. By inserting Eq. (71) for Eq. (2) and equating like coefficients to zero, we get:
Set-1:

$$a_1 = -\frac{a_9^{1/3}}{2^{2/3}}, \qquad a_3 = a_9, \qquad a_7 = -\frac{a_9^{1/3}}{2^{2/3}}, \qquad d_3 = 0, \tag{72}$$

putting Eq. (72) in Eq. (71), yields

$$\mu(x, y, t) = d_1 e^{p(a_2 y - \frac{a_9^{1/3}}{2^{2/3}} x + a_9 t)} + d_2 e^{-p(a_2 y - \frac{a_9^{1/3}}{2^{2/3}} x + a_9 t)} + d_4 \sinh(a_8 y - \frac{a_9^{1/3}}{2^{2/3}} x + a_9 t).$$

Set-2:

$$a_2 = \iota a_5, \qquad a_3 = -a_1^3 - 3\iota a_1^2 a_4 + 3a_1 a_4^2 + \iota a_4^3 - \iota a_6, \qquad d_4 = 0,$$
$$d_3 = -\frac{2\sqrt{d_1 d_2}(\sqrt{-3\iota a_1^3 - 3a_1^2 a_4 - 3\iota a_1 a_4^2 + a_4^3 - a_6})}{\sqrt{4a_4^3 - a_6}}, \tag{73}$$

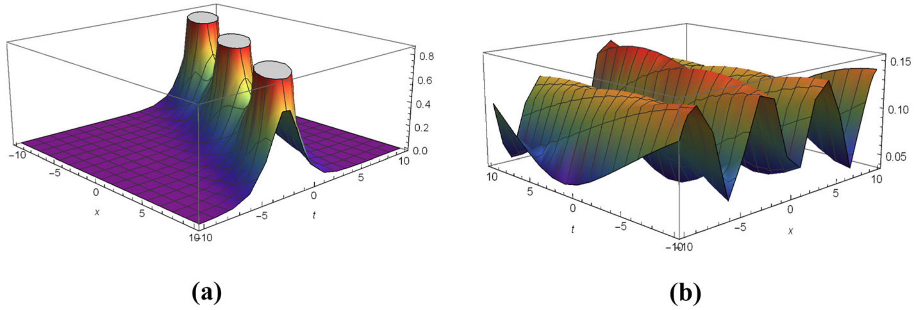


Fig. 12 **a** Two-wave solution to Set-1 for $a_9 = -0.5$, $a_8 = 0.9$, $d_2 = 2.5$, $d_1 = -0.1$, $d_4 = 1.2$, $p = -0.9$, $a_2 = -0.1$, $y = 1$. **b** Two-wave solution to Set-2 for $a_4 = -0.5$, $a_5 = -1.9$, $a_6 = 0.08$, $a_1 = 0.02$, $a_4 = 0.05$, $a_3 = -0.1$, $d_2 = -5.4$, $d_1 = 0.3$, $p = -0.1$, $y = 1$

putting Eq. (73) in Eq. (71), yields

$$\begin{aligned}\mu(x, y, t) = {} & d_1 e^{p(a_1 x + \iota a_5 y + (-a_1^3 - 3\iota a_1^2 a_4 + 3a_1 a_4^2 + \iota a_4^3 - \iota a_6)t)} \\ & + d_2 e^{-p(a_1 x + \iota a_5 y + (-a_1^3 - 3\iota a_1^2 a_4 + 3a_1 a_4^2 + \iota a_4^3 - \iota a_6)t)} \\ & - \frac{2\sqrt{d_1 d_2}(\sqrt{-3\iota a_1^3 - 3a_1^2 a_4 - 3\iota a_1 a_4^2 + a_4^3 - a_6})}{\sqrt{4a_4^3 - a_6}} \sin(a_5 x + a_6 y + a_7 t)\end{aligned} \tag{74}$$

Three-Wave Solutions

To construct the three-wave solutions of Eq. (1) we consider the following ansatz function,

$$\mu(x, y, t) = a_1 e^{\xi} + e^{-\xi} + a_2 \cos(\alpha) + a_3 \sin(\beta), \tag{75}$$

where $\xi = k_1 x + l_1 y + m_1 t$, $\alpha = k_2 x + l_2 y + m_2 t$ and $\beta = k_3 x + l_3 y + m_3 t$. By inserting Eq. (75) for Eq. (2) and equating like coefficients to zero, we get:
Set-1:

$$k_1 = -\iota k_3, \qquad l_1 = 0, \qquad m_1 = -4\iota k_3^3, \qquad m_3 = 4k_3^3, \qquad a_2 = 0, \tag{76}$$

putting Eq. (76) in Eq. (75), yields

$$\mu(x, y, t) = a_1 e^{-\iota k_3 - 4\iota k_3^3} + a_3 \sin(k_3 x + l_3 y + 4k_3^3). \tag{77}$$

Set-2:

$$m_1 = -4k_1^3, \qquad m_2 = -k_2, \qquad m_3 = 4k_3^3, \qquad l_1 = l_2 = l_3 = 0, \tag{78}$$

putting Eq. (78) in Eq. (75), yields

$$\mu(x, y, t) = a_1 e^{k_1 x - 4k_1^3 t} + a_2 \cos(k_2 x - k_2 t) + a_3 \sin(k_3 x + 4k_3^3 t). \tag{79}$$

Set-3:

$$m_3 = 4k_3^3, \qquad l_1 = l_2 = l_3 = 0, \tag{80}$$

putting Eq. (80) in Eq. (75), yields

$$\mu(x, y, t) = a_1 e^{k_1 x - m_3 t} + a_2 \cos(k_2 x - m_2 t) + a_3 \sin(k_3 x + 4k_3^3 t). \tag{81}$$

Extended Transformed Rational Function Methodology

As complexiton solutions include novel sorts of traveling wave speeds, it is hard to construct them for nonlinear partial differential equations. We improve the transform rational function technique in the manner described below to produce complexitons. [39, 40]. Steps involved in the extended transform rational function method are described as:

Step A: Consider the Hirota bilinear form of Eq. (1) as below

$$H(\Omega_x, \Omega_t, \ldots)\mu \cdot \mu = 0. \tag{82}$$

Hirota operators Ω_x, Ω_t,... are represented as

$$\Omega_s^n \mu(s) \cdot \nu(s) = (\partial_s - \partial_{s'}|_{s'=s} = \partial_{s'}^n \mu(s+s')\nu(s-s')|_{s'=0}, \tag{83}$$

where $n \geq 1$.

Step B: Suppose

$$\mu = \frac{u(\delta_1, \delta_2)}{v(\delta_1, \delta_2)}, \tag{84}$$

where $u(\delta_1, \delta_2)$ are polynomials and δ_1, δ_2 satisfies the following relation

$$\delta_1'' = \frac{d^2\delta_1}{d\varsigma_1^2} = -\delta_1, \tag{85}$$

$$\delta_2'' = \frac{d^2\delta_2}{d\varsigma_2^2} = \delta_2, \tag{86}$$

where $\varsigma_1 = p_1 x + bq_1 y + r_1 t + k_1$ and $\varsigma_2 = p_2 x + bq_2 y + r_2 t + k_2$ and constants p_i, q_i, r_i and k_i $(i = 1, 2)$ can be derived.

Step C: By selecting appropriate $u(\delta_1, \delta_2)$ and $v(\delta_1, \delta_2)$, Eq. (83) can be converted into an algebraic expressions comprising p_i, q_i and r_i. By resolving these algebraic expressions we can achieve complexiton solutions to Eq. (1). The bilinear representation of the considered problem is given by Eq. (2). Utilizing the extended transformed rational function method, we suppose that

$$u(\delta_1, \delta_2) = S\delta_1 + T\delta_2, \quad v(\delta_1, \delta_2) = 1, \tag{87}$$

such that,

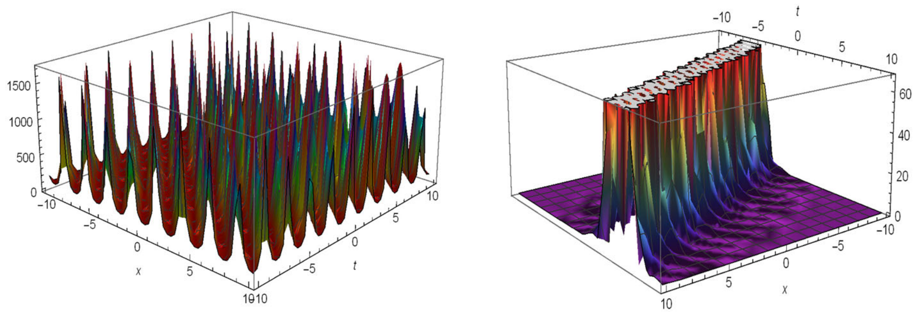
$$\mu = S\delta_1 + T\delta_2, \tag{88}$$

where δ_1 and δ_2 are defined by (85) and (86) and S, T are to be determined.
It is well-known that (85), (86) verifies the following solutions

$$\delta_1 = \sin \nu_1 \quad \text{or} \quad \delta_1 = \cos \nu_1, \tag{89}$$

$$\delta_2 = \sinh \nu_2 \quad \text{or} \quad \delta_1 = \cosh \nu_2, \tag{90}$$

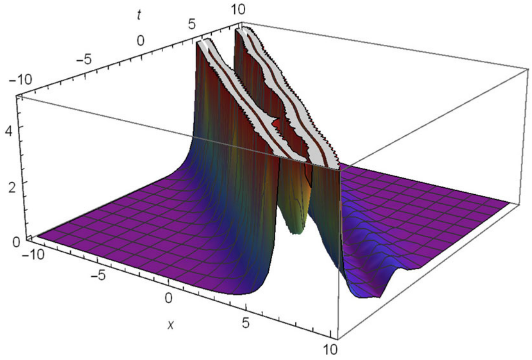
respectively.



(a) **(b)**

Fig. 13 **(a)** Three-wave solution to Set-1 for $a_1 = -0.2, k_1 = -0.7, k_2 = 9.1, k_3 = 7.5, l_1 = 2.1, l_3 = 4.1, m_3 = -0.4, y = 1$. **(b)** Three-wave solution to Set-2 for $a_1 = -0.2, k_1 = -0.7, k_2 = 9.1, k_3 = 7.5, l_1 = 2.1, l_3 = 4.1, m_3 = -0.4, y = 1$

Fig. 14 Three-wave solution to Set-3 for $a_1 = 0.2, a_2 = 0.3, k_1 = 0.7, k_2 = 1.1, k_3 = -0.5, m_1 = 0.8, m_2 = -8.2, m_3 = 0.1, y = 1$



Substituting (88) into (2) and collecting coefficients of δ_1^2, δ_2^2, $\delta_1\delta_2$, $\delta_1^{'}\delta_2^{'}$, constant term and comparing them to zero. Based on the result $\delta_1^{'2} = 1 - \delta_1^2$ and $\delta_2^{'2} = 1 + \delta_2^2$, we derive a system of equations.

$$STbp_1{}^4 - 6\,STbp_1{}^2p_2{}^2 + STbp_2{}^4 - STbp_1r_1 + STbp_2r_2 = 0, \tag{91}$$

$$4\,STbp_1{}^3p_2 - 4\,STbp_1p_2{}^3 - STbp_1r_2 - STbp_2r_1 = 0, \tag{92}$$

$$v4\,S^2bp_1{}^4 - 4\,T^2bp_2{}^4 - S^2bp_1r_1 - T^2bp_2r_2 = 0. \tag{93}$$

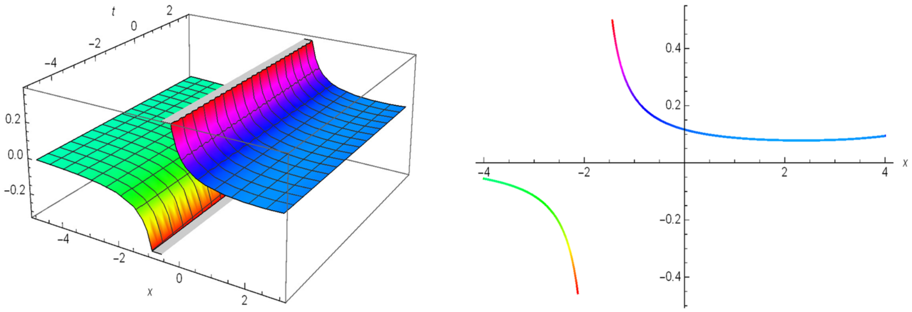
Solving Eqs. (91)–(93), we obtain

$$S = \pm T\frac{p_2}{p_1}, \qquad r_1 = p_1^3 - 3p_1p_2^2, \qquad r_2 = p_2(3p_1^2 - p_2^2). \tag{94}$$

Inserting (89), (90) with (94) into (88) respectively, the solutions to Eq. (1) are shown visually by Figs. 14 and 15.

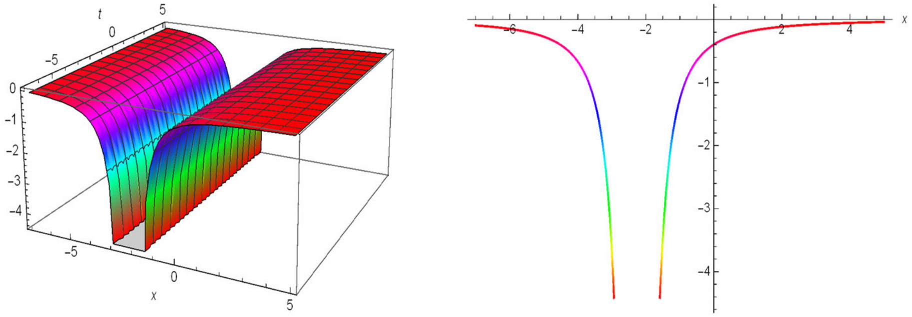
$$u(x, y, t) = 2(\ln\mu(x, y, t))_{xy}, \quad v(x, y, t) = 2(\ln\mu(x, y, t))_{xx}, \tag{95}$$

$$\mu(x, y, t) = S\Bigg(\sin(p_1x + bp_1y + (p_1^3 + 3p_1p_2^2)t + k_1)$$



(a) **(b)**

Fig. 15 Complexiton solution to Eq. (1) for $k_1 = -\frac{1}{3}$, $p_1 = -\frac{1}{7}$, $p_2 = \frac{1}{7}$, $b = \frac{-11}{5}$, $k_2 = \frac{1}{3}$, $y = 0$



(a) **(b)**

Fig. 16 Complexiton solution to Eq. (1) acquired for $p_1 = \frac{1}{7}$, $k_1 = -\frac{1}{3}$, $p_2 = \frac{1}{9}$, $b = -2$, $k_2 = -\frac{1}{3}$, $y = 0$

$$\pm\frac{p_1}{p_2}\sinh(p_2x + bp_2y + p_2(3p_1^2 - p_2^2)t + k_2)\Big), \tag{96}$$

or

$$\mu(x, y, t) = S\Big(\cos(p_1x + bp_1y + (p_1^3 + 3p_1p_2^2)t + k_1) \\ \pm\frac{p_1}{p_2}\sinh(p_2x + bp_2y + p_2(3p_1^2 - p_2^2)t + k_2)\Big). \tag{97}$$

Results and Discussion

To begin, the auto-Bäcklund and Cole–Hopf transformations have been established for the (2+1)-dimensional KdV equation, resulting in the generation of numerous soliton-like solutions. The fundamental significance of soliton solutions and the concept of auto-Bäcklund transformations in the realm of nonlinear integrable systems holds wide-reaching importance in diverse branches of physics and mathematics. By carefully selecting parameters, we have visually portrayed the dynamics of various bright, dark, singular, exponential, and periodic solutions derived from the auto-Bäcklund transformation through 3-D and 2-D plots in figs.

1, 2, 3, 4, 5, 6, 7 and 8. Additionally, we have examined the interaction properties of different wave structures using the hirota bilinear form. Notably, Figs. 910, 11, 12, 13 and 14 exhibit lump, lump–kink, breather, two-wave, and three-wave solutions concurrently, employing the previously discussed parameters.

Moving on, we have applied the extended transformed rational function method to construct complexiton solutions. These solutions involve intricate complex-valued functions that manifest highly localized behavior and can arise in various physical systems. The importance of complexiton solutions lies in their capacity to elucidate and account for specific rare and extreme events encountered in diverse scientific and engineering domains. The visualization of complexiton solutions is illustrated through 3-D and 2-D plots in Figs. 15 and 16. In summary, soliton, complexiton, and breather solutions represent pivotal concepts in the study of nonlinear wave phenomena. Each category of solution provides distinct insights and bears significant implications across a broad spectrum of disciplines encompassing physics, mathematics, and engineering. These solutions offer profound understandings of intricate system behaviors, furnish tools for communication, energy manipulation, and risk evaluation, and propel advancements in the realms of mathematical and computational methodologies.

Conclusion

This article examined the classical (2+1)-dimensional coupled KdV equation. The extended homogeneous balance technique has produced the auto-Bäcklund and Cole–Hopf transformations, as well as several analytical solutions. 3-D and 2-D wave profiles of the traveling and periodic wave solutions have been portrayed. Moreover, the collisions between the lump and other analytical solutions have been examined in this research. Using Hirota bilinear technique, lump, lump–kink, breather, two-wave and three-wave solutions have been explored. Considering the bilinear form of the underlying problem, complexitons have been extracted by the extended transform rational function method. The numerical simulations of the extracted solutions are exhibited by identifying appropriate values for the parameter. The outcomes of this research undoubtedly added new knowledge and innovative conclusions to the (2 + 1)-dimensional KdV equation.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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