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New conservation laws and exact solutions of the special case of the fifth-order KdV equation

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A B S T R A C T

The current study deals with the Kaup-Kupershmidt (KK) equation to construct formal Lagrangian, conservation laws, and exact solutions. KK is basically a special case of the 5th-order KdV equation. The conservation laws obtained by using the conservation theorem are trivial conservation laws. In addition, exact solutions are found via the modified simple equation (MSE) method. For a suitable value of solutions, the 3D surfaces have been plotted using MAPLE. These plots giving novel exact solutions are made to reveal important wave characteristics. Our obtained results in this work concerning our investigated equation are essential to explain many physical and oceanographic applications involving ocean gravity waves and many other related phenomena.

1. Introduction

In physics, the notion of conservation laws has been considered profoundly. Whatever the physical laws tackled: solid-state physics, classical mechanics, quantum mechanics, fluid mechanics, general relativity, or quantum field theory have always been important ingredients to describe nature [1–3]. The existence of conservation laws is mathematically based on Lie symmetries. This relationship was suggested by Emmy Noether [4,5].

Conservation laws play an important role in mathematical and physical problems. If the conservation laws of equations can be found, it can be said that their integrability is quite possible. Obtaining conservation laws for a considered system is frequently a challenging procedure. Emmy Noether has proposed a theorem in which the conservation laws can be found when Lagrange equations exist. According to Emmy Noether's theorem, conservation laws can be found with the help of each Lie symmetry generator and Lagrangian equation [6,7].

The proposed conservation theorem by Ibragimov is related to Lie symmetry generators, formal Lagrangian, and adjoint equations. According to this theorem, we can obtain conservation laws with every Lie symmetry generator. Obtained conservation laws are named trivial or non-trivial [8–11].

The constructed exact solutions for nonlinear partial differential equations are very important due to their essential role in mathematics. Several techniques have been utilized to attain exact solutions to these equations such as the Hirota's bilinear method [12–17], the inverse scattering method [18], the homogeneous balance method [19], the tanh method [20], the sine-cosine method [21,22], the exp-function method [23], the auxiliary equation method [24], the Kudryashov method [25], the extended sinh-Gordon equation expansion method [26], the first integral method [27], the ansatz method [28,29], the (G'/G)-expansion method [30,31] and so on (see [32–43]). While there are some interesting studies that are important to the field of ocean engineering such as the numerical solution of KdV and Burgers equations formulated in the context of Caputo fractional derivative (CpFrD) via the technique of local meshless collocation [60], the solution of potential KdV and Benjamin equations formulated in the context of CpFrD via the technique of $q-$homotopy analysis transform [61], novel exact solutions to some interesting nonlinear equations such as the fifth-order KdV equation, modified KdV-Zakharov-Kuznetsov equation, and

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Jimbo-Miwa equation using the technique of (G'/G)-expansion [62], and new solutions to the generalized Hirota-Satsuma coupled KdV formulated in the context of conformable derivative via the technique of (G'/G)-expansion [63], our research work is considered as original and novel where conservation laws and exact solutions are obtained to the fifth-order KdV equation. For more information about conformable and fractional derivatives and integrals with applications, refer to [65–71]. All our results are essential to understand nonlinear phenomena in dispersive waves which are important in the field of ocean engineering because investigating the interaction solutions to the studied model in nonlinear dispersive waves can enable other researchers to do further studies on the effects of dispersive wave in coastal processes and waves' interactions with complex structures in the nearshore areas [64].

The generalization of the fifth-order Korteweg–de Vries (fifth-order KdV) equation is given by:

$$u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + + u_{xxxxx} = 0, \tag{1.1}$$

where α, β and γ are arbitrary nonzero and real parameters. This equation describes the motion of long waves which is important for mathematical models. If we substitute $\alpha = 10, \beta = 25, \gamma = 20$ in Eq. (1.1), we obtain the Kaup-Kupershmidt (KK) equation, which is considered as a special case of the 5th-order KdV equation [44].

This paper is devoted to the fifth-order Kaup-Kupershmidt (KK) equation which is an essential equation in mathematical physics and engineering.

$$u_t + 10 u u_{xxx} + 25 u_x u_{xx} + 20 u^2 u_x + u_{xxxxx} = 0. \tag{1.2}$$

KdV type equations of fifth-order take place in describing many distinct wave phenomena, like the gravity-capillary waves and magneto-sound propagation in plasmas and propagation of shallow water waves over a flat surface [45]. This equation is known as the initial equation of the hierarchy of integrable equations with Lax operator. Eq. (1.2) is completely integrable [46] and has bilinear representations. The exact solutions to this equation have been obtained by several scientists (refer to [47,48]). Salas et al. [49] have applied the projective Riccati equations method and Cole-Hopf transformation. Goodarzian et al. [50] investigated exact solutions by applying the Exp-function method. Feng and Li founded the exact solutions [51] by using the Fan sub-equation method. Shakeel and Mohyud-Din [52] and Roshid et al [53]. applied some different methods.

This manuscript is organized as follows: First, some preliminaries of the new conservation theorem have been given in sec. 2. Then, we have adopted a theorem to the fifth-order KK equation in sec. 3. In the next section, some preliminaries of the MSE method have been introduced. The MSE method has been applied to the given equation. Also, we have plotted the graphics of all founded solutions that have been given by setting some special values for the parameters in sec. 5. The graphics have been plotted by MAPLE. Conclusions have been given in the last section.

2. Some preliminaries on conservation laws

In this section, some needed preliminaries will be given about new conservation theorem. We first deal with the k-th order partial differential equations:

$$P(x, u, u_1, u_2, \ldots, u_k) = 0, \tag{2.1}$$

here: $x = (x_1, x_2, \ldots, x_m)$ are m independent variables, $u = (u^1, u^2, \ldots, u^n)$ are dependent variables. u_α denotes the derivative with respect to x. Here, for $\alpha = 1, 2, \ldots, k$ and $i_\alpha = 1, 2, \ldots, m$. Lie symmetry generator can be written as follows:

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{2.2}$$

where ξ^i and η^α are called infinitesimals, $i = 1, 2, \ldots, m$ and $\alpha = 1, 2, \ldots, n$. Infinitesimals are functions that contain dependent and independent variables. In this paper, only Lie point symmetries are obtained. The k-th prolongation of the Lie symmetry generator is expressed as:

$$X^{(k)} = X + \eta_i^{(1)\alpha} \frac{\partial}{\partial u_i^\alpha} + \ldots + \eta_{i_1 i_2 \ldots i_k}^{(k)\alpha} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}^\alpha}, k \geq 1, \tag{2.3}$$

where

$$\eta_i^{(1)\alpha} = D_i \eta^\alpha - \left(D_i \xi^j\right) u_j^\alpha$$

$$\eta_{i_1 i_2 \ldots i_k}^{(k)\alpha} = D_{i_k} \eta_{i_1 i_2 \ldots i_{k-1}}^{(k-1)\alpha} - \left(D_{i_k} \xi^j\right) u_{i_1 i_2 \ldots i_{k-1} j}^\alpha,$$

here: $i, j = 1, 2, \ldots, m$ and $\alpha = 1, 2, \ldots, n$ and $i_p = 1, 2, \ldots, m$ for $p = 1, 2, \ldots, k.D_i$ is called the total derivative operator. Formal Lagrangian is given by:

$$L = w^\beta P, \tag{2.4}$$

where $w = \left(w^1, w^2, \ldots, w^\beta\right)$ is an adjoint variable, and so adjoint equation can be founded as follows [54]:

$$P_\alpha^*\left(x, u, w, \ldots, u_{(k)}, w_{(k)}\right) \equiv \frac{\delta\left(w^\beta P\right)}{\delta u^\alpha} = 0 \quad \alpha = 1, 2, \ldots, m \tag{2.5}$$

where $\frac{\delta}{\delta u^\alpha}$ is variational derivative with:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=0}^{\infty} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha} \quad (\alpha = 1, \ldots, m).$$

P_α^* admits the same symmetries with given equation. If we obtain: Eq. (2.1) by substituting $w = u$ in Eq. (2.5), then Eq. (2.1) is said to be a self adjoint equation.

Conserved quantities can be obtained by the following formula:

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s} (W^\alpha) \frac{\delta L}{\delta u_{i i_1 \ldots i_s}^\alpha} \quad (i = 1, \ldots, m), \tag{2.6}$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ and $\alpha = 1, \ldots, n$, T^i. The obtained conservation laws involve w so that we can say that obtained conservation laws are infinite conservation laws. The finite number of conservation laws can be found by selecting a special w.

For the conservation laws to be trivial, the following equality must be ensured [55–57]:

$$D_i\left(T^i\right) = 0. \tag{2.7}$$

3. Applying conservation theorem to the Kaup-Kupershmidt (KK) equation

New conservation laws will be obtained in this section for a given equation. The formal Lagrangian for Eq. (1.2) can be constructed in the following form:

$$L = w(t, x)\left(u_t + 20u^2 u_x + 25 u_x u_{xx} + 10 u u_{xxx} + u_{xxxxx}\right), \tag{3.1}$$

where w is the new adjoint variable. The adjoint equation can be obtained by substituting Eq. (3.1) in Eq. (2.5) as follows:

$$P^* = -w_t - 20u^2 w_x - 5u_{xx} w_x - 5u_x w_{xx} - 10 u w_{xxx} - w_{xxxxx}. \tag{3.2}$$

Then, it is concluded that Eq. (1.2) is not verified by substituting $w = u$ in Eq. (3.2). Consequently, we reach the end of that Eq. (1.2) which is not self-adjoint. Also, $w = 1$ satisfies Eq. (3.2).

KK equation has three Lie-point symmetry generators as follows:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \tag{3.3}$$

Let us obtain the conservation laws of Eq. (1.2).

Case 1: We deal with:

$$X_1 = \frac{\partial}{\partial t} \tag{3.4}$$

$$\begin{aligned} \xi^x &= 0, \\ \xi^t &= 1, \\ \eta &= 0. \end{aligned} \tag{3.5}$$

For X_1, conserved vectors' formulations can be obtained by using Eq. (2.6) as follows:

$$\begin{aligned} T^t &= \xi^t L + W\left[\tfrac{\partial L}{\partial u_t}\right], \\ T^x &= \xi^x L + W\left[\tfrac{\partial L}{\partial u_x} - D_x\left(\tfrac{\partial L}{\partial u_{xx}}\right) + D_x^2\left(\tfrac{\partial L}{\partial u_{xxx}}\right) + D_x^4\left(\tfrac{\partial L}{\partial u_{xxxxx}}\right)\right] \\ &\quad + D_x(W)\left[\tfrac{\partial L}{\partial u_{xx}} - D_x\left(\tfrac{\partial L}{\partial u_{xxx}}\right) - D_x^3\left(\tfrac{\partial L}{\partial u_{xxxxx}}\right)\right] \\ &\quad + D_x^2(W)\left[\tfrac{\partial L}{\partial u_{xxx}} + D_x^2\left(\tfrac{\partial L}{\partial u_{xxxxx}}\right)\right] + D_x^3\left[-D_x\left(\tfrac{\partial L}{\partial u_{xxxxx}}\right)\right] \\ &\quad + D_x^4\left[\tfrac{\partial L}{\partial u_{xxxxx}}\right]. \end{aligned} \tag{3.6}$$

We can determine W by using (3.5) as follows:

$$W = -u_t. \tag{3.7}$$

Therefore, conserved vectors can be found as follows:

$$\begin{aligned} T_1^t &= w\left(20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{xxxxx}\right), \\ T_1^x &= -20u_tu^2w - 10u_tu_{xx}w + 5u_tu_xw_x - 10u_tuw_{xx} \\ &\quad - u_tw_{xxxx} - 15u_{xt}u_xw + 10u_{xt}uw_x + u_{xt}w_{xxx} \\ &\quad - 10u_{xxt}uw - u_{xt}w_{xx} + u_{xxxt}w_x - u_{xxxxt}w, \end{aligned} \tag{3.8}$$

for Eq. (1.2). Then, we substitute $w = 1$ in Eq. (3.8) to obtain following finite conservation laws:

$$\begin{aligned} \widetilde{T_1^t} &= 20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{xxxxx}, \\ \widetilde{T_1^x} &= -20u_tu^2 - 10u_tu_{xx} - 15u_{xt}u_x - 10u_{xxt}u - u_{xxxxt}. \end{aligned} \tag{3.9}$$

We find:

$$D_t\left(\widetilde{T_1^t}\right) + D_x\left(\widetilde{T_1^x}\right) = 0, \tag{3.10}$$

by substituting Eq. (3.9) in Eq. (2.7).

Case 2: Similarly, we deal with X_2 as follows:

$$X_2 = \frac{\partial}{\partial x}. \tag{3.11}$$

Now, conservation laws that correspond to symmetry X_2 can be constructed as follows:

$$\begin{aligned} \xi^x &= 1, \\ \xi^t &= 0, \\ \eta &= 0, \end{aligned} \tag{3.12}$$

can be obtained From X_2. Then, we find:

$$W = -u_t. \tag{3.13}$$

by using $W = \eta - \xi^x u_x - \xi^t u_t$ and Eq. (3.12). Hence, we obtain the following conserved vectors by substituting Eq. (3.13) in (3.6):

$$\begin{aligned} T_2^t &= -u_xw, \\ T_2^x &= wu_t + 5u_x^2w_x - 10u_xuw_{xx} - u_xw_{xxxx} \\ &\quad + 10u_{xx}uw_x + u_{xx}w_{xxx} - u_{xxx}w_{xx} + u_{xxxx}w_x. \end{aligned} \tag{3.14}$$

Similarly, we find:

$$\begin{aligned} \widetilde{T_2^t} &= -u_x, \\ \widetilde{T_2^x} &= u_t. \end{aligned} \tag{3.15}$$

by substituting $w = 1$ in Eq. (3.14), we conclude that Eq. (2.7) is satisfied for (3.15). Therefore, $\widetilde{T_2^t}$ and $\widetilde{T_2^x}$ are conservation laws for Eq. (1.2).

Case 3: Finally, the following Lie-point symmetry will be used:

$$X_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \tag{3.16}$$

Eq. (1.2) satisfies invariance test for X_3. We can obtain the conserved vectors by applying the same procedure as follows:

$$\begin{aligned} \widetilde{T_3^t} &= 5tu_{xxxxx} + 50tuu_{xxx} + 125tu_xu_{xx} + 100tu^2u_x - 2u - xu_x, \\ \widetilde{T_3^x} &= -6u_{xxxx} - 5tu_{xxxxt} - 45u_x^2 - 40u^3 - 100tu_tu^2 - 50tu_tu_{xx} \\ &\quad - 75tu_{xt}u_x - 50tu_{xxt}u - 60uu_{xx} + xu_t. \end{aligned} \tag{3.17}$$

Also, it is shown that

$$D_t\left(\widetilde{T_3^t}\right) + D_x\left(\widetilde{T_3^x}\right) = 0, \tag{3.18}$$

by using Eq. (2.7) for (3.17). Therefore, we conclude that the invariance condition is satisfied.

4. Some preliminaries on modified simple equation (MSE) method

The algorithm of the MSE method is provided step-by-step in this section.

1. Let us think that a nonlinear partial differential equation is given as follows:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0. \tag{4.1}$$

2. To solve the above equation, we consider the following form of wave transformation because this transformation is needed to construct exact solutions by helping us overcome the challenges of solving PDEs by reducing into ordinary differential equations (ODEs) which can be easily solved:

$$u(t, x) = u(\xi), \quad \xi = x - ct, \tag{4.2}$$

where c is the wave speed. Eq. (4.1) can be reduced as an ODE via this transformation

$$Q(u, u', u'', u''', \ldots) = 0. \tag{4.3}$$

Eq. (4.3) should be integrated as much as possible.

According to MSE method, we investigate the solutions of ODE Eq. (4.3) in terms of $\frac{\Psi'(\xi)}{\Psi(\xi)}$ as follows [58,59]:

$$u(\xi) = \sum_{n=0}^{m} a_n\left[\frac{\Psi'(\xi)}{\Psi(\xi)}\right]^n, \quad a_n = \text{const.}, a_m \neq 0. \tag{4.4}$$

Here, $\Psi(\xi)$ is a function to be determined. ($\Psi'(\xi) \neq 0$).

By equating the highest power of the nonlinear term(s) and the highest power of the highest order derivative of Eq. (4.3), the positive integer m in Eq. (4.4) can be determined.

If we substitute Eq. (4.4) into Eq. (4.3), we should collect all coefficients: $\Psi^j(\xi)$ $(j = 0, -1, -2, \ldots)$. Each equation in the obtained determining equation system must equal zero. In addition, we find the solutions of obtained determining equation system with the help of symbolic computation. Then, we substitute them into Eq. (4.4) for finding the exact solutions of Eq. (4.1).

5. Applying MSE method to the kaup-Kupershmidt (KK) equation

The exact solutions of Eq. (1.2) by MSE are obtained in this section. Then, all graphs of obtained exact solutions are given. Let us begin with the traveling wave transformation Eq. (4.2). This transformation reduces Eq. (1.2) to following ODE:

$$-cu' + u^{(5)} + 10uu''' + 25u'u'' + 20u^2u' = 0. \tag{5.1}$$

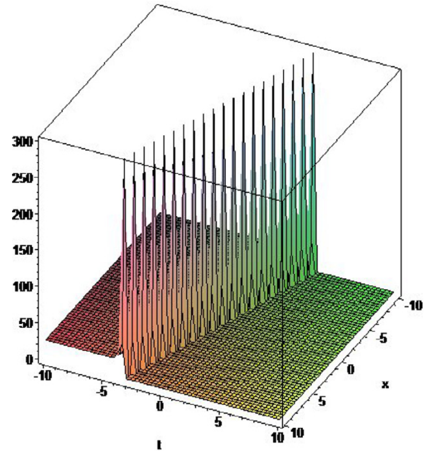

Fig. 1. The graph of u_1 in Eq. (5.10) where $c = -3, c_1 = -2, c_2 = -3$.

If we integrate Eq. (5.2) with respect to ξ once, we establish:

$$-cu + u^{(4)} + 10uu'' + \frac{15}{2}(u')^2 + \frac{20}{3}u^3 = 0. \tag{5.2}$$

The balance is obtained by $m = 2$. Based on the MSE method, the solution of Eq. (5.2) is given by:

$$u(\xi) = a_0 + a_1\left(\frac{\Psi'(\xi)}{\Psi(\xi)}\right) + a_2\left(\frac{\Psi'(\xi)}{\Psi(\xi)}\right)^2. \tag{5.3}$$

By substituting Eq. (5.3) into Eq. (5.2), we get:

$$\begin{aligned}
\Psi^0(\xi) \quad &: \quad ca_0 - a_0^2 - a_0 = 0,\\
\Psi^{-1}(\xi) \quad &: \quad -a_1\Phi' + a_1\Phi''' - 2a_0a_1\Phi' + ca_1\Phi' = 0,\\
\Psi^{-2}(\xi) \quad &: \quad 2a_2(\Phi'')^2 - a_2(\Phi')^2 - 2a_0a_2(\Phi')^2 - a_1^2(\Phi')^2\\
&\qquad -3a_1\Phi''\Phi' + 2a_2\Phi'\Phi''' + ca_2(\Phi')^2 = 0,\\
\Psi^{-3}(\xi) \quad &: \quad -10a_2(\Phi')^2\Phi'' - 2a_1(\Phi')^3a_2 + 2a_1(\Phi')^3 = 0,\\
\Psi^{-4}(\xi) \quad &: \quad -a_2^2(\Phi')^4 + 6a_2(\Phi')^4 = 0.
\end{aligned} \tag{5.4}$$

We obtain that from the above equation system the following:

$$a_0 = 0, a_0 = c - 1, \tag{5.5}$$

and

$$a_2 = 6. \tag{5.6}$$

Next, we replace (5.5) and (5.6) with the other equations in the system. Since a_0 has two different values, we have two different cases. Let us now discuss both of these cases.

Case1: When $a_0 = 0$, Eq. (5.4) turns into:

$$\begin{aligned}
\Psi^1(\xi) \quad &: \quad -a_1\Psi' + a_1\Psi''' + ca_1\Psi' = 0,\\
\Psi^2(\xi) \quad &: \quad 12(\Psi'')^2 - 6(\Psi')^2 - a_1^2(\Psi')^2\\
&\qquad -3a_1\Psi''\Psi' + 12\Psi'\Psi''' + 6c(\Psi')^2 = 0,\\
\Psi^3(\xi) \quad &: \quad -60(\Psi')^2\Psi'' - 10a_1(\Psi')^3 = 0.
\end{aligned} \tag{5.7}$$

From Eq. (5.7), we find:

$$a_1 = \pm 6\sqrt{1-c}, \tag{5.8}$$

and

$$\Psi(\xi) = \mp\frac{c_1 e^{\sqrt{1-c}\xi}}{\sqrt{1-c}} + c_2, \tag{5.9}$$

where $c \neq 1$. Therefore, $u(\xi)$ takes the following form:

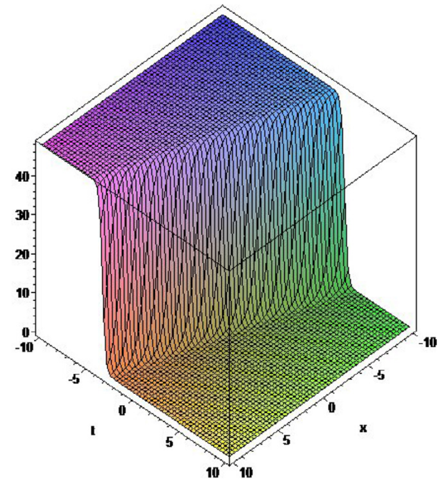

Fig. 2. The graph of u_2 in Eq. (5.10) where $c = -3, c_1 = -2, c_2 = -3$.

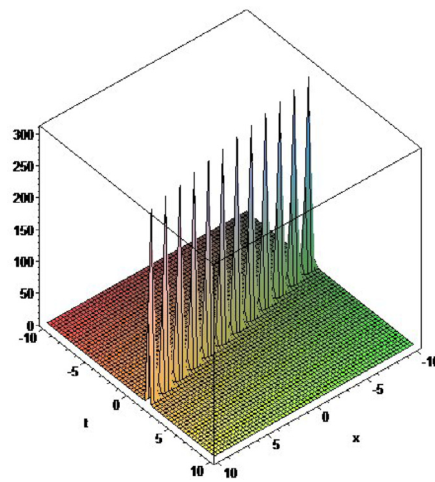

Fig. 3. The graph of u_3 in Eq. (5.14) where $c = 5, c_1 = -2, c_2 = -3$.

$$\begin{aligned}
u_{1,2}(\xi) = &\frac{6\sqrt{1-c}c_1(\cosh(\sqrt{1-c}\xi) - \sinh(\sqrt{1-c}\xi))}{\mp\frac{c_1(\cosh(\sqrt{1-c}\xi)-\sinh(\sqrt{1-c}\xi))}{\sqrt{1-c}} + c_2}\\
&+\frac{6c_1^2(\cosh(\sqrt{1-c}\xi) - \sinh(\sqrt{1-c}\xi))^2}{\left(\mp\frac{c_1(\cosh(\sqrt{1-c}\xi)-\sinh(\sqrt{1-c}\xi))}{\sqrt{1-c}} + c_2\right)^2},
\end{aligned} \tag{5.10}$$

where $\xi = x - ct$. Fig. 1 is the graph of the solution u_1 where this solution is categorized as multi solitons interaction, and Fig. 2 is the graph of the solution u_2 where this solution is categorized as anti-kink soliton.

Case2: When $a_0 = c - 1$, Eq. (5.4) turns into:

$$\begin{aligned}
&\Psi^1(\xi) : -a_1\Psi' + a_1\Psi''' + ca_1\Psi' + 2(1-c)a_1\Psi' = 0,\\
&\Psi^2(\xi) : 12(\Psi'')^2 - 6(\Psi')^2 - a_1^2(\Psi')^2 - 3a_1\Psi''\Psi'\\
&\qquad +12\Psi'\Psi''' + 6c(\Psi')^2 - 12(1-c)(\Psi')^2 = 0,\\
&\Psi^3(\xi) : -60(\Psi')^2\Psi'' - 10a_1(\Psi')^3 = 0.
\end{aligned} \tag{5.11}$$

Then, we similarly solve the above system to obtain:

$$a_1 = \pm 6\sqrt{c-1}, \tag{5.12}$$

and

$$\Psi(\xi) = \mp\frac{c_1 e^{-\sqrt{c-1}\xi}}{\sqrt{c-1}} + c_2, \tag{5.13}$$

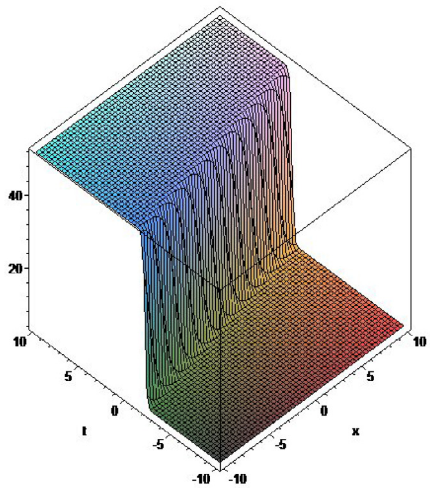

Fig. 4. The graph of u_4 in Eq. (5.14) where $c = 5, c_1 = -2, c_2 = -3$.

where $c \neq 1$. Finally, to complete the determination of the solution of Eq. (1.2), we substitute the values into Eq. (5.3), and it derives:

$$u_{3,4}(\xi) = c - 1 + \frac{6\sqrt{c-1}c_1(\cosh(\sqrt{c-1}\xi) - \sinh(\sqrt{c-1}\xi))}{\mp\frac{c_1(\cosh(\sqrt{c-1}\xi)-\sinh(\sqrt{c-1}\xi))}{\sqrt{c-1}} + c_2} + \frac{6c_1^2(\cosh(\sqrt{c-1}\xi) - \sinh(\sqrt{c-1}\xi))^2}{\left(\mp\frac{c_1(\cosh(\sqrt{c-1}\xi)-\sinh(\sqrt{c-1}\xi))}{\sqrt{c-1}} + c_2\right)^2}, \tag{5.14}$$

where $\xi = x - ct$. The graphs of solutions: u_3 and u_4 are given in Fig. 3 which represent multi solitons interaction, and Fig. 4 represents anti-kink soliton.

6. Conclusion

The conserved quantities and exact solutions of the Kaup-Kupershmidt equation are covered in this study. Firstly, we have given the required information about the new conservation theorem and MSE method. Also, we have implemented these methods to the considered system. Then, we have derived a formal Lagrangian and adjoint equation. Conservation laws are derived for every five Lie symmetry generators. Consequently, we can state that $w = 1$ is a solution to the adjoint equation. Since the divergence condition is satisfied, the obtained conservation laws are trivial. Besides, we have obtained exact solutions of the equation mentioned and have plotted the 3D under suitable values of constants. The used methodology is shown to prove a productive approach to solve the nonlinear partial differential equations in mathematical physics. We can conclude that the obtained solutions in our work are totally different in comparison to the existing ones in the literature. Furthermore, according to the best of our knowledge, the obtained exact solutions are very useful in various areas of applied mathematics to interpret some physical and ocean engineering phenomena. Some other new exact solution strategies can be applied to the investigated equation in future research works.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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