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Dynamical investigation of time-fractional order Phi-4 equations

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Abstract
In this manuscript, Optimal Homotopy Asymptotic Method (OHAM) is used to find the approximate solutions of time fractional Phi-4 nonlinear partial differential equations. Approximate first order results are acquired through OHAM and are compared with the exact solutions. It has been noticed that the obtained results from OHAM have large convergence rate for time-Fractional Order Partial Differential Equations. The solutions are plotted and therelative errors are tabulated.

Keywords Fractional calculus · Time-fractional order Phi-4 partial differential equations · Approximate solutions · Optimal homotopy asymptotic method

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1 Introduction

The nonlinear evolution equations are greatly significant owing to their extensive solicitations. In contemporary technological advancements, nonlinearities are one of the greatest imposing arenas of study. Nonlinearities happen in real life in many disciplines of study, like plasmas, fluidics, particle movements, plasticity, reaction-kinetics, environmental sciences, optics, chip-technology, biomedical, etc., all are fundamentally nonlinear systems and thus ruled by nonlinear equalities. Nonlinear evolution equations (NLEEs) are repeatedly used to model wave-like disturbances. Different types of modelling like traffic flow, viscoelasticity, fluid flow, signal processing etc. belonging to real world problems, result in FPDEs. The use of NLEEs in physical circumstances, such as the above discussed technological fields, is very important. The study of these wavy disturbances and oscillations in technological disciplines is growing gradually, it is significant to pursue new methods to solve the traveling wave NLEEs (Tariq and Akram 2017; Razazadeh et al. 2018; Akter and Akbar 2015). A Phi-4 NPDE is defined as:

$$\frac{\partial^2 w(y,t)}{\partial t^2} - \frac{\partial^2 w(y,t)}{\partial y^2} + m^2 w + cw^3 = 0, \tag{1}$$

where c, m are real constants and $t,\ y$ are time and space variables. Phi-4 PDEs have played a vital role in nuclear and particle physics in recent years.

A Phi-4 time fractional partial differential equation is as follows:

$$\frac{\partial^\alpha w(y,t)}{\partial t^\alpha} - \frac{\partial^2 w(y,t)}{\partial y^2} + m^2 w + cw^3 = 0, y \in [0,1], t > 0, m > 0,\ c > 0, 1 < \alpha \leq 2, \tag{2}$$

where $\partial^\alpha/\partial t^\alpha$ denotes the Caputo or Riemann-Liouville non-integral order derivative operator, $w(y, t)$ is the transversal displacement of the beam.

In fractional calculus, the idea of differentiation and integration to non-integer order has been studied. Fractional calculus is not a new topic: in reality it has almost the same history as classical calculus. Fractional calculus is a generalized form of classical calculus. The question of the one-half order derivative was first raised by L'Hospital on 1695. Nowadays, fractional calculus is undergoing rapid development, with more and more convincing applications in the real world. The theoretical explanation of the subject has been studied in detail and developed by Oldham and Spanier (1974), Miller and Ross (1993) and Podlubny (1999) . In the last many years many mathematicians and scientists have observed that the role of non-integer operators is very important in expressing the properties of physical phenomena. Additionally relative study has been done between classical models and fractional models. It was concluded that sometimes fractional models are not always more efficient than classical models. Recently, a large number of researchers have introduced many methods to find analytical solutions of NLEEs such as the generalized Kudryashov method (Tuluce Demiray and Bulut 2017), modified extended tanh function method (Zahran and Khater 2016), exp function method (Khan and Akbar 2014), extended trial equation method (Demiray and Bulut 2015), sine-cosine method (Bibi and Mohyud-Din 2014), G'/G-expansion method (Younis and Rizvi 2015) and two-dimensional linear voltra integral equations of the first kind by HPM in Eslami and Mirzazadeh (2014).

Some Abbreviations and Nomenclature	
OHAM	Optimal homotopy asymptotic method
m, c	Real constants
t, y	Time and space variables
FPDEs	Fractional order Partial differential equations
NLEEs	Nonlinear evolution equations
NLPDEs	Nonlinear Partial differential equations
μ, p	Real numbers
$w(y,t)$	Transversal displacement of the beam
$R(y,t)$	Residual of Phi-4 equation
$\frac{\partial^\alpha}{\partial t^\alpha}$	Non-integral order derivative operator

For the solution of nonlinear problems, the idea of homotopy has been combined with perturbation methods. Liao (1992) did the fundamental work by utilizing the homotopy analysis method. For the first time in 1998, the homotopy perturbation method was presented by He (1998). A novel technique which is known as OHAM was invented by Marinca and Herisanu (2014, 2008) and Marinca and Herisanu (2010). The benefit of OHAM is that it is established in convergence criteria similar to HAM but more pliable. In various research papers (Iqbal et al. 2010; Iqbal and Javed 2011; Iqbal et al. 2015; Sarwar et al. 2015; Sarwar and Rashidi 2016; Alkhalaf 2017) have verified the usefulness generality of this method, and achieved trust worthy solutions, and presented significant applications in engineering and science. The concept of OHAM has been articulated in this paper. It provides solutions to linear, nonlinear, time dependent, time fractional and space fractional differential equations and PDEs. The arrangement of the paper is as follows. Basic definitions of fractional calculus are provided in Sects. 2, 3 is dedicated to the scheme of the method, Sect. 4 includes model problems, Sect. 5 represents the results and discussions, in Sect. 6 includes conclusion.

2 Basic definitions of fractional order derivatives and integrals

Definition 1 The Riemann-Liouville integral operator of a function $f \in C_\mu$ (function space), $\mu \in R$, of fractional order $\alpha > 0$ is defined as:

$$
\begin{aligned}
&{}^{RL}J_{a,y}^{-\alpha} f(y) = \frac{1}{\Gamma(\alpha)} \int_a^y (y-\mu)^{\alpha-1} f(\mu)\, d\mu,\ y > \alpha,\\
&k-1 < \alpha < k,\ k \in Z^+.
\end{aligned}
\tag{3}
$$

Definition 2 The Caputo derivative operator of a function $f(y)$ of fractional order $\alpha > 0$ is defined as:

$$
\begin{aligned}
&{}^{C}D_{a,y}^{\alpha} f(y) = \frac{1}{\Gamma(k-\alpha)} \int_a^y (y-\mu)^{k-\alpha-1} f^{\,k}(\mu)\, d\mu, y > a,\\
&k-1 < \alpha < k,\ k \in Z^+.
\end{aligned}
\tag{4}
$$

3 OHAM procedure for time fractional order Phi-4 equation

According to the OHAM (Iqbal et al. 2015; Sarwar et al. 2015), scheme has been extended for time fractional Phi-4 partial differential equations (tFPDEs) in the upcoming steps:

Step-(i): Compose the time fractional order Phi-4 governing equation as:

$$\frac{\partial^\alpha w(y,t)}{\partial t^\alpha} - \frac{\partial^2 w(y,t)}{\partial y^2} + m^2 w + cw^3 - f(y,t) = 0,\ y \in [0,1],\ t > 0. \tag{5}$$

Step-(ii): Make an optimal homotopy for time fractional order phi-4 partial differential equation, $\Psi(y,t;p) : \Omega \times [0,1] \longrightarrow R$, which satisfies:

$$\begin{aligned}(1-p)&\left(\frac{\partial^\alpha \Psi(y,t)}{\partial t^\alpha} - f(y,t)\right)\\ &- H(y,p;c)\left(\frac{\partial^\alpha \Psi(y,t)}{\partial t^\alpha} - \frac{\partial^2 \Psi(y,t)}{\partial y^2} + m^2\Psi + c\Psi^3 - f(y,t)\right) = 0,\end{aligned}$$

where $y \in \Omega$ and $p \in [0,1]$ is an embedding parameter, for $p \neq 0$, $H(y,p;c)$ is a nonzero auxiliary function and $H(y,0;c) = 0$, when p increases in the interval $[0,1]$, the solution $\Psi(y,t)$ certifies a rapidly convergence to the exact solution.

$$H(y,p;c) = pk_1(y,c_i) + p^2k_2(y,c_i) + p^3k_3(y,c_i) + \ldots + p^m k_m(y,c_i), \tag{6}$$

where $c_i; i = 1,2,3,\ldots,m$ are auxiliary control parameters of convergence and $k_i(y)$, $i = 1,2,3,\ldots,m$ can be function on the variables . The selections of $k_m(y,c_i)$ may be polynomial, exponential and so on. The selection of functions is very important, because the rate of convergence of the solution really depends on the functions.

Step-(iii): Extend $\Psi(y,t;p,c)$ in Taylor's series about p, to develop an approximate results as:

$$\Psi(y,t;p,c_i) = w_0(y,t) + \sum_{k=1}^{m} w_k(y,t;c_i)p^k, i = 1,2,\ldots,m. \tag{7}$$

It has been cleared that the rate of convergence of (7) depends upon auxiliary constants c_i. It is observed that, if the series is convergent at $p = 1$, then one has:

$$\tilde{w}(y,t,c_i) = w_0(y,t) + \sum_{k=1}^{m} w_k(y,t;c_i), i = 1,2,\ldots,m. \tag{8}$$

Step-(iv): Comparing the coefficients of like powers of p after substituting equation (7) into equation (6), we get zeroth, first, second and higher-order problems:

$$p^0 : \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - f(y,t) = 0, \tag{9}$$

$$\begin{aligned}p^1 :\ &\frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} + f(y,t)\\ &- c_1\left(\frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - \frac{\partial^2 w_0(y,t)}{\partial y^2} + mw_0 + cw_0^3 - f(y,t)\right) = 0,\end{aligned} \tag{10}$$

$$
\begin{aligned}
p^2 :\ & \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} - c_1\left(\frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} - \frac{\partial^2 w_1(y,t)}{\partial y^2} + mw_1 + cw_1^3\right) \\
& - c_2\left(\frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} + \frac{\partial^2 w_0(y,t)}{\partial y^2} + mw_0 + cw_0^3 + f(y,t)\right) = 0,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
p^3 :\ & \frac{\partial^\alpha w_3(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} - c_1\left(\frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} - \frac{\partial^2 w_2(y,t)}{\partial y^2} + mw_2 + cw_2^3\right) \\
& - c_2(\frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} - \frac{\partial^2 w_1(y,t)}{\partial y^2} + mw_1 + cw_1^3) \\
& - c_3\left(\frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} + \frac{\partial^2 w_0(y,t)}{\partial y^2} + mw_0 + cw_0^3 - f(y,t)\right) = 0.
\end{aligned} \tag{12}
$$

and so on.

Step-(v): A series of solutions of equations (9)-(12) are obtained by using Riemann-Liouville integral operator J^αgiven in equation (3).

$$
w_0(y,t) = J^\alpha\left[f(y,t)\right], \tag{13}
$$

$$
\begin{aligned}
w_1(y,t) = J^\alpha[& \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - f(y,t) \\
& + c_1\left(\frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - \frac{\partial^2 w_0(y,t)}{\partial y^2} + mw_0 + cw_0^3 - f(y,t)\right)],
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
w_2(y,t) = J^\alpha[& \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} + c_1\left(\frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} + \frac{\partial^2 w_1(y,t)}{\partial y^2} + mw_1 + cw_1^3\right) \\
& + c_2\left(\frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} + \frac{\partial^2 w_0(y,t)}{\partial y^2} + mw_0 + cw_0^3 + f(y,t)\right)],
\end{aligned} \tag{15}
$$

and so on.

Step-(vi): Put equation (8) into equation (5), outcome will be residual:

(*i*) Create the $\Phi(c_i)$

$$
\Phi(c_i) = \int_0^t \int_\Omega R^2(y,t;c_i)\,dydt, \tag{16}
$$

where residual of the problem Phi-4, is $R(y,t;c_i)$.

(*ii*) For the determination of auxiliary constants c_i, Least Square method is used as follows:

$$
\frac{\partial \Phi}{\partial c_1} = \frac{\partial \Phi}{\partial c_2} = \dots = \frac{\partial \Phi}{\partial c_m} = 0. \tag{17}
$$

$w(y,t;c_i)$ will be the exact solution, If $R(y,t;c_i) = 0$. Normally it doesn't happen, in nonlinear problems.

Step-(vii): Auxiliary constants ($c_{i's}$), after using in equation (8), we can get the rapidly convergent approximate solutions.

Step-(viii): Accuracy of the method by:

(i) Error norm L_2

$$L_2 = \left\|w^{exact} - w_N\right\|_2 \simeq \sqrt{\frac{b-a}{N}\sum_{i=0}^{N}\left|w_i^{exact} - \left(w_N\right)_i\right|^2}. \tag{18}$$

(ii) Error norm L_∞

$$L_\infty = \left\|w^{exact} - w_N\right\|_\infty \simeq \max_i \left|w_i^{exact} - \left(w_N\right)_i\right|. \tag{19}$$

There are also other approximate methods sharing the same idea of optimization, such as the Optimal Homotopy Perturbation Method presented in the paper (Ghani et al. 2016) and the Optimal Auxiliary Functions Method presented in the paper (Herisanu et al. 2019).

4 Applications of fractional order Phi-4 equation

In this section, consider the two models for (OHAM) algorithm. which represents the accuracy, validity and effectiveness of the extended (OHAM) algorithm.

4.1 Problem-i

Let us consider the following type Phi-4 time fractional order equation (Ehsani and Ehsani 2013):

$$\frac{\partial^\alpha w(y,t)}{\partial t^\alpha} - \frac{\partial^2 w(y,t)}{\partial y^2} + w + w^3 = 0, y \in [0, 1.28], t > 0, 1 < \alpha \leq 2, \tag{20}$$

Initial and boundary conditions of Phi-4 equation are:

$$w(y,0) = A\left(1 + \cos\left[\frac{2\pi y}{1.28}\right]\right),$$
$$w_t(y,0) = 0.$$

Exact solution of problem-i (for second order PDE) is:

$$w_{exact}(y,t) = A\left\{1 + \cos\left(\frac{2\pi y}{1.28}\right)\right\}e^{\left(\frac{-1}{2}t\right)}\left\{\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right\}.$$

In the following p^0, p^1, p^2 and p^3 are zeroth, first, second and third-order problems:

$$p^0 : \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} = 0, \tag{21}$$

$$
\begin{aligned}
p^1 &: -c_1 w_0(y,t) - c_1 w_0^3(y,t) + c_1 \frac{\partial^2 w_0(y,t)}{\partial y^2} - \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - \\
& c_1 \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} + \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} = 0,
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
p^2 &: -c_2 w_0(y,t) - c_2 w_0^3(y,t) - c_1 w_1(y,t) - 3c_1 w_0^2(y,t) w_1(y,t) + \\
& c_2 \frac{\partial^2 w_0(y,t)}{\partial y^2} + c_1 \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} - c_2 \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} \\
& - c_1 \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} + \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} = 0,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
p^3 &: -c_3 w_0(y,t) - c_3 w_0^3(y,t) - c_2 w_1(y,t) - 3c_2 w_0^2(y,t) w_1(y,t) - \\
& 3c_1 w_0(y,t) w_1^2(y,t) - c_1 w_2(y,t) - 3c_1 w_0^2(y,t) w_2(y,t) + \\
& c_3 \frac{\partial^2 w_0(y,t)}{\partial y^2} + c_2 \frac{\partial^2 w_1(y,t)}{\partial y^2} + c_1 \frac{\partial^2 w_2(y,t)}{\partial y^2} - c_3 \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - \\
& c_2 \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} - c_1 \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} + \frac{\partial^\alpha w_3(y,t)}{\partial t^\alpha}.
\end{aligned} \tag{24}
$$

In the following w_0 and w_1 are zeroth-order and first-order solutions, by using these two solutions in Sect. 3, equation (8), we get w solution:

$$
w_0 = A\left\{1 + \cos\left(\frac{2\pi y}{1.28}\right)\right\}, \tag{25}
$$

$$
\begin{aligned}
w_1 = &At^\alpha \; [(24.095713869847067 \cos\left[4.908738521234052y\right] + \\
&(1 + \cos\left[4.908738521234052y\right]) + A^2(1 + \cos\left[4.908738521234052y\right])^3)c_1/\Gamma(1+\alpha)]
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
w = A \; [&1 + \cos(4.908738521234052y) + t^\alpha(24.095713869847067 \\
&\cos\left[4.908738521234052y\right] + (1 + \cos\left[4.908738521234052y\right]) \\
&+ A^2(1 + \cos\left[4.908738521234052y\right])^3)c_1/\Gamma(1+\alpha)].
\end{aligned} \tag{27}
$$

4.2 Problem-ii

Let us consider the following type Phi-4 time fractional order model (Ehsani and Ehsani 2013)

$$
\frac{\partial^\alpha w(y,t)}{\partial t^\alpha} - \frac{\partial^2 w(y,t)}{\partial y^2} - w + w^3 = 0, y \in [0,1], t > 0, 1 < \alpha \le 2. \tag{28}
$$

Initial and boundary conditions of Phi-4 equations are:

$$
w(y,0) = 0, \; w_t(y,0) = y.
$$

In the following p^0, p^1, p^2 and p^3 are zero, first, second and third-order problems.

$$p^0 : \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} = 0, \tag{29}$$

$$\begin{aligned} p^1 &: c_1 w_0(y,t) - c_1 w_0^3(y,t) + c_1 \frac{\partial^2 w_0(y,t)}{\partial y^2} - \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - \\ & c_1 \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} + \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} = 0, \end{aligned} \tag{30}$$

$$\begin{aligned} p^2 &: c_2 w_0(y,t) - c_2 w_0^3(y,t) + c_1 w_1(y,t) - 3c_1 w_0^2(y,t) w_1(y,t) + c_2 \frac{\partial^2 w_0(y,t)}{\partial y^2} \\ &+ c_1 \frac{\partial^2 w_1(y,t)}{\partial y^2} - c_2 \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} \\ &- c_1 \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} + \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} p^3 &: c_3 w_0(y,t) - c_3 w_0^3(y,t) + c_2 w_1(y,t) - 3c_2 w_0^2(y,t) w_1(y,t) - 3c_1 \\ & w_0(y,t) w_1^2(y,t) + c_1 w_2(y,t) - 3c_1 w_0(y,t) w_1^2(y,t) + c_3 \frac{\partial^2 w_0(y,t)}{\partial y^2} \\ &+ c_2 \frac{\partial^2 w_1(y,t)}{\partial y^2} + c_1 \frac{\partial^2 w_2(y,t)}{\partial y^2} - c_3 \frac{\partial^\alpha w_0(y,t)}{\partial t^\alpha} - c_2 \frac{\partial^\alpha w_1(y,t)}{\partial t^\alpha} \\ &- \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} - c_1 \frac{\partial^\alpha w_2(y,t)}{\partial t^\alpha} + \frac{\partial^\alpha w_3(y,t)}{\partial t^\alpha} = 0. \end{aligned} \tag{32}$$

In the following w_0 and w_1 are zeroth-order, first-order solutions, by using these two solutions in Sect. 3, equation (8), we get w solution.

$$w_0 = t\,y, \tag{33}$$

$$w_1 = \frac{1}{\Gamma[4+\alpha]} \left(t^{1+\alpha} y \left(6t^2 y^2 - (2+\alpha)(3+\alpha) \right) c_1 \right), \tag{34}$$

$$w = \frac{ty}{\Gamma[4+\alpha]} \left(1 + t^\alpha \left(6t^2 y^2 - (2+\alpha)(3+\alpha) \right) c_1 \right). \tag{35}$$

5 Results and discussions with physical understanding

There are different types of traveling wave solution exhibiting various physical phenomenon. In the solitary wave theory, the traveling waves depends on the variation of physical parameter. If varying physical parameters, the solution of Phi-4 equation is of complex in nature then the wave propagation is characterized by the factor $|u(x,t)|$. For some special values of physical parameters, the traveling wave solution originated is explicit type. The dispersion term present in Phi-4 equation effects in the solution. If the dispersion term is linear then it destroys the solitary effect of waves as they stat propagating at different velocity groups. While if the dispersion term is non linear type, as we have mentioned

Table 1 For different values of α, first order auxiliary constant of model-1

α	1.5	1.75	2
c_1	−0.07021634384803148	−0.0870365696197838	−0.1141573047418468

Table 2 Results of model-1 and absolute error with different values of α.

y	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 2$	Exact Solution	Abs.Error
0	1.99994	1.99999	2.00000	2.00000	9.46471×10^{-7}
0.1	1.88187	1.88191	1.88192	1.88192	7.60155×10^{-7}
0.2	1.55554	1.55556	1.55557	1.55557	2.90222×10^{-7}
0.3	1.09801	1.09802	1.09802	1.09802	2.75783×10^{-7}
0.4	0.617331	0.617319	0.617317	0.617316	7.86216×10^{-7}
0.5	0.22702	0.226995	0.226991	0.226989	1.16299×10^{-6}
0.6	0.0192542	0.0192219	0.0192161	0.0192147	1.35743×10^{-6}
0.7	0.0430981	0.0430667	0.043061	0.0430596	1.33519×10^{-6}
0.8	0.292921	0.292898	0.292894	0.292893	1.10077×10^{-6}
0.9	0.709725	0.709717	0.709716	0.709715	6.93069×10^{-7}
1	1.19508	1.19509	1.19509	1.19509	1.63388×10^{-7}
1.1	1.63436	1.63439	1.63439	1.63439	3.98075×10^{-7}
1.2	1.92383	1.92387	1.92388	1.92388	8.25302×10^{-7}

in equation, it prevent the formulation of solitary wave due to shifting of pulse energy to higher level. The interesting fact become, as in case of Phi-4 equation, both propagating term u_{xx} and nonlinear term u^3 is present. Their interaction with other solitons makes the problem more interesting and worthwhile. The Phi-4 equation has many solutions. In the last section, we explain the OHAM to find the solution of our model problem.

The OHAM algorithm is presented in Sect. 3 for time fractional order Phi-4 equation and description of the formulation in the examples of Sect. 4, provides extremely valid results without spatial discretization for the problems. There is no need to measure higher-order solutions while applying OHAM. All nonlinear time-fractional solutions of the Phi-4 equations are unique, and to our knowledge can not be found in literature.

The values of the auxiliary constant c_1 for dissimilar values of α, $\alpha = 1.5, 1.75$, and 2 are represented in the following Tables 1 and 3 of problems i and ii. Table of auxiliary constant c_1 represents the order of solution. Table 2 of problems-i is represents approximate results, exact results and absolute error for dissimilar values of α at fixed time $t = 0.001$, $A = 1$, and also error norms of problems i are $L_2 = 9.1652 \times 10^{-7}$, $L_\infty = 2.0000 \times 10^{-6}$. Tables 4, 5, 6 and 7 of problems ii represents the comparison between current OHAM and other three methods (Homotopy perturbation method, Adomian decomposition method, Homotopy analysis method) (Ehsani and Ehsani 2013), also represents the validity and accuracy of the method.

In the above, Figs. 1 and 2 of model-1 represents the $3D$, exact and approximate results for fixed value of $\alpha = 2$. Figures 3 and 4 of model- 1 represents the $3D$and$2D$ results respectively for different values of α, $\alpha = 1.5, 1.75$ and 2. Similarly of model-2

Fig. 1 3D graphic of exact solution for problem-i with single value of $\alpha = 2$

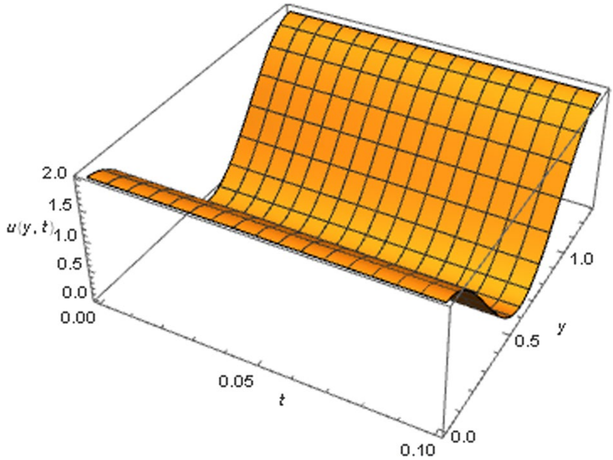
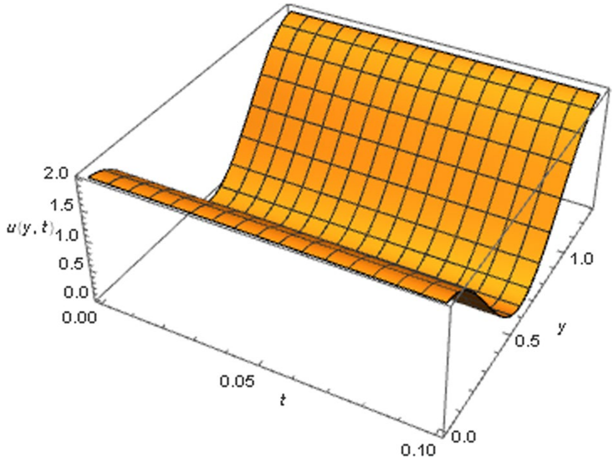


Fig. 2 3D graphics of approximate Solution for problem-i with single value $\alpha = 2$



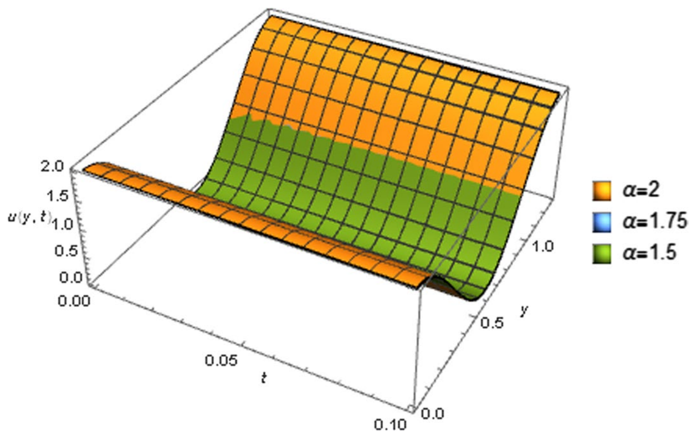


Fig. 3 3D graphic of approximate solutions for problem-i with dissimilar values of α

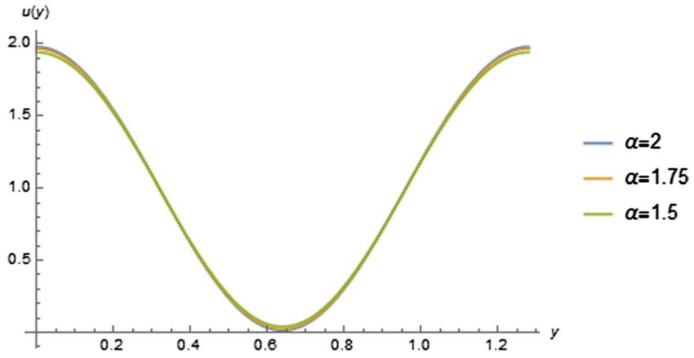


Fig. 4 2D graphic of approximate solutions for problem-i with dissimilar values of α

Table 3 First order auxiliary constant of model-2 with different values of α

α	1.5	1.75	2
c_1	−0.6064100515789376	−0.7218567254596747	−0.8172519816389054

Table 4 Results of model-2 with different values of α

y	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
0	0	0	0
0.1	0.00100018	0.00100005	0.00100001
0.2	0.00200036	0.0020001	0.00200003
0.3	0.00300055	0.00300015	0.00300004
0.4	0.00400073	0.00400021	0.00400005
0.5	0.00500091	0.00500026	0.00500007
0.6	0.00600109	0.00600031	0.00600008
0.7	0.00700128	0.00700036	0.0070001
0.8	0.00800146	0.00800041	0.00800011
0.9	0.00900164	0.00900046	0.00900012
1	0.0100018	0.0100005	0.0100001
1.1	0.011002	0.0110006	0.0110001
1.2	0.0120022	0.0120006	0.0120002
1.3	0.0130024	0.0130007	0.0130002

Fig. 5 represents the $3D$ approximate result for fixed value $\alpha = 2$ and Figs. 6,7 represents the $3D$, $2D$ results for different values of α. All the above Figs. represents the accuracy, validity and effectiveness of the extended OHAM algorithm. It is cleared that as we proceed along the domain, we obtain consistent validity. In the above discussion Table2 shows the best agreement between the approximate solutions and exact solution. Tables 4,5, 6 and 7 represents latent strength of OHAM.

Table 5 Comparison of OHAM of model-2 with other methods at $y = 0.3, \alpha = 2$

t	HPM	ADM	HAM	Present
0.006	0.00180001	0.00180001	0.00180001	0.00180001
0.01	0.00300005	0.00300005	0.00300005	0.00300004
0.05	0.0150063	0.0150063	0.0150063	0.0150063
0.1	0.03005001	0.03005	0.03005	0.0300409
0.15	0.045169	0.0451688	0.39851187	0.0451378
0.2	0.0604003	0.0604003	0.0604007	0.0603265
0.25	0.0757822	0.0757822	0.0757823	0.0756374

Table 6 Comparison of OHAM of model-2 with other methods at $y = 0.6, \alpha = 2$

t	HPM	ADM	HAM	Present
0.006	0.00360002	0.00360002	0.00360002	0.00360002
0.01	0.0060001	0.0060001	0.0060001	0.00600008
0.05	0.0300125	0.0300125	0.0300125	0.0300102
0.1	0.0600999	0.0600999	0.0601	0.0600816
0.15	0.090337	0.0903371	0.0903374	0.0902752
0.2	0.120798	0.120798	0.120799	0.120651
0.25	0.151556	0.151557	0.151558	0.151268

Table 7 Comparison of OHAM of model-2 with other methods at $y = 1.3, \alpha = 2$

t	HPM	ADM	HAM	Present
0.006	0.00780005	0.00780005	0.00780005	0.00780004
0.01	0.0130002	0.0130002	0.0130002	0.0130002
0.05	0.0650271	0.0650271	0.0650271	0.0650221
0.1	0.130216	0.0130216	0.0130216	0.130176
0.15	0.195724	0.195724	0.195724	0.195591
0.2	0.261701	0.261701	0.261703	0.261388
0.25	0.328286	0.328288	0.32829	0.327679

6 Conclusion

The extended approximate technique has been effectively applied to discover the new travelling wave solutions of quite significant nonlinear dispersive equations, namely, the time-fractional Phi-4 equations. The results of applications, shows the best agreement between approximate solutions and exact solutions. This work illustrates that OHAM rapidly convergent. Therefore OHAM demonstrates its power and latent strength for the solutions of nonlinear models in applications of real life.

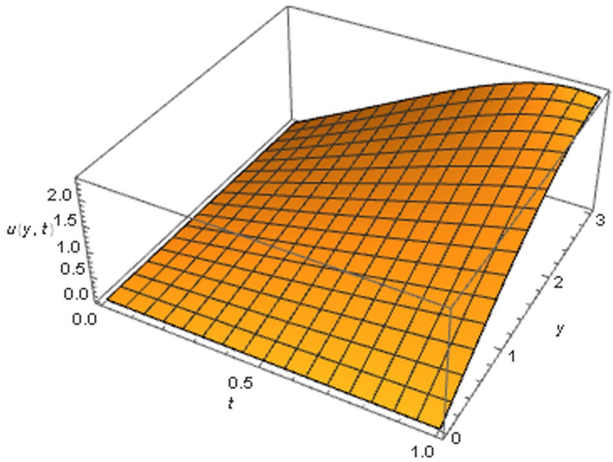


Fig. 5 3D graphic of exact solution for problem-ii with single value of $\alpha = 2$

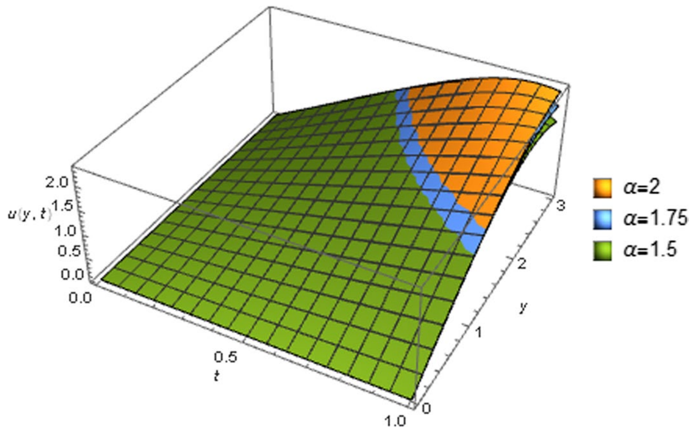


Fig. 6 3D graphic of exact solution for problem-ii with different values of α

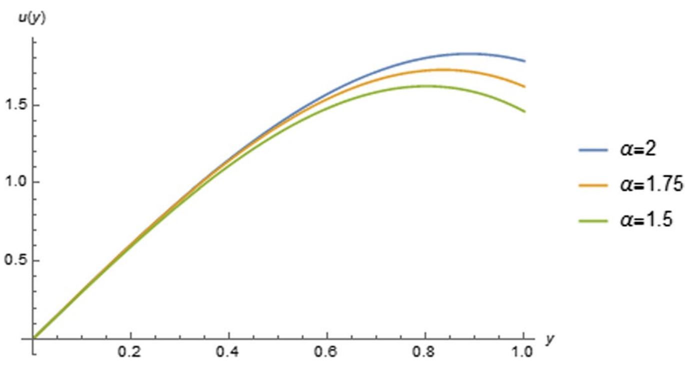


Fig. 7 2D graphic of exact solution for problem-ii with different values of α

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