




Complexiton and resonant multi-solitons of a (4 + 1)-dimensional Boiti–Leon–Manna–Pempinelli equation

Nauman Raza¹ · Melike Kaplan² · Ahmad Javid³ · Mustafa Inc^{4,5,6} 

Received: 18 September 2021 / Accepted: 21 December 2021 / Published online: 4 January 2022
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Abstract

This work is concerned with the extraction of auto-Bäcklund transformations of (4 + 1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation in accordance with the extended homogeneous balance (HB) method incorporating Maple. Subsequently, these transformations are used to study analytic explicit solutions of this equation. Also, based on the Hirota bilinear form and the extended transformed rational function method, complexiton solutions have been found of the (4 + 1)-dimensional BLMP equation through a direct symbolic computation with Maple. This method is the improved form of the transformed rational function method. The obtained complexiton solutions, including trigonometric and hyperbolic trigonometric solutions, have been verified utilizing Hirota bilinear forms. Some of the solutions are depicted in 3D graphs to understand physical properties. The reported results are new and have applications in several scientific fields such as incompressible fluids.

Keywords Auto-Bäcklund transformations · Complexiton solutions · Symbolic computation · Extended transformed rational function method · The extended homogeneous balance method

✉ Mustafa Inc
minc@firat.edu.tr

¹ Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore, Pakistan

² Department of Mathematics, Faculty of Arts & Sciences, Kastamonu University, Kastamonu, Turkey

³ School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), Islamabad 44000, Pakistan

⁴ Department of Computer Engineering, Biruni University, Istanbul, Turkey

⁵ Department of Mathematics, Science Faculty, Firat University, Elazig, Turkey

⁶ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

1 Introduction

A subclass of nonlinear partial differential equations (PDEs) that describe the development of a system with respect to time is called nonlinear evolution equations (NLEEs) (Ablowitz and Clarkson 1991). The importance of NLEEs is inevitable due to their applications in various scientific fields such as plasma physics, fluid mechanics, quantum field theory, optical sciences and oceanography (Lambert and Springael 2008; Weiss et al. 1983; Hirota 2004; Jhangeer et al. 2020; Raza et al. 2019a, b; Gao et al. 2020c; Rezazadeh et al. 2019; Korkmaz et al. 2018). These models can further be categorized into non-integrable and integrable NLEEs. The class of integrable NLEEs have gained importance over the other due to their solvability (Xu and Wazwaz 2019).

Higher dimensional NLEEs have engaged researchers all over the globe. A wide range of solutions such as quasiperiodic, rogue wave and lump wave solutions are reported for NLEEs in higher dimensions (Fokas 2016; Xu and Huang 2013; Wazwaz and El-Tantawy 2017; Chow et al. 2017; Dai et al. 2016). Integrable NLEE models having three or more space dimensions are proposed recently (Lou 1998; Fokas 2006; Xu 2008). The $(4 + 1)$ -dimensional Fokas equation (Yang and Yan 2009) is obtained from $(2 + 1)$ -dimensional Kadomtsev–Petviashvili (KP) equation by Lax Pair extension. This equation has been part of comprehensive investigations including results on point symmetry, potential symmetry, rogue waves, multi solitons and much more (Yang and Yan 2009; Lee et al. 2010; Cheng and Zhang 2017; Wazwaz 2019). We are motivated by the fact that there are a plenty of higher dimensional models such as proposed by Xu and Wazwaz (2019)

$$h_{yt} + h_{zt} + h_{st} + a_1 h_{xxy} + a_1 h_{xxz} + a_1 h_{xxs} + a_2 h_x(h_{xy} + h_{xz} + h_{xs}) + a_2 h_{xx}(h_y + h_z + h_s) = 0. \quad (1)$$

Here t represents time while x, y, z, s represent space dimensions. Furthermore, a_1 and a_2 represent non-zero constants of arbitrary nature. Equation (1) is derived from (KDV) equation in $(4 + 1)$ -dimensions. As this equation transforms to BLMP equation in lower dimensions, it is known as BLMP equation in $(4 + 1)$ -dimensions.

Recently, Kuo (2021) studied Eq. (1) resonant multi-solitons using linear superposition principle while Hoessini et al. (2020) investigated rational solutions and their evolutionary behavior for Eq. (1). In this work, we find new and interesting kinds of complexitons and multi wave solutions for this important Eq. (1) which have not been reported before. The importance of this study lies in the fact that Eq. (1) generalizes many physical models with diverse scientific applications. For appropriate parametric choices, Eq. (1) gives important $(3+1)$ and $(2 + 1)$ -dimensional models with a lot of physical applications. For example, if $u = u(x, y, t)$ and $a_1 = a_2 = -1$, Eq. (1) transforms to

$$h_{yt} - h_{xxy} - h_{xx}h_y - h_x h_{xy} = 0, \quad (2)$$

which is the well known BLMP equation (Boiti et al. 1986) and can be used to study incompressible fluids. Some relevant work on nonlinear physics can also be found in (Liu 2018a, b, c; Liu and He 2018; Liu et al. 2018; Yin et al. 2018; Liu 2019; Gao et al. 2020a, b; Du et al. 2020; Chen et al. 2020; Wang et al. 2020; Akbar et al. 2021; Khater et al. 2021; Akinyemi et al. 2021a, b, c, d).

This paper comprises of five sections. After the introductory first section, second and third are dedicated to the application of Auto-Bäcklund transformations and extended transformed rational function method for the $(4 + 1)$ -dimensional BLMP equation respectively. The fourth

section is devoted to discuss the obtained outcomes and results which is followed by concluding remarks.

2 Auto-Bäcklund transformations for the BLMP equation in (4 + 1) dimensions

In accordance with extended homogeneous balancing technique (Shang 2007; Shang et al. 2011; Kaplan and Ozer 2018a), we consider solution of Eq. (1) as

$$h(x, y, z, s, t) = f'(u)u_x + h_0(x, y, z, s, t). \tag{3}$$

Here f, u are to be found, and $h_0(x, y, z, s, t)$ is a solution of Eq. (1). Putting Eq. (3) into (1).

For further simplification, we assume

$$f = c \ln u. \tag{4}$$

Here c is a constant with arbitrary value. Then we get an equation with nonlinear terms of derivatives of f .

$$\begin{aligned} (f')^2 &= -cf'', (f'')^2 = -\frac{c}{6}f^{(4)}, \\ f'f''' &= -\frac{c}{3}f^{(4)}, f'f'' = -\frac{c}{2}f''', \\ f''f''' &= -\frac{c}{12}f^{(5)}. \end{aligned} \tag{5}$$

This equation can be rewritten in terms of $f^{(0)}f', f'', f''', f^{(4)}$ and $f^{(5)}$ by using the equalities of Eq. (5). Then comparing the coefficients of f''', f'', f', f^0 to zero yields the following differential equations:

$$f^{(5)} : -\frac{c}{6}a_2u_x^4(u_y + u_z + u_s) + a_1u_x^4(u_y + u_z + u_s) = 0. \tag{6}$$

We find

$$c = \frac{6a_1}{a_2}, \tag{7}$$

from Eq. (6). By substituting the value of c , we find the following system. The coefficient of the $f^{(4)}$ is vanished and

$$\begin{aligned} f''' : & (a_2h_{0x}u_x + u_t + a_1u_{xxx})(u_y + u_z + u_s) - 3a_1u_{xx}(u_{xy} + u_{xz} + u_{xs}) \\ & + 3a_1u_x(u_{xy} + u_{xz} + u_{xs}) + a_2u_x^2(h_{0y} + h_{0z} + h_{0s}) = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} f'' : & (u_{xt} + a_2h_{0xx}u_x + a_2h_{0x}u_{xx} + a_1u_{xxxx})(u_y + u_z + u_s) \\ & + u_x(u_{yt} + u_{zt} + u_{st}) + (u_t - 2a_1u_{xxx} + 2a_2h_{0x}u_x)(u_{xy} + u_{xz} + u_{xs}) \\ & + 3a_2u_xu_{xx}(h_{0y} + h_{0z} + h_{0s}) + 4a_1u_x(u_{xxy} + u_{xxz} + u_{xzs}) = 0, \end{aligned} \tag{9}$$

$$\begin{aligned}
 f' : & u_{xyt} + u_{xzt} + u_{st} + a_1(u_{xxxxy} + u_{xxxzx} + u_{xxxxs}) \\
 & + a_2h_{0xx}(u_{xy} + u_{xz} + u_{xs}) + a_2u_{xxx}(h_{0y} + h_{0z} + h_{0s}) \\
 & + a_2u_{xx}(h_{0xy} + h_{0xz} + h_{0xs}) + a_2h_{0x}(u_{xxy} + u_{xxz} + u_{xxs}) = 0,
 \end{aligned}
 \tag{10}$$

$$\begin{aligned}
 f^{(0)} : & h_{0yt} + h_{0zt} + h_{0st} + a_1h_{0xxxxy} + a_1h_{0xxxz} + a_1h_{0xxxs} \\
 & + a_2h_{0xx}(h_{0xy} + h_{0xz} + h_{0xs}) + a_2h_{0xx}(h_{0y} + h_{0z} + h_{0s}) = 0.
 \end{aligned}
 \tag{11}$$

Since $h_0(x, y, z, s, t)$ is a solution of Eq. (1), Eq. (11) is expected. If we rearrange Eq. (10), the following equation is obtained.

$$\frac{\partial}{\partial x} \left(u_{yt} + u_{zt} + u_{st} + a_1(u_{xxxxy} + u_{xxxz} + u_{xxxs}) + a_2h_{0xx}(u_{xy} + u_{xz} + u_{xs}) + a_2u_{xxx}(h_{0y} + h_{0z} + h_{0s}) \right) = 0.
 \tag{12}$$

If we differentiate and then regulate Eq. (8), the following equation is found.

$$\begin{aligned}
 & (a_2h_{0xx}u_x + a_2h_{0x}u_{xx} + u_{xt} + a_1u_{xxxx})(u_y + u_z + u_s) \\
 & + (a_2h_{0x}u_x + u_t - 2a_1u_{xxx})(u_{xy} + u_{xz} + u_{xs}) + 3a_1u_x(u_{xxy} + u_{xxz} + u_{xxs}) \\
 & + 2a_2u_xu_{xx}(h_{0y} + h_{0z} + h_{0s}) + a_2u_x^2(h_{0xy} + h_{0xz} + h_{0xs}) = 0.
 \end{aligned}
 \tag{13}$$

Then, by subtracting Eq. (9) from Eq. (13) and dividing the obtained relation to u_x , the following equation is obtained.

$$\begin{aligned}
 & u_{yt} + u_{zt} + u_{st} + a_1(u_{xxxxy} + u_{xxxz} + u_{xxxs}) + a_2h_{0x}(u_{xy} + u_{xz} + u_{xs}) \\
 & + a_2u_{xx}(h_{0y} + h_{0z} + h_{0s}) - a_2(h_{0xy} + h_{0xz} + h_{0xs}) = 0.
 \end{aligned}
 \tag{14}$$

Then we observe that the preceding conditions are satisfied if the subsequent equations hold. By using Eqs. (12)–(14) we found the following equations

$$h_{0xy} + h_{0xz} + h_{0xs} = 0,
 \tag{15}$$

$$\begin{aligned}
 & u_{yt} + u_{zt} + u_{st} + a_1(u_{xxxxy} + u_{xxxz} + u_{xxxs}) \\
 & + a_2h_{0x}(u_{xy} + u_{xz} + u_{xs}) + a_2u_{xx}(h_{0y} + h_{0z} + h_{0s}) = c_1,
 \end{aligned}
 \tag{16}$$

where c_1 is the integration constant. We find an Auto-Bäcklund transformation for the BLMP equation in (4 + 1) dimensions as

$$h(x, y, z, s, t) = \frac{6a_1}{a_2} \frac{u_x}{u} + h_0,
 \tag{17}$$

where $u(x, y, z, s, t)$ and $h_0(x, y, z, s, t)$ satisfy Eqs. (8), (15) and (16).

A number of novel wave profiles to Eq. (1) from Eq. (17) can be retrieved from Eqs. (8), (15) and (16) for any solution h_0 of Eq. (1).

(i) Setting $h_0 = c$ in Eq. (17),

$$h(x, y, z, s, t) = \frac{6a_1}{a_2} \frac{u_x}{u} + c,
 \tag{18}$$

is found, while $u(x, y, z, s, t)$ satisfies the following equations

$$\begin{aligned} & (u_t + a_1 u_{xxx})(u_y + u_z + u_s) - 3a_1 u_{xx}(u_{xy} + u_{xz} + u_{xs}) \\ & + 3a_1 u_x(u_{xy} + u_{xz} + u_{xs}) = 0, \end{aligned} \quad (19)$$

$$u_{yt} + u_{zt} + u_{st} + a_1(u_{xxy} + u_{xxz} + u_{xss}) = 0. \quad (20)$$

Here the integration constant is set as zero.

(ii) Setting $h_0 = 0$ in Eq. (17), we retrieve Cole–Hopf transformation

$$h(x, y, z, s, t) = \frac{6a_1 u_x}{a_2 u}, \quad (21)$$

while $u(x, y, z, s, t)$ satisfies Eqs. (19) and (20).

2.1 Soliton solutions of (4 + 1)-dimensional BLMP equation

In the current section, the Auto-Bäcklund transformations found in the previous section and Maple packet programme to construct various soliton solutions to Eq. (1) is used.

To find such wave profiles, we consider the Auto-Bäcklund transformation given in (17) and the constant solution h_0 .

(i) We consider $u(x, y, z, s, t)$ in the subsequent form due to homogeneity of partial derivatives in Eqs. (19) and (20),

$$\begin{aligned} u(x, y, z, s, t) = & a \cosh(kx + ly + mz + ns + rt + \xi_0) \\ & + b \sinh(kx + ly + mz + ns + rt + \xi_0) + p, \end{aligned} \quad (22)$$

while l, b, a, k, m, n, p and r are to be found and ξ_0 is a constant of integration. Putting Eq. (22) into Eqs. (19) and (20) gives an algebraic system of equations whose solution is categorized in a couple of cases:

Case 1

$$a = b, r = -a_1 k^3. \quad (23)$$

Here a_1 and b, k are arbitrary constants with non-zero values. So, we get

$$\begin{aligned} u_1(x, y, z, s, t) = & b \cosh\left(kx + ly + mz + ns - a_1 \frac{k^3}{2} t + \xi_0\right) \\ & + b \sinh\left(kx + ly + mz + ns - a_1 \frac{k^3}{2} t + \xi_0\right) + p. \end{aligned} \quad (24)$$

By putting Eq. (24) into Eq. (17), we retrieve soliton solution of (4 + 1)-dimensional BLMP Eq. (1) as given below

$$\begin{aligned} h_1(x, y, z, s, t) = & \frac{6a_1}{a_2} \\ & \frac{bk \sinh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) + bk \cosh(kx + ly + mz + ns - a_1 k^3 t + \xi_0)}{b \cosh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) + b \sinh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) + p} + C_1. \end{aligned} \quad (25)$$

Case 2

$$a = -b, r = -a_1 k^3. \tag{26}$$

Here a_1 and b, k are arbitrary constants with non-zero values. Then, we get

$$u_2(x, y, z, s, t) = -b \cosh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) + b \sinh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) + p. \tag{27}$$

By putting Eq. (27) into Eq. (17), we retrieve soliton solution for the (4 + 1)-dimensional BLMP equation Eq. (1) as follows

$$h_2(x, y, z, s, t) = \frac{6a_1}{a_2} \frac{bk \cosh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) - bk \sinh(kx + ly + mz + ns - a_1 k^3 t + \xi_0)}{b \sinh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) - b \cosh(kx + ly + mz + ns - a_1 k^3 t + \xi_0) + p} + C_1. \tag{28}$$

Case 3

$$l = -m - n. \tag{29}$$

Here m and n are arbitrary constants with non-zero values. Then, we get

$$u_3(x, y, z, s, t) = a \cosh(kx - (m + n)y + mz + ns + rt + \xi_0) + b \sinh(kx - (m + n)y + mz + ns + rt + \xi_0) + p \tag{30}$$

By putting Eq. (30) into Eq. (17), we find solitary wave solution for the BLMP equation in (4 + 1) dimensions given in Eq. (1) as follows

$$h_3(x, y, z, s, t) = \frac{6a_1}{a_2} \frac{ak \sinh(kx - (m + n)y + mz + ns + rt + \xi_0) + bk \cosh(kx - (m + n)y + mz + ns + rt + \xi_0)}{a \cosh(kx - (m + n)y + mz + ns + rt + \xi_0) + b \sinh(kx - (m + n)y + mz + ns + rt + \xi_0) + p} + C_1. \tag{31}$$

Case 4

$$a = -\frac{b((e^{kx-(m+n)y+mz+ns-4a_1k^3t+\xi_0})^2 - 1)}{(e^{kx-(m+n)y+mz+ns-4a_1k^3t+\xi_0})^2 + 1}, r = -4a_1 k^3. \tag{32}$$

Here b, k, m, p and n are constants with non-zero values of arbitrary nature. Thus, we retrieve

$$u_4(x, y, z, s, t) = -\frac{b((e^{kx-(m+n)y+mz+ns-4a_1k^3t+\xi_0})^2 - 1)}{(e^{kx-(m+n)y+mz+ns-4a_1k^3t+\xi_0})^2 + 1} \cosh\left(kx - (m + n)y + mz + ns - a_1 \frac{k^3}{2} t + \xi_0\right) + b \sinh\left(kx - (m + n)y + mz + ns - a_1 \frac{k^3}{2} t + \xi_0\right) + p. \tag{33}$$

By putting Eq. (33) into Eq. (17), we find soliton solution for the (4 + 1)-dimensional BLMP equation Eq. (1).

(ii) We next consider $u(x, y, z, s, t)$ be of the form

$$u(x, y, z, s, t) = a \cos(kx + ly + mz + ns + rt + \xi_0) + b \sin(kx + ly + mz + ns + rt + \xi_0) + p, \tag{34}$$

where $m, n, n, r, a, b, k, l, p$ and ξ_0 are the unknown constants while ξ_0 is a constant of integration having arbitrary value. Putting Eq. (22) into Eqs. (19) and (20) give a system of equations whose solutions gives:

Case 5

$$l = -m - n \tag{35}$$

Here m and n are constants with arbitrary value. Then, we obtain

$$u_5(x, y, z, s, t) = a \cos(kx - (m + n)y + mz + ns + rt + \xi_0) + b \sin(kx - (m + n)y + mz + ns + rt + \xi_0) + p. \tag{36}$$

By putting Eq. (36) into Eq. (17), we retrieve periodic wave for Eq. (1) as follows

$$h_5(x, y, z, s, t) = \frac{6a_1}{a_2} \frac{bk \cos(kx - (m + n)y + mz + ns + rt + \xi_0) - ak \sin(kx - (m + n)y + mz + ns + rt + \xi_0)}{a \cos(kx - (m + n)y + mz + ns + rt + \xi_0) + b \sin(kx - (m + n)y + mz + ns + rt + \xi_0) + p} + C_1. \tag{37}$$

Here C_1 is integration constant.

It is observed that for $c = 0$ in retrieved solutions, the exact solutions of nonlinear evolution equation in (2 + 1)-dimensions can easily be verified Cole–Hopf transformation Eq. (21).

3 Extended transformed rational function method

The extended transformed rational function method have been proposed by Zhang and Ma in 2014 (Zhang and Ma 2014) to find complexiton solutions containing hyperbolic and trigonometric functions. This method is the improved form of the transformed rational function method which is constructed by Ma and Lee in 2009 (Ma and Lee 2009). Both these methods have been applied to various equations (Yasar et al. 2019; Kaplan and Ozer 2018b; Unsal 2018).

3.1 Description of the extended transformed rational function method

The extended transformed rational function method is described as follows:

Suppose that a NPDE in Hirota bilinear form, considering variables x and t as independent and f as dependent variable:

$$P(D_x, D_t, \dots)f \cdot f = 0, \tag{38}$$

where D_x, D_t, \dots , are Hirota’s differential operators defined by

$$D_y^p f(y) \cdot g(y) = (\partial_y - \partial_{y'})^p f(y)g(y') \Big|_{y'=y} = \partial_{y'}^p f(y + y')g(y - y') \Big|_{y'=0}, \quad p \geq 1. \tag{39}$$

Then, we seek the solution of Eq. (38) the following form:

$$f = \frac{p(\rho_1, \rho_2)}{q(\rho_1, \rho_2)} \tag{40}$$

where $p(\rho_1, \rho_2)$ and $q(\rho_1, \rho_2)$ are polynomials and ρ_1 and ρ_2 admit the following differential equations ,

$$\rho_1'' = \frac{d^2 \rho_1}{d\xi_1^2} = -\rho_1, \tag{41}$$

$$\rho_2'' = \frac{d^2 \rho_2}{d\xi_2^2} = \rho_2, \tag{42}$$

where $\zeta_1 = k_1x + \omega_1t + c_1$ $\zeta_2 = k_2x + \omega_2t + c_2$, where ω_1, ω_2, k_1 and k_2 are unknowns to be determined and c_1 and c_2 are constants having arbitrary value.

Here one can easily find the solutions of Eqs. (41) and (42) as follows:

$$\begin{aligned} \rho_1 &= \pm \sin \zeta_1 \text{ or } \rho_1 = \pm \cos \zeta_1, \\ \rho_2 &= \pm \sinh \zeta_2 \text{ or } \rho_2 = \pm \cosh \zeta_2, \end{aligned} \tag{43}$$

respectively. The following equalities can be obtained from Eq. (43)

$$\rho_1^2 = 1 - \rho_1'^2 \text{ and } \rho_2^2 = 1 + \rho_2'^2. \tag{44}$$

Then, by choosing suitable $p(\rho_1, \rho_2)$ and $q(\rho_1, \rho_2)$, Eq. (38) can be converted into an algebraic equation contains k_i and ω_i . By using symbolic computation programs like Maple or Matlab, analytic complexiton solutions to Eq. (38) can be verified.

3.2 Application

In this section, implementation of the extended transformed rational function method to (4 + 1)-dimensional BLMP equation is given.

To find complexiton solutions, we consider

$$p(\rho_1, \rho_2) = A\rho_1 + B\rho_2, \quad q(\rho_1, \rho_2) = 1 \tag{45}$$

and

$$\begin{aligned} \zeta_1 &= k_1x + bk_1y + ck_1z + dk_1s + \omega_1t + c_1, \\ \zeta_2 &= k_2x + bk_2y + ck_2z + dk_2s + \omega_2t + c_2, \end{aligned} \tag{46}$$

where $A, B, k_i,$ and ω_i are to be determined.

3.3 (4 + 1)-Dimensional BLMP equation

Considering the Hirota bilinear form of Eq. (1) and along with

$$h = \frac{6a_1 f_x}{a_2 f}, \tag{47}$$

the following equation is obtained

$$\begin{aligned} &2[ff_{yt} - f_y f_t + ff_{zt} - f_z f_t + ff_{st} - f_s f_t \\ &+ a_1(ff_{xxxy} - f_{xx} f_y - 3f_x f_{xxy} + 3f_{xx} f_{xy}) \\ &+ a_1(ff_{xxxz} - f_{xxx} f_z - 3f_x f_{xxz} + 3f_{xx} f_{xz}) \\ &+ a_1(ff_{xxxs} - f_{xxx} f_s - 3f_x f_{xxs} + 3f_{xx} f_{xs})] = 0. \end{aligned} \tag{48}$$

Substituting Eq. (45) into Eq. (48) along with Eq. (44), a polynomial of $\rho_1, \rho_2, \rho'_1, \rho'_2, \rho_1^2$ and ρ_2^2 is obtained and then comparing coefficients of this polynomial, a system of algebraic equations is retrieved as:

$$\begin{aligned} &a_1 Ak_1^4 cB + a_1 Bk_2^4 bA - 6a_1 Ak_1^2 Bbk_2^2 - Ack_1 \omega_1 B + a_1 Bdk_2^4 A \\ &- Adk_1 \omega_1 B + a_1 Bk_2^4 cA + a_1 Ak_1^4 bB - 6a_1 Ak_1^2 Bck_2^2 + ABbk_2 \omega_2 \\ &- BABk_1 \omega_1 + a_1 Adk_1^4 B + Bck_2 \omega_2 A - 6a_1 Ak_1^2 Bdk_2^2 + Bdk_2 \omega_2 A = 0, \\ &- Bdk_2 A \omega_1 - Ack_1 B \omega_2 - Abk_1 B \omega_2 - 4a_1 Bk_2^3 Ack_1 \\ &+ 4a_1 Ak_1^3 Bbk_2 - Bck_2 A \omega_1 - 4a_1 Bk_2^3 Abk_1 - Adk_1 B \omega_2 \\ &- Bbk_2 A \omega_1 + 4a_1 Ak_1^3 Bdk_2 - 4a_1 Bk_2^3 Adk_1 + 4a_1 Ak_1^3 Bck_2 = 0, \\ &- 4a_1 B^2 k_2^4 b - 4a_1 B^2 k_2^4 c - 4a_1 B^2 dk_2^4 + 4a_1 A^2 dk_1^4 \\ &+ 4a_1 A^2 k_1^4 b - A^2 bk_1 \omega_1 - A^2 dk_1 \omega_1 - B^2 dk_2 \omega_2 \\ &+ 4a_1 A^2 k_1^4 c - A^2 ck_1 \omega_1 - B^2 bk_2 \omega_2 - B^2 ck_2 \omega_2 = 0. \end{aligned} \tag{49}$$

From the solutions of the system above with thanks to Maple, we get two different cases:

Case 1

$$\begin{aligned} &A = \frac{k_2}{k_1} B, \\ &\omega_1 = (k_1^2 - 3k_2^2)a_1 k_1, \omega_2 = (-k_2^2 + 3k_1^2)a_1 k_2. \end{aligned} \tag{50}$$

Therefore the solution of Eq. (48) takes the following form:

$$\begin{aligned} f_1(x, y, z, s, t) = &\frac{k_2}{k_1} B \sin(k_1 x + bk_1 y + ck_1 z + dk_1 s + (k_1^2 - 3k_2^2)a_1 k_1 t + c_1) \\ &+ B \sinh(k_2 x + bk_2 y + ck_2 z + dk_2 s + (-k_2^2 + 3k_1^2)a_1 k_2 t + c_2), \end{aligned} \tag{51}$$

and

$$\begin{aligned} f_2(x, y, z, s, t) = &\frac{k_2}{k_1} B \cos(k_1 x + bk_1 y + ck_1 z + dk_1 s + (k_1^2 - 3k_2^2)a_1 k_1 t + c_1) \\ &+ B \sinh(k_2 x + bk_2 y + ck_2 z + dk_2 s + (-k_2^2 + 3k_1^2)a_1 k_2 t + c_2). \end{aligned} \tag{52}$$

The corresponding complexiton solution of the (4 + 1)-dimensional BLMP equation can be written as

$$\begin{aligned}
 h_1(x, y, z, s, t) = & \frac{6a_1}{a_2} k_1 \cos(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + k_2 \cosh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) / \\
 & \sin(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) \\
 & + C_1,
 \end{aligned} \tag{53}$$

and

$$\begin{aligned}
 h_2(x, y, z, s, t) = & \frac{6a_1}{a_2} k_2 \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) \\
 & - k_1 \sin(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) / \\
 & \sin(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) \\
 & + C_1,
 \end{aligned} \tag{54}$$

Case 2

$$\begin{aligned}
 A = & -\frac{k_2}{k_1} B, \\
 \omega_1 = & (k_1^2 - 3k_2^2)a_1k_1, \omega_2 = a_1k_2(-k_2^2 + 3k_1^2).
 \end{aligned} \tag{55}$$

Hence the solution of Eq. (48) takes the subsequent form:

$$\begin{aligned}
 f_3(x, y, z, s, t) = & -\frac{k_2}{k_1} B \sin(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + B \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2),
 \end{aligned} \tag{56}$$

and

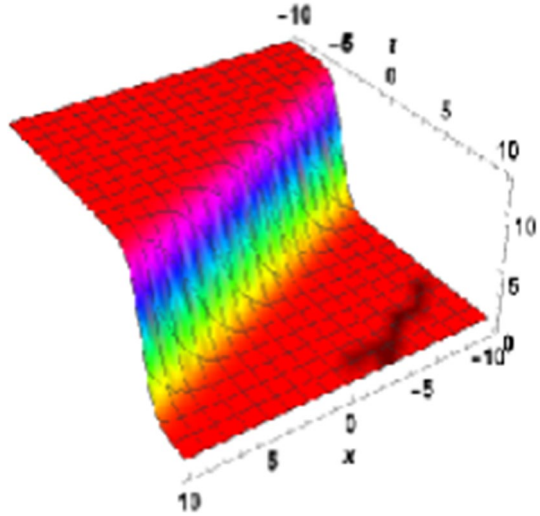
$$\begin{aligned}
 f_4(x, y, z, s, t) = & -\frac{k_2}{k_1} B \cos(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + B \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2).
 \end{aligned} \tag{57}$$

The corresponding complexiton solution of the (4 + 1)-dimensional BLMP equation can be written as

$$\begin{aligned}
 h_3(x, y, z, s, t) = & -\frac{6a_1}{a_2} k_1 \cos(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + k_2 \cosh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) / \\
 & \sin(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) \\
 & + C_1,
 \end{aligned} \tag{58}$$

and

Fig. 1 The 3D plot of Eq. (25) where $b = 2, k = 1, l = 2, m = 1, n = 2, p = 1, a_1 = 2, a_2 = 1, \xi_0 = 1, C_1 = 1, y = 0, z = 0, s = 0$



$$\begin{aligned}
 h_4(x, y, z, s, t) = & -\frac{6a_1}{a_2}k_2 \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) \\
 & -k_1 \sin(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) / \\
 & \sin(k_1x + bk_1y + ck_1z + dk_1s + (k_1^2 - 3k_2^2)a_1k_1t + c_1) \\
 & + \sinh(k_2x + bk_2y + ck_2z + dk_2s + (-k_2^2 + 3k_1^2)a_1k_2t + c_2) \\
 & + C_1,
 \end{aligned}
 \tag{59}$$

4 Discussion of results and conclusions

This paper studies (4 + 1)-dimensional BLMP equation with three different kinds of integration algorithms. In second section, auto-Bäcklund transformations of the aforementioned equation are retrieved. The obtained solutions are given in five different cases. These hyperbolic solutions are given in Eqs. (24), (27), (30), (32) and (36). Furthermore, 3D graphs of these hyperbolic solutions are given in Figs. 1, 2, 3, 4 and 5 respectively for appropriate choice of parameters. In 3rd section, the extended rational function method is capitalized to investigate complexiton solutions of Eq. (1) based on Hirota bilinear forms. These complexiton solutions are reported in two cases. The wave profiles of these

Fig. 2 The 3D plot of Eq. (28) where $b = 2, k = 1, l = 2, m = 1, n = 2, p = 1, \xi_0 = 1, a_1 = 2, a_2 = 1, C_1 = 1, y = 0, z = 0, s = 0$

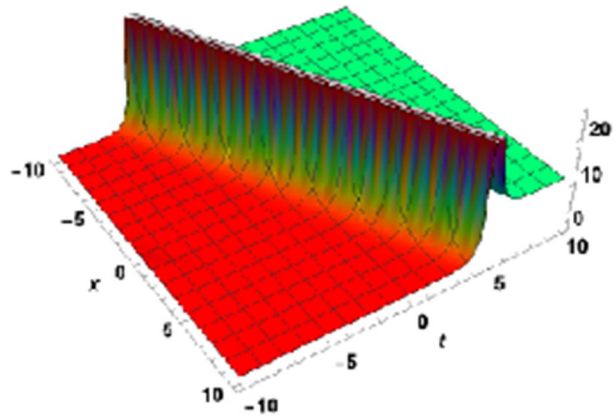
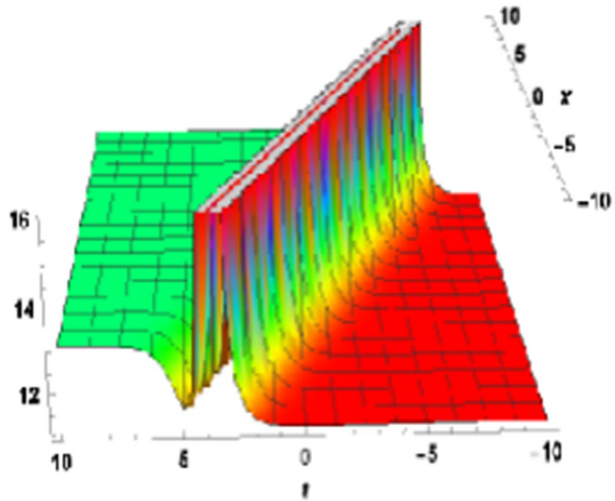


Fig. 3 The 3D plot of Eq. (31) where $k = 1, m = 1, n = 2, r = 2, p = 1, \xi_0 = 1, a_1 = 2, a_2 = 1, a = 1, b = 2, C_1 = 1, y = 0, z = 0, s = 0$



complexitons are given in Eqs. (53), (57) and (58) respectively along with there 3D graphs given in Figs. 6, 7 and 8 respectively.

Briefly highlighting our results, we have studied different kinds of solutions including exponential, trigonometric and hyperbolic solutions of (4 + 1)-dimensional BLMP equation by using two different methods. Firstly, we obtained auto-Bäcklund transformations of the considered equations which in turn give the analytic solutions of aforementioned equation. Secondly, by using extended transformed rational function method, we have obtained complexiton solutions of the considered equations. These newly reported results can be useful in different sectors of communication engineering.

Fig. 4 The 3D plot of the obtained solution where $k = 1, l = 2, m = 1, n = 2, p = 1, \xi_0 = 1, a_1 = 2, a_2 = 1, b = 2, C_1 = 1, y = 0, z = 0, s = 0$

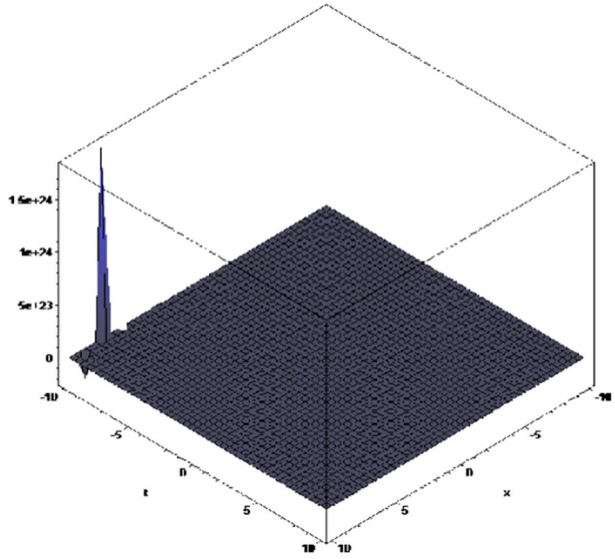


Fig. 5 The 3D plot of Eq. (37) where $k = 1, m = 1, n = 2, r = 2, p = 1, \xi_0 = 1, a_1 = 2, a_2 = 1, a = 1, b = 2, C_1 = 1, y = 0, z = 0, s = 0$

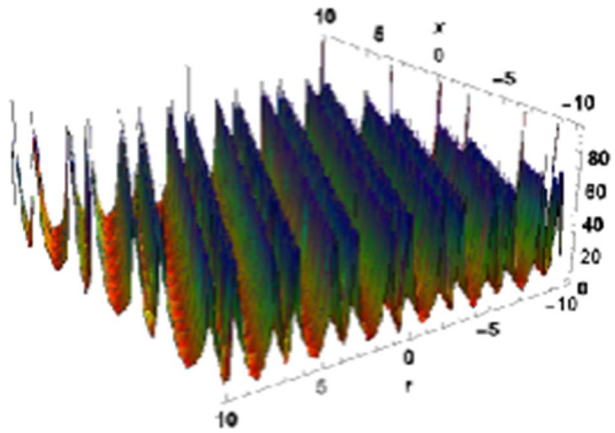


Fig. 6 The 3D plot of Eq. (54) where $k_1 = 1, k_2 = 2, B = 3, b = 2, c = 2, d = 4, c_1 = 2, c_2 = 1, a_1 = 2, a_2 = 1, C_1 = 0, y = 0, z = 0, s = 0$

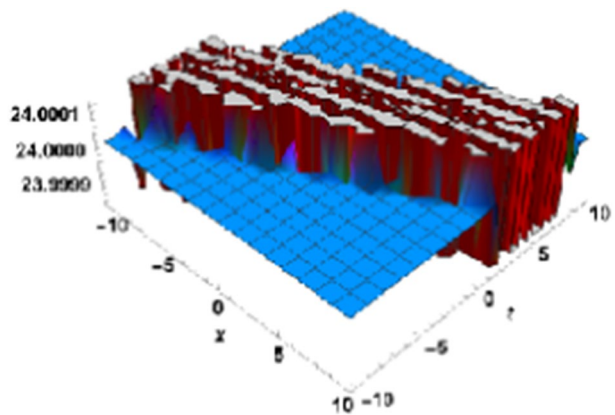
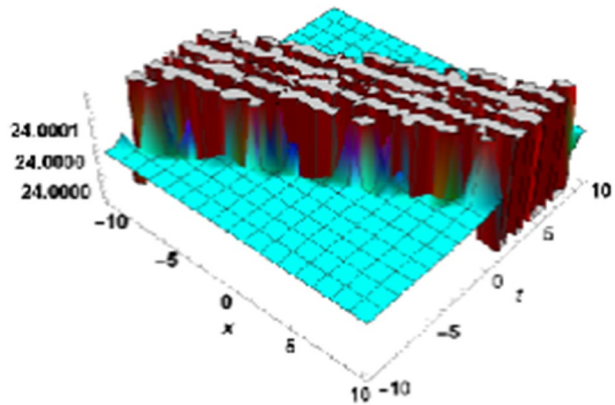


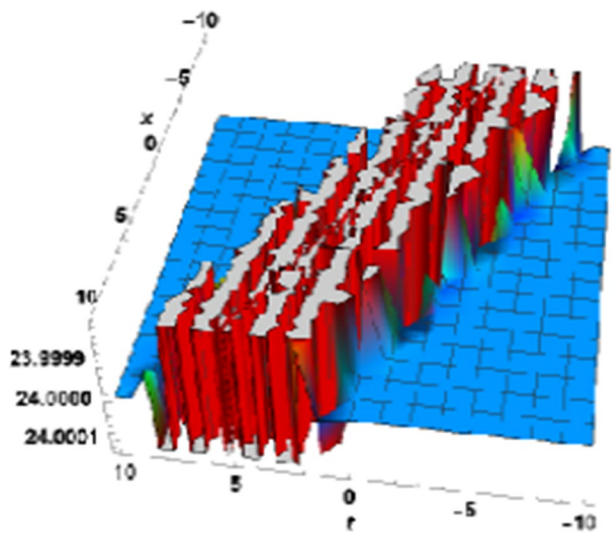
Fig. 7 The 3D plot of Eq. (58)

$k_1 = 1, k_2 = 2, B = 3, b = 2, c = 2,$
 $d = 4, c_1 = 2, c_2 = 1, a_1 = 2,$
 $a_2 = 1, C_1 = 0, y = 0, z = 0, s = 0$

**Fig. 8** The 3D plot

of Eq. (59) where

$k_1 = 1, k_2 = 2, B = 3, b = 2,$
 $c = 2, d = 4, c_1 = 2, c_2 = 1, a_1 = 2,$
 $a_2 = 1, C_1 = 0, y = 0, z = 0, s = 0$



References

- Ablowitz, M.J., Clarkson, P.A.: *Solitons. Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, Cambridge (1991)
- Akbar, M.A., Akinyemi, L., Yao, S.W., Jhangeer, A., Rezazadeh, H., Khater, M.M.A., Ahmad, H., Inc, M.: Soliton solutions to the Boussinesq equation through sine-Gordon method and Kudryashov method. *Res. Phys.* **25**, 104228 (2021)
- Akinyemi, L., Rezazadeh, H., Yao, S.W., Akbar, M.A., Khater, M.M.A., Jhangeer, A., Inc, M., Ahmed, H.: Nonlinear dispersion in parabolic law medium and its optical solitons. *Res. Phys.* **26**, 104411 (2021a)
- Akinyemi, L., Rezazadeh, H., Shi, Q.S., Inc, M., Khater, M.M.A., Ahmed, H., Jhangeer, A., Akbar, M.A.: New optical solitons of perturbed nonlinear Schrödinger–Hirota equation with spatio-temporal dispersion. *Res. Phys.* **29**, 104656 (2021b)
- Akinyemi, L., Hoessini, K., Salahshour, S.: The bright and singular solitons of (2 + 1)-dimensional nonlinear Schrödinger equation with spatio-temporal dispersions. *Optik* **242**, 167120 (2021c)

- Akinyemi, L., Senol, M., Akpan, U., Oluwasegun, K.: The optical soliton solutions of generalized coupled nonlinear Schrödinger–Korteweg–de Vries equations. *Opt. Quant. Elect.* **53**, 394 (2021d)
- Boiti, M., Leon, J.P., Manna, M., Pempinelli, F.: On the spectral transform of a Korteweg–de Vries equation in two spatial dimension. *Inverse Probl.* **2**, 271–279 (1986)
- Chen, S.S., Tian, B., Chai, J., Wu, X.Y., Du, Z.: Lax pair, binary Darboux transformations and dark-soliton interaction of a fifth-order defocusing nonlinear Schrödinger equation for the attosecond pulses in the optical fiber communication. *Wave. Random Complex Med.* **30**, 389–402 (2020)
- Cheng, L., Zhang, Y.: Lump-type solutions for the $(4 + 1)$ -dimensional Fokas equation via symbolic computations. *Mod. Phys. Lett. B* **31**, 1750224 (2017)
- Chow, K.W., Fan, E.G., Yuen, M.W.: The analytical solutions for the N-dimensional damped compressible Euler equation. *Stud. Appl. Math.* **138**, 294–316 (2017)
- Dai, C.Q., Wang, Y.Y., Liu, J.: Spatiotemporal Hermite–Gaussian solitons of a $(3 + 1)$ -dimensional partially nonlocal nonlinear Schrödinger equation. *Nonlinear Dynam.* **84**, 1157 (2016)
- Du, X.X., Tian, B., Qu, Q.X., Yuan, Y.Q., Zhao, X.H.: Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov–Kuznetsov equation in an electron-positron-ion magnetoplasma. *Chaos Solitons Fract.* **134**, 109709 (2020)
- Fokas, A.S.: Integrable nonlinear evolution partial differential equations in $4 + 2$ and $3 + 1$ dimensions. *Phys. Rev. Lett.* **96**, 190201 (2006)
- Fokas, A.S.: Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation. *Nonlinearity* **29**, 319 (2016)
- Gao, X.Y., Guo, Y.J., Shan, W.R.: Shallow water in an open sea or a wide channel: Auto- and non-auto-Bäcklund transformations with solitons for a generalized $(2 + 1)$ -dimensional dispersive long-wave system. *Chaos Solitons Fract.* **138**, 109950 (2020a)
- Gao, X.Y., Guo, Y.J., Shan, W.R.: Water-wave symbolic computation for the Earth, Enceladus and Titan: The higher-order Boussinesq–Burgers system, auto- and non-auto-Bäcklund transformations. *Appl. Math. Lett.* **104**, 106170 (2020b)
- Gao, W., Rezazadeh, H., Pinar, Z., Baskonus, H.M., Sarwar, S., Yel, G.: Novel explicit solutions for the nonlinear Zoomeron equation by using newly extended direct algebraic technique. *Opt. Quant. Electron.* **52**(1), 1–13 (2020c)
- Hirota, R.: *The Direct Method in Soliton Theory*. Cambridge University Press, Cambridge (2004)
- Hoessini, K., Ma, W.X., Ansari, R., Mirzazadeh, M., Pouyanmehr, R.R., Samadani, F.: Evolutionary behavior of rational wave solutions to the $(4 + 1)$ -dimensional Boiti–Leon–Manna–Pempinelli equation. *Phys. Scr.* **95**, 065208 (2020)
- Jhangeer, A., Raza, N., Rezazadeh, H., Seadawy, A.: Nonlinear self-adjointness, conserved quantities, bifurcation analysis and travelling wave solutions of a family of long-wave unstable lubrication model. *Pramana* **94**(1), 1–9 (2020)
- Kaplan, M., Ozer, M.N.: Auto-Bäcklund transformations and solitary wave solutions for the nonlinear evolution equation. *Opt. Quant. Electron.* **50**, 33 (2018a)
- Kaplan, M., Ozer, M.N.: Multiple-soliton solutions and analytical solutions to a nonlinear evolution equation. *Opt. Quant. Electron.* **50**, 2 (2018b)
- Khater, M.M.A., Akbar, M.A., Akinyemi, L., Jhangeer, A., Rezazadeh, H., Inc, M.: Bifurcation of new optical solitary wave solutions for the nonlinear long-short wave interaction system via two improved models of (G'/G) expansion method. *Opt. Quant. Elect.* **53**, 507 (2021)
- Korkmaz, A., Hepson, O.E., Hosseini, K., Rezazadeh, H., Eslami, M.: Sine-Gordon expansion method for exact solutions to conformable time fractional equations in RLW-class. *J. King Saud Univ. Sci.* **32**(1), 567–574 (2018)
- Kuo, C.K.: Novel resonant multi-soliton solutions and inelastic interactions to the $(3 + 1)$ - and $(4 + 1)$ -dimensional Boiti–Leon–Manna–Pempinelli equations via the simplified linear superposition principle. *Eur. Phys. J. Plus* **77**, 136 (2021)
- Lambert, F., Springael, J.: Soliton equations and simple combinatorics. *Acta Appl. Math.* **102**, 147–178 (2008)
- Lee, J., Sakthivel, R., Wazzan, L.: Exact traveling wave solutions of $(4 + 1)$ -dimensional nonlinear Fokas equation. *Mod. Phys. Lett. B* **24**, 1011–1021 (2010)
- Liu, J.G.: Double-periodic soliton solutions for the $(3 + 1)$ -dimensional Boiti–Leon–Manna–Pempinelli equation in incompressible fluid. *Appl. Math. Comput.* **75**(10), 3604–3613 (2018a)
- Liu, J.G.: Interaction behaviors for the $(2 + 12 + 1)$ -dimensional Sawada–Kotera equation. *Nonlinear Dynam.* **93**(2), 741–747 (2018b)
- Liu, J.G.: Lump-type solutions and interaction solutions for the $(2 + 1)$ -dimensional generalized fifth-order KdV equation. *Appl. Math. Lett.* **86**, 36–41 (2018c)
- Liu, J.G.: Collisions between lump and soliton solutions. *Appl. Math. Lett.* **92**, 184–189 (2019)

- Liu, J.G., He, Y.: Abundant lump and lump-kink solutions for the new $(3 + 1)$ -dimensional generalized Kadomtsev–Petviashvili equation. *Nonlinear Dynam.* **92**, 1103–1108 (2018)
- Liu, J.G., Zhou, L., He, Y.: Multiple soliton solutions for the new $(2 + 1)$ -dimensional Korteweg–de Vries equation by multiple exp-function method. *Appl. Math. Lett.* **80**, 71–78 (2018)
- Lou, S.Y.: Searching for higher dimensional integrable models from lower ones via Painleve analysis. *Phys. Rev. Lett.* **80**, 5027–5031 (1998)
- Ma, W.X., Lee, J.H.: A transformed rational function method and exact solutions to the 3+1 dimensional Jimbo–Miwa equation. *Chaos Solitons Fractals* **42**(3), 1356–1363 (2009)
- Raza, N., Jhangeer, A., Rezazadeh, H., Bekir, A.: Explicit solutions of the $(2 + 1)$ -dimensional Hirota–Macari system arising in nonlinear optics. *Int. J. Mod. Phys. B* **33**(30), 1950360 (2019a)
- Raza, N., Afzal, U., Butt, A.R., Rezazadeh, H.: Optical solitons in nematic liquid crystals with Kerr and parabolic law nonlinearities. *Opt. Quant. Electron.* **51**(4), 107 (2019b)
- Rezazadeh, H., Korkmaz, A., Eslami, M., Mirhosseini-Alizamini, S.M.: A large family of optical solutions to Kundu–Eckhaus model by a new auxiliary equation method. *Opt. Quant. Electron.* **51**(3), 84 (2019)
- Shang, Y.: Backlund transformation, Lax pairs and explicit exact solutions for the shallow water waves equation. *Appl. Math. Comput.* **187**, 1286–1297 (2007)
- Shang, Y., Huang, Y., Yuan, W.: Backlund transformations and abundant exact explicit solutions of the Sharma–Tasso–Olver equation. *Appl. Math. Comput.* **217**, 7172–7183 (2011)
- Unsal, O.: Application of extended transformed rational function method to some $(3+1)$ dimensional nonlinear evolution equations. *Karaelmas Fen ve Müh. Derg.* **8**(2), 433–437 (2018)
- Wang, M., Tian, B., Sun, Y., Zhang, Z.: Lump, mixed lump-stripe and rogue wave-stripe solutions of a $(3 + 1)$ -dimensional nonlinear wave equation for a liquid with gas bubbles. *Comput. Math. Appl.* **79**, 576–587 (2020)
- Wazwaz, A.M.: A variety of multiple-soliton solutions for the integrable $(4 + 1)$ -dimensional Fokas equation. *Waves Random Complex Med.* (2019). <https://doi.org/10.1080/17455030.2018.1560515>
- Wazwaz, A.M., El-Tantawy, S.A.: New $(3 + 1)$ -dimensional equations of Burgers type and Sharma–Tasso–Olver type: multiple-soliton solutions. *Nonlinear Dynam.* **87**, 2457–2461 (2017)
- Weiss, J., Tabor, M., Carnevale, G.: The Painleve property for partial differential equations. *J. Math. Phys.* **24**, 522–526 (1983)
- Xu, G.Q.: Searching for Painleve integrable conditions of nonlinear PDEs with constant parameters using symbolic computation. *Comput. Phys. Commun.* **178**, 505–517 (2008)
- Xu, G.Q., Huang, X.Z.: New variable separation solutions for two nonlinear evolution equations in higher dimensions. *Chin. Phys. Lett.* **30**, 030202 (2013)
- Xu, G.Q., Wazwaz, A.M.: Integrability aspects and localized wave solutions for a new $(4 + 1)$ -dimensional Boiti–Leon–Manna–Pempinelli equation. *Nonlinear Dynam.* **98**, 1379–1390 (2019)
- Yang, Z.Z., Yan, Z.Y.: Symmetry groups and exact solutions of new $(4 + 1)$ -dimensional nonlinear Fokas equation. *Commun. Theor. Phys.* **51**, 876–880 (2009)
- Yasar, E., Yildirim, Y., Adem, A.R.: Extended transformed rational function method to nonlinear evolution equations. *Int. J. Nonlinear Sci. Numer. Simul.* **20**(6), 691–701 (2019)
- Yin, Y.H., Ma, W.X., Liu, J.G., Lu, X.: Diversity of exact solutions to a $(3 + 1)$ -dimensional nonlinear evolution equation and its reduction. *Comp. Math. Appl.* **76**(6), 1375–1383 (2018)
- Zhang, H., Ma, W.X.: Extended transformed rational function method and applications to complexiton solutions. *Appl. Math. Comput.* **230**, 509–515 (2014)

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