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A novel exploration for traveling wave solutions to the integrable equation of wave packet envelope

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ABSTRACT

In this paper, with the aid of symbolic computation, different types of traveling wave solutions to a model involving an integrable equation for wave packet envelope have been presented. The $\exp(-\Phi(\xi))$-expansion and modified Kudryashov procedures have been adopted and finally 3D and 2D graphics of the obtained solutions have been plotted. The obtained results are including dark, breather, bright, and periodic soliton solutions.

1. Introduction

Exact solutions of nonlinear partial differential equations (NPDEs) describe crucial physical and mathematical phenomena. A countless number of interesting manuscripts have been published to work on these solutions.[1–15] For example, Javid and Raza have employed the modified simple equation procedure to find singular and dark soliton solutions of the (1+2)-dimensional nonlinear Schrodinger's equation.[2] Ma has obtained rational solutions to the KdV equation via linear superposition principle.[3] Kumar et al. have obtained some novel exact solutions of generalized Schrödinger–Boussinesq equations.[4] Yao et al. have obtained dark, bright, combined bright–dark, and singular optical solitons of higher-order Sasa–Satsuma equation by using the improved generalized Riccati equation mapping technique.[5] Ye et al. have derived dark vector soliton solutions for complex mKdV systems.[6] Kaplan and Ozer have employed Auto-Backlund transformations to find different explicit solutions to nonlinear evolution equations.[7] Akbulut et al. have considered the Phi-4 equation to find conservation laws and traveling wave solutions.[8] Liu and Osman have founded lump wave and periodic wave solutions of the 3D variable-coefficient generalized shallow water wave equation.[9] Wang and Ma have constructed lump solutions to nonlinear PDEs involving Hirota derivative.[10] Kaplan et al. have given the sensitivity analysis of Oskolov type equations.[16] Younis et al. have investigated the integrability of Schrö dinger–Poisson dynamical system.[17] Tang et al. have solved the generalized Hirota equation via the exponential rational function procedure and Jacobi elliptic function rational expansion technique.[18] Manukure et al. have founded lump solutions to a (2+1)-dimensional extended KP equation.[19] Zada et al. have employed the auxiliary function method to find approximate-analytical solutions to partial differential equations.[11] Kaabar have used the double Laplace transform method.[12] Az-Zo'bi et al. have applied the simplest equation technique.[13] Bhanotar and Kaabar have considered triple Laplace transform decomposition procedure.[14] Abu-Shady and Kaabar have employed enhanced homotopy perturbation method.[15]

In this manuscript, we consider the following NPDE which is used for describing traveling wave solutions in optical fibers. Let $q(x,t)$ is a complex-valued function and a_1, a_2, a_3, a_4, a_5, and a_6 are parameters.

$$iq_t+a_1q_{xx}+ia_2q_{xxx}+a_3q_{xxxx}+a_4\left|q\right|^4q+a_5\left|q\right|^2q+a_6\left(\left|q\right|^2\right)_{xx}q=0. \quad (1.1)$$

In this equation if we choose $a_2 = a_3 = a_4 = a_6 = 0$, the nonlinear Schrödinger equation is obtained. If $a_2 = a_3 = a_6 = 0$ is chosen cubic–quintic nonlinear Schrödinger equation is founded.[20,21] This equation describes the boson gas with two and three-body interactions,[22] nuclear hydrodynamics with Skyrme forces,[23] and the optical pulse propagations in dielectric media of non-Kerr type.[24] Therefore, we can consider Eq. (1.1) as a generalization of the NLSE. It could be utilized for describing nonlinear processes in optics. Kudryashov has obtained the Lax pair for this equation and verified the general solution of this equation via the ultraelliptic integrals.[25]

To obtain different types of traveling wave solutions to a model involving an integrable equation for wave packet envelope, we will applied the $\exp(-\Phi(\xi))$ and modified Kudryashov procedures. The $\exp(-\Phi(\xi))$ -expansion technique has been considered by many scientist to derive exact solutions to NPDEs and nonlinear fractional differential equations. For example, Roshid et al. have derived traveling wave solutions to the simplified MCH equation.[26] Khan and Akbar applied this method to the Vakhnenko–Parkes equation.[27] Also, Kabir et al. have utilized the modified Kudryashov technique to the seventh-order Sawada–Kotera and Kuramoto–Sivashinsky equations.[28] Ege and Misirli have considered the (2+1) dimensional Nizhnik–Nokikov–Veselov system via this method.[29]

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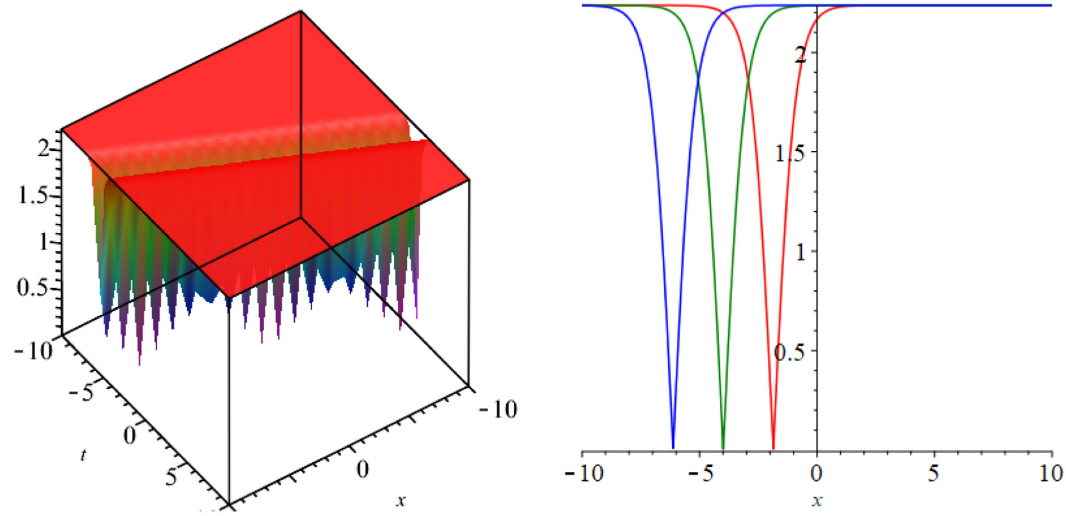
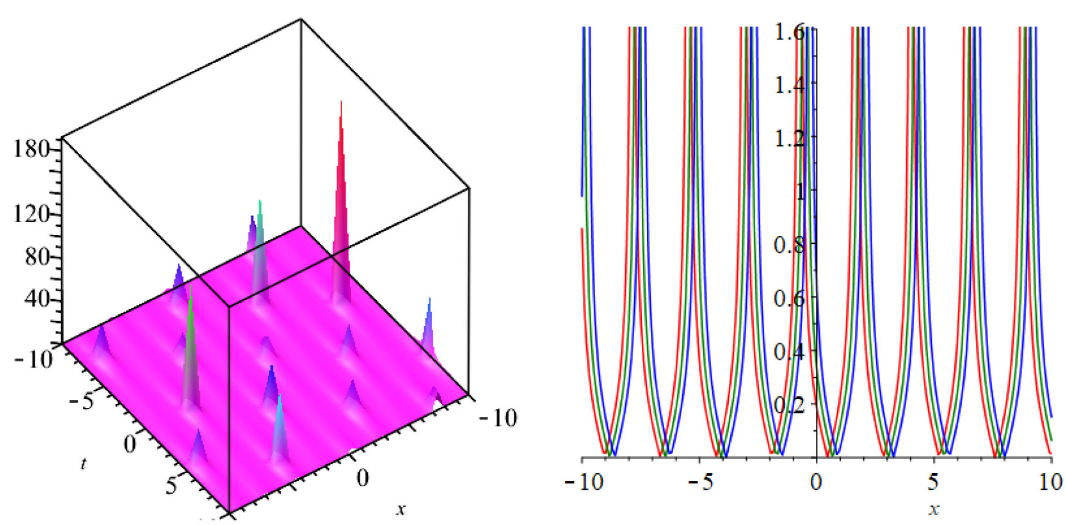


(a) (b)

Fig. 1. (a) The 3D surfaces of the dark wave soliton of $|q_{1,1}(x,t)|$ when $\alpha_1 = 2, l = 1, a_6 = 0.2, C = 1, \lambda = 3, \mu = 1$, (b) 2D plot of $|q_{1,1}(x,t)|$ when $\alpha_1 = 2, l = 1, a_6 = 0.2, C = 1, \lambda = 3, \mu = 1$, The red line is drawn when $t = 0$, the green line is drawn when $t = 1$, the blue line is drawn when $t = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) (b)

Fig. 2. (a) The 3D surfaces of the periodic wave solution $|q_{1,2}(x,t)|$ when $\alpha_1 = 0.2, l = 1, a_6 = 0.2, C = 1, \lambda = 1, \mu = 2$, (b) 2D plot of $|q_{1,2}(x,t)|$ when $\alpha_1 = 0.2, l = 1, a_6 = 0.2, C = 1, \lambda = 1, \mu = 2$, The red line is drawn when $t = 0$, the green line is drawn when $t = 5$, the blue line is drawn when $t = 10$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

This paper is organized as follows. In Section 2, we first give adopted algorithms, namely the $\exp(-\Phi(\xi))$-expansion and modified Kudryashov methods. In Section 3, we get the exact traveling wave solutions to a model involving an integrable equation for wave packet envelope and graph the obtained solutions (Figs. 1–7). Finally, we give the conclusions.

2. Adopted algorithms

Let us present the preliminaries to reduce a NPDE to an ordinary differential equation (ODE). Assume the following NPDE

$$P\left(q, q_t, q_x, q_{tt}, q_{xx}, q_{xt}, \dots\right) = 0, \tag{2.1}$$

where P is a polynomial of q and its partial derivatives, and q is a complex-valued function.

We will take a transformation as:

$$q(x,t) = u(\xi)\, e^{i(lx-\omega t)}, \ \xi = x - c_1 t. \tag{2.2}$$

Here ω, l, and c_1 are constants. By the application of Eq. (2.2) to Eq. (2.1) and equating the real and imaginary parts to zero, we obtain two equation systems. Then, we solve these systems to find conditions for parameters and use the values of the parameters. Therefore, we find an ODE as follows which will be integrated with respect to ξ possible times.

$$Q\left(u, u', u'', u''', \dots\right) = 0, \tag{2.3}$$

where the partial derivative is given with respect to ξ.[30]

Next, we will describe the modified Kudryashov and $\exp(-\Phi(\xi))$ procedures.

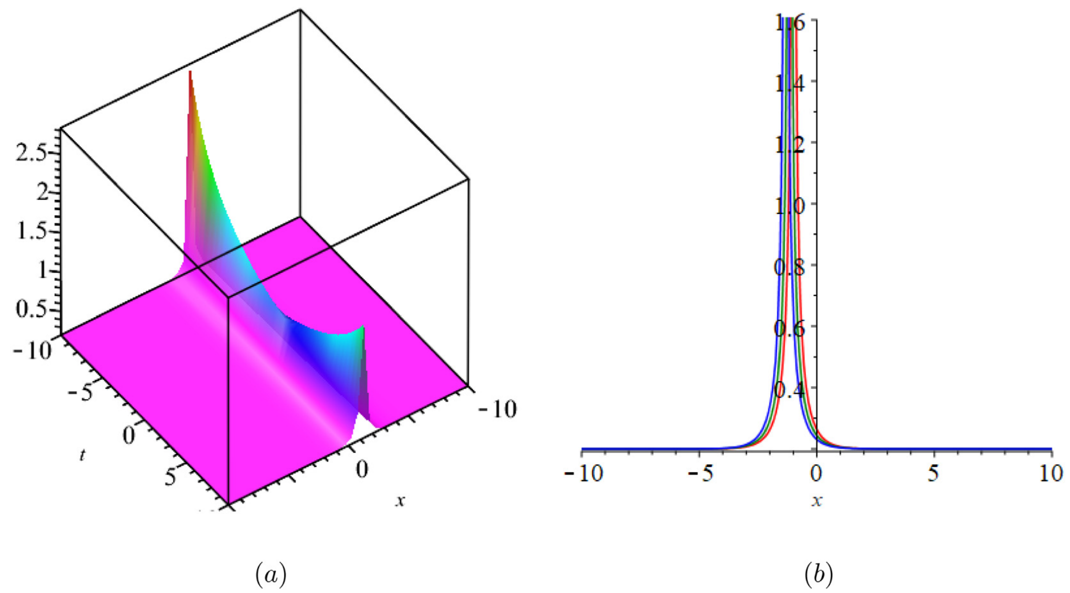


Fig. 3. (a) The 3D surfaces of the bright soliton $|q_{1,3}(x,t)|$ when $\alpha_1 = 0.2, l = 1, a_6 = 0.2, C = 1, \lambda = -2$, (b) 2D plot of $|q_{1,3}(x,t)|$ when $\alpha_1 = 0.2, l = 1, a_6 = 0.2, C = 1, \lambda = -2$, The red line is drawn when $t = 0$, the green line is drawn when $t = 10$, the blue line is drawn when $t = 20$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

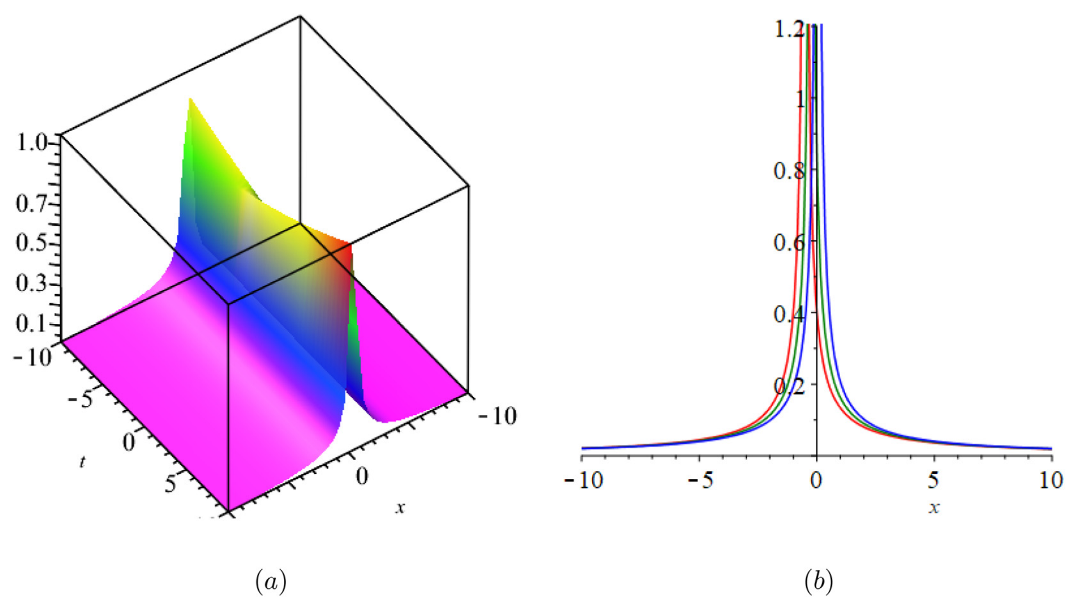


Fig. 4. (a) The 3D surfaces of the bright soliton solution $|q_{1,4}(x,t)|$ when $\alpha_1 = 0.2, l = -1, a_6 = -0.2, C = 1, \lambda = -4, \mu = 4$, (b) 2D plot $|q_{1,4}(x,t)|$ when $\alpha_1 = 0.2, l = -1, a_6 = -0.2, C = 1, \lambda = -4, \mu = 4$, The red line is drawn when $t = 0$, the green line is drawn when $t = 50$, the blue line is drawn when $t = 100$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2.1. The $\exp(-\Phi(\xi))$-expansion procedure

The $\exp(-\Phi(\xi))$-expansion procedure is presented.[26,31] The desired solution of Eq. (2.3) presented as:

$$u(\xi) = \sum_{n=0}^{m} \alpha_n (\exp(-\Phi(\xi)))^n, \tag{2.4}$$

where α_n $(\alpha_m \neq 0)$ are constants to be calculated later and $\Phi(\xi)$ is the solution of:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda. \tag{2.5}$$

From the solutions of Eq. (2.5), we can clearly obtain:

1 When $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$\Phi_1(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{2.6}$$

2 When $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$\Phi_2(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{2.7}$$

3 When $\lambda \neq 0$, $\mu = 0$, and $\lambda^2 - 4\mu > 0$,

$$\Phi_3(\xi) = -\ln\left(\frac{\lambda}{\cosh(\lambda(\xi + C)) + \sinh(\lambda(\xi + C)) - 1}\right). \tag{2.8}$$

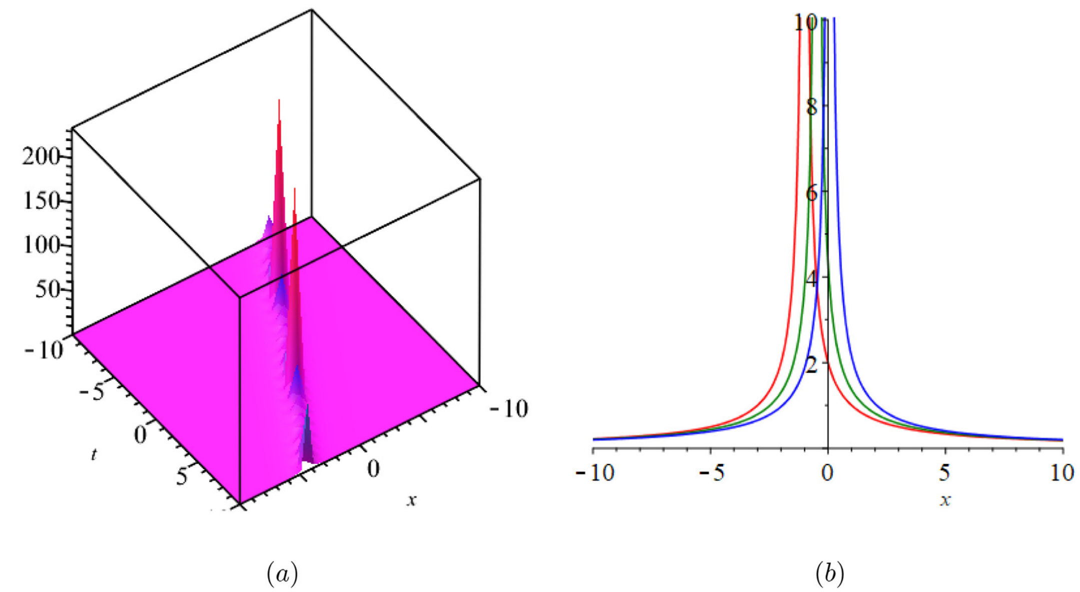


Fig. 5. (a) The 3D surfaces of the breather wave soliton $|q_{1,5}(x,t)|$ when $\alpha_1 = 2, l = 1, a_6 = 0.2, C = 1$, (b) 2D plot of $|q_{1,5}(x,t)|$ when $\alpha_1 = 2, l = 1, a_6 = 0.2, C = 1$, Red line is drawn when $t = 0$, the green line is drawn when $t = 1$, the blue line is drawn when $t = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

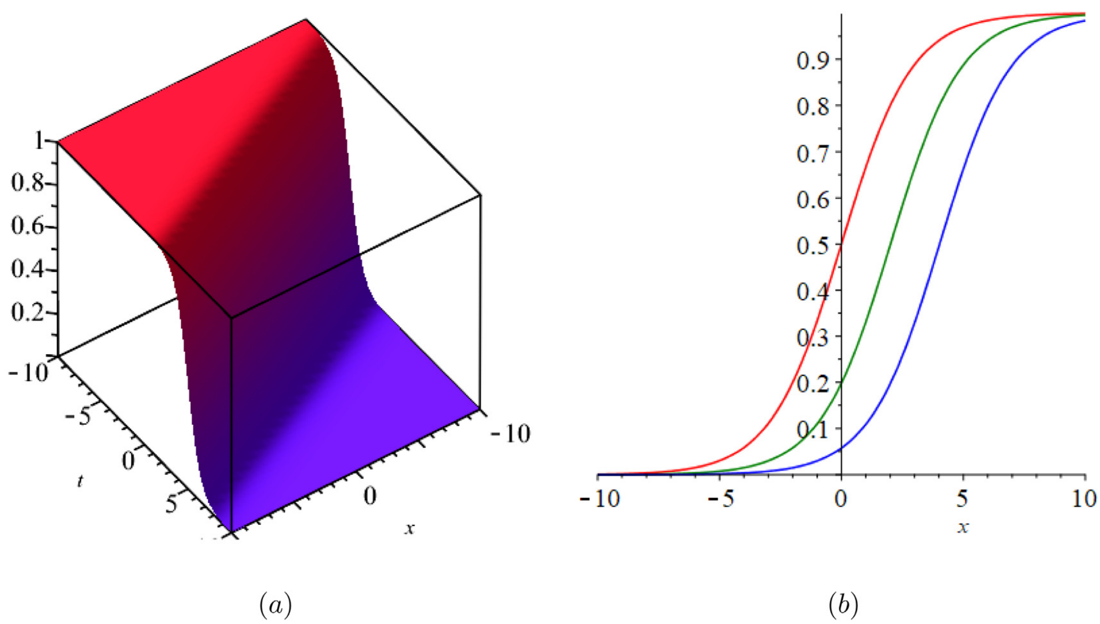


Fig. 6. (a) The 3D surfaces of the kink soliton $|q_1(x,t)|$ when $\alpha_1 = 1, l = 0.5, a_6 = 2, a = 0.5, \delta = 1$, (b) 2D plot of $|q_1(x,t)|$ when $\alpha_1 = 1, l = 0.5, a_6 = 2, a = 0.5, \delta = 1$, The red line is drawn when $t = 0$, the green line is drawn when $t = 1$, the blue line is drawn when $t = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4 When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi_4(\xi) = \ln\left(-\frac{2(\lambda(\xi + C) + 2)}{\lambda^2(\xi + C)}\right). \tag{2.9}$$

5 When $\lambda = 0$, $\mu = 0$, and $\lambda^2 - 4\mu = 0$,

$$\Phi_5(\xi) = \ln(\xi + C). \tag{2.10}$$

Here C is the constant of integration. We may also balance the highest order linear term with the highest order nonlinear term in Eq. (2.3) to find the balancing number m.

Sequentially, we substitute Eq. (2.4) into Eq. (2.3) and collect all terms with the same order of $\exp(-\Phi(\xi))^n (n = 0, 1, 2, \dots)$ together. Then, we derive a polynomial in $\exp(-\Phi(\xi))$ and vanish the coefficients to get a set of algebraic equations for $\alpha_n, l, \lambda, \mu, \omega$ and c_1. Finally, we can construct a variety of exact solutions for Eq. (2.1) from the solutions of this system.

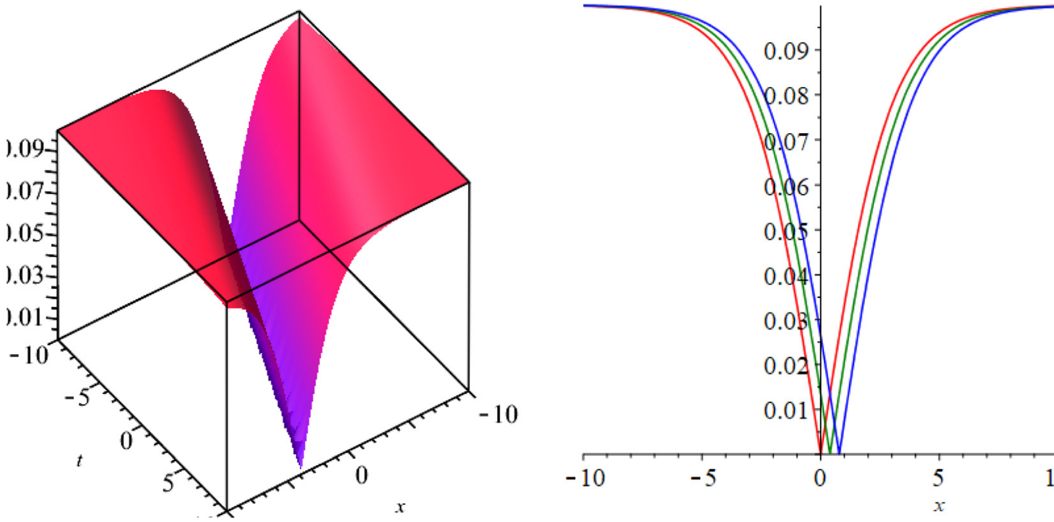
2.2. The modified Kudryashov procedure

We will assume the exact solution of Eq. (2.3) by

$$u(\xi) = \sum_{n=0}^{m} \alpha_i (\phi(\xi))^n, \tag{2.11}$$

to employ the modified Kudryashov method. Here α_n $(n = 0, 1, \dots, m)$ are constants and they will be determined later and α_m cannot be zero. m is called the balancing term. $\phi(\xi)$ is given by

$$\phi(\xi) = \frac{1}{1 + \delta a^{\xi}}, \tag{2.12}$$



(a) (b)

Fig. 7. (a) The 3D surfaces of the dark soliton $|q_2(x,t)|$ when $\alpha_1 = 0.2, l = 3, a_4 = 0.1, a_5 = 0.2, a_6 = 0.2, a = 2, \delta = 1$, (b) 2D plot of $|q_2(x,t)|$ when $\alpha_1 = 0.2, l = 3, a_4 = 0.1, a_5 = 0.2, a_6 = 0.2, a = 2, \delta = 1$, The red line is drawn when $t = 0$, the green line is drawn when $t = 1$, the blue line is drawn when $t = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and Eq. (2.12) satisfies following first order ODE

$$\phi'(\xi) = \phi(\xi)(\phi(\xi) - 1)\ln a, \tag{2.13}$$

Substituting Eq. (2.11) into Eq. (2.3) along with Eq. (2.13) then accepting all the coefficients of $\phi^n(\xi)$ is zero, we get a set of over-determined nonlinear algebraic equations for $\alpha_n (n = 0, \ldots, m), a, l, \omega, c_1$. If we solve obtained determining equation system with the help of Maple, we obtain a variety of exact solutions.[32–34]

3. Implementations

By substituting Eq. (2.2) into Eq. (1.1) and equaling the real and imaginary parts to zero yields:

$$\left(a_2 + 4a_3 l\right) u_{\xi\xi\xi} + \left(2a_1 l - c_1 - 3a_2 l^2 - 4a_3 l^3\right) u_\xi = 0. \tag{3.1}$$

$$\begin{aligned} &a_3 u_{\xi\xi\xi\xi} + 2a_6 u^2 u_{\xi\xi} + 2a_6 u_\xi^2 u + \left(a_1 - 3a_2 l - 6a_3 l^2\right) u_{\xi\xi} \\ &+ a_4 u^5 + a_5 u^3 + \left(\omega - a_1 l^2 + a_2 l^3 + a_3 l^4\right) u = 0. \end{aligned} \tag{3.2}$$

From Eq. (3.1), one may find:

$$a_2 = -4a_3 l, c_1 = 2a_1 l + 8a_3 l^3. \tag{3.3}$$

By applying Eq. (3.3) in Eq. (3.2),

$$\begin{aligned} &a_3 u_{\xi\xi\xi\xi} + 2a_6 u^2 u_{\xi\xi} + 2a_6 u_\xi^2 u + \left(a_1 + 6a_3 l^2\right) u_{\xi\xi} + a_4 u^5 + a_5 u^3 \\ &+ \left(\omega - a_1 l^2 - 3a_3 l^4\right) u = 0, \end{aligned} \tag{3.4}$$

is founded. Next, the $\exp(-\Phi(\xi))$ and modified Kudryashov procedures will be presented.

3.1. The $\exp(-\Phi(\xi))$ procedure

Noticing the homogeneous balance process between $u_{\xi\xi\xi\xi}$ and u^5 in Eq. (3.4), the balancing number is founded as 1 and the desired solution becomes

$$u(\xi) = \alpha_0 + \alpha_1 \exp(-\Phi(\xi)), \tag{3.5}$$

substituting Eq. (3.5) into Eq. (3.4) gives the following system:

$$\begin{aligned} &-a_1 l^2 \alpha_0 + 2a_6 \alpha_0^2 \alpha_1 \mu\lambda - 3a_3 l^4 \alpha_0 + a_5 \alpha_0^3 + 6a_3 l^2 \alpha_1 \mu\lambda + a_4 \alpha_0^5 \\ &+ \omega\alpha_0 + 2a_6 \alpha_1^2 \alpha_0 \mu^2 + a_3 \alpha_1 \mu\lambda^3 + a_1 \alpha_1 \mu\lambda + 8a_3 \alpha_1 \mu^2 \lambda = 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &5a_4 \alpha_0^4 \alpha_1 + 2a_6 \alpha_1^3 \mu^2 + a_1 \alpha_1 \lambda^2 + 22a_3 \alpha_1 \mu\lambda^2 + 2a_1 \alpha_1 \mu + 2a_6 \alpha_0^2 \alpha_1 \lambda^2 \\ &+ 3a_5 \alpha_0^2 \alpha_1 + a_3 \alpha_1 \lambda^4 + 8a_6 \alpha_0 \alpha_1^2 \mu\lambda - 3a_3 l^4 \alpha_1 \\ &+ 16a_3 \alpha_1 \mu^2 + 12a_3 l^2 \alpha_1 \mu + 6a_3 l^2 \alpha_1 \lambda^2 + \omega\alpha_1 + 4a_6 \alpha_0^2 \alpha_1 \mu - a_1 l^2 \alpha_1 = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &60a_3 \alpha_1 \mu\lambda + 10a_4 \alpha_0^3 \alpha_1^2 + 18a_3 l^2 \alpha_1 \lambda \\ &+ 3a_1 \alpha_1 \lambda + 3a_5 \alpha_0 \alpha_1^2 + 6a_6 \alpha_0^2 \alpha_1 \lambda + 6a_6 \alpha_1^3 \mu\lambda \\ &+ 12a_6 \alpha_0 \alpha_1^2 \mu + 15a_3 \alpha_1 \lambda^3 + 6a_6 \alpha_0 \alpha_1^2 \lambda^2 = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &40a_3 \alpha_1 \mu + 50a_3 \alpha_1 \lambda^2 + 8a_6 \alpha_1^3 \mu \\ &+ 4a_6 \alpha_1^3 \lambda^2 + 4a_6 \alpha_0^2 \alpha_1 + 10a_4 \alpha_0^2 \alpha_1^3 \\ &+ 12a_3 l^2 \alpha_1 + a_5 \alpha_1^3 + 2a_1 \alpha_1 + 16a_6 \alpha_0 \alpha_1^2 \lambda = 0, \end{aligned} \tag{3.9}$$

$$60a_3 \alpha_1 \lambda + 10a_6 \alpha_0 \alpha_1^2 + 10a_6 \alpha_1^3 \lambda + 5a_4 \alpha_0 \alpha_1^4 = 0, \tag{3.10}$$

$$24a_3 \alpha_1 + a_4 \alpha_1^5 + 6a_6 \alpha_1^3 = 0. \tag{3.11}$$

By solving this system gets:

Solution 1:

$$\begin{aligned} &a_1 = -\frac{a_6 \alpha_1^2 \lambda^2}{3} + \frac{4a_6 \alpha_1^2 \mu}{3} + \alpha_1^2 a_6 l^2, \alpha_0 = \frac{\alpha_1 \lambda}{2}, a_3 = -\frac{a_6 \alpha_1^2}{6}, \\ &a_4 = -\frac{2a_6}{\alpha_1^2}, a_5 = -a_6(-\lambda^2 + 4\mu), \\ &\omega = \frac{\alpha_1^2 a_6 (24\mu\lambda^2 - 48\mu^2 + 32l^2\mu - 3\lambda^4 - 8l^2\lambda^2 + 12l^4)}{24}. \end{aligned} \tag{3.12}$$

In this case, we find the solution as as given Box I.

Solution 2:

$$\begin{aligned} &\alpha_0 = \frac{\alpha_1 \lambda}{2}, a_4 = -\frac{6(a_6 \alpha_1^2 + 4a_3)}{\alpha_1^4}, \\ &a_5 = -\frac{2(a_6 \alpha_1^2 \lambda^2 + 20a_3 \mu + 6a_3 l^2 + a_1 + 4a_6 \alpha_1^2 \mu - 5a_3 \lambda^2)}{\alpha_1^2}, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &\omega = -2a_1 \mu + \frac{a_1 \lambda^2}{2} - a_3 \lambda^4 + 3a_3 l^4 - 2a_6 \alpha_1^2 \mu^2 + a_1 l^2 \\ &-16a_3 \mu^2 - \frac{a_6 \alpha_1^2 \lambda^4}{8} + a_6 \alpha_1^2 \lambda^2 \mu + 8a_3 \mu\lambda^2 + 3a_3 l^2 \lambda^2 - 12a_3 l^2 \mu. \end{aligned} \tag{3.19}$$

In this case, we find the solution as as given Box II.

1.1. When $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$

$$q_{1,1}(x,t) = e^{i\left(lx - \left(\frac{\alpha_1^2 a_6 (24\mu\lambda^2 - 48\mu^2 + 32l^2\mu - 3\lambda^4 - 8l^2\lambda^2 + 12l^4)}{24}\right)t\right)} \left(\frac{\alpha_1\lambda}{2} - \frac{2\mu\alpha_1}{\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x - \left(\frac{2}{3}a_6\alpha_1^2 l^3 - \frac{2}{3}la_6\alpha_1^2\lambda^2 + \frac{8}{3}la_6\alpha_1^2\mu\right)t + C\right)\right) + \lambda}\right). \tag{3.13}$$

2. When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$,

$$q_{1,2}(x,t) = e^{i\left(lx - \left(\frac{\alpha_1^2 a_6 (24\mu\lambda^2 - 48\mu^2 + 32l^2\mu - 3\lambda^4 - 8l^2\lambda^2 + 12l^4)}{24}\right)t\right)} \left(\frac{\alpha_1\lambda}{2} + \frac{2\mu\alpha_1}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\left(x - \left(\frac{2}{3}a_6\alpha_1^2 l^3 - \frac{2}{3}la_6\alpha_1^2\lambda^2 + \frac{8}{3}la_6\alpha_1^2\mu\right)t\right) + C\right)\right) - \lambda}\right). \tag{3.14}$$

1.3. When $\lambda \neq 0$, $\mu = 0$, and $\lambda^2 - 4\mu > 0$

$$q_{1,3}(x,t) = e^{i\left(lx - \left(\frac{\alpha_1^2 a_6 (-3\lambda^4 - 8l^2\lambda^2 + 12l^4)}{24}\right)t\right)} \left(\frac{\alpha_1\lambda}{2} + \frac{\alpha_1\lambda}{\cosh(\lambda(\left(x - \left(\frac{2}{3}a_6\alpha_1^2 l^3 - \frac{2}{3}la_6\alpha_1^2\lambda^2\right)t\right) + C)) + \sinh(\lambda(\left(x - \left(\frac{2}{3}a_6\alpha_1^2 l^3 - \frac{2}{3}la_6\alpha_1^2\lambda^2\right)t\right) + C)) - 1}\right). \tag{3.15}$$

1.4. When $\lambda \neq 0$, $\mu \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$q_{1,4}(x,t) = e^{i\left(lx - \left(\frac{\alpha_1^2 a_6 (4\lambda^2 l^2 + 3l^4)}{6}\right)t\right)} \left(\frac{\alpha_1\lambda}{2} - \frac{\alpha_1(\lambda^2(\xi + C))}{2(\lambda(\xi + C) + 2)}\right). \tag{3.16}$$

1.5. When $\lambda^2 - 4\mu = 0$, $\mu = 0$ and $\lambda = 0$,

$$q_{1,5}(x,t) = e^{i\left(lx - \frac{l^4\alpha_1^2 a_6}{2}t\right)} \frac{\alpha_1}{x - \frac{2}{3}a_6\alpha_1^2 l^3 t + C}. \tag{3.17}$$

Box I.

Solution 3:

$$a_1 = \frac{\alpha_1^2(-20\mu + 5\lambda^2 + 6l^2)a_6}{6}, a_3 = -\frac{a_6\alpha_1^2}{6}, \quad a_4 = -\frac{2a_6}{\alpha_1^2}, a_5 = -\frac{8a_6(-\lambda^2 + 4\mu)}{3}, \tag{3.25}$$

$$\omega = \frac{\alpha_1^2(-4\lambda^4 - 64\mu^2 + 32\mu\lambda^2 + 5l^2\lambda^2 - 20l^2\mu + 3l^4)a_6}{6}, \tag{3.26}$$

$$\alpha_0 = \alpha_1\left(\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\right). \tag{3.27}$$

3.1. When $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$q_{3,1}(x,t) = e^{i\left(lx - \left(\frac{\alpha_1^2(-4\lambda^4 - 64\mu^2 + 32\mu\lambda^2 + 5l^2\lambda^2 - 20l^2\mu + 3l^4)a_6}{6}\right)t\right)} \left(\alpha_1\left(\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right) - \frac{2\mu\alpha_1}{\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(x - \frac{a_6\alpha_1^2}{3}\left(2l^3 + 5l\lambda^2 - 20\mu l\right)t + C)) + \lambda}\right). \tag{3.28}$$

3.2. When $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$q_{3,2}(x,t) = e^{i\left(lx - \left(\frac{\alpha_1^2(-4\lambda^4 - 64\mu^2 + 32\mu\lambda^2 + 5l^2\lambda^2 - 20l^2\mu + 3l^4)a_6}{6}\right)t\right)} \left(\alpha_1\left(\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right) + \frac{2\mu\alpha_1}{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(x - \frac{a_6\alpha_1^2}{3}\left(2l^3 + 5l\lambda^2 - 20\mu l\right)t + C)) - \lambda}\right). \tag{3.29}$$

3.3. When $\lambda \neq 0$, $\mu = 0$, and $\lambda^2 - 4\mu > 0$,

$$q_{3,3}(x,t) = e^{i\left(lx - \left(\frac{\alpha_1^2(-4\lambda^4 + 5l^2\lambda^2 + 3l^4)a_6}{6}\right)t\right)} \left(\alpha_1\left(\frac{\lambda}{2} + \frac{\sqrt{\lambda^2}}{2}\right) + \frac{\alpha_1\lambda}{\cosh(\lambda(x - \frac{a_6\alpha_1^2}{3}(2l^3 + 5l\lambda^2)t + C)) + \sinh(\lambda(x - \frac{a_6\alpha_1^2}{3}(2l^3 + 5l\lambda^2)t + C)) - 1}\right). \tag{3.30}$$

3.4. When $\lambda \neq 0$, $\mu \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$q_{3,4}(x,t) = e^{i\left(lx - \frac{\alpha_1^2 l^4 a_6}{2}t\right)} \left(\frac{\alpha_1\lambda}{2} - \frac{\alpha_1(\lambda^2(x - \frac{2a_6\alpha_1^2 l^3 t}{3} + C))}{2(\lambda(x - \frac{2a_6\alpha_1^2 l^3 t}{3} + C) + 2)}\right). \tag{3.31}$$

3.5. When $\lambda = 0$, $\mu = 0$, and $\lambda^2 - 4\mu = 0$,

$$q_{3,5}(x,t) = e^{i\left(lx - \frac{\alpha_1^2 l^4 a_6}{2}t\right)} \frac{\alpha_1}{x - \frac{2a_6\alpha_1^2 l^3 t}{3} + C}. \tag{3.32}$$

3.2. The modified Kudryashov method

According the method, we can choose exact solution of Eq. (3.4):

$$u(\xi) = \alpha_0 + \alpha_1\phi(\xi). \tag{3.33}$$

2.1. When $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$q_{2,1}(x,t) = e^{i\left(lx-\left(-2a_1\mu+\frac{a_1\lambda^2}{2}-a_3\lambda^4+3a_3l^4-2a_6\alpha_1^2\mu^2+a_1l^2-16a_3\mu^2-\frac{a_6\alpha_1^2\lambda^4}{8}+a_6\alpha_1^2\lambda^2\mu+8a_3\mu\lambda^2+3a_3l^2\lambda^2-12a_3l^2\mu\right)t\right)} \left(\frac{\alpha_1\lambda}{2} - \frac{2\mu\alpha_1}{\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-\left(2a_1l+8a_3l^3\right)t+C))+\lambda}\right). \tag{3.20}$$

2.2. When $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$q_{2,2}(x,t) = e^{i\left(lx-\left(-2a_1\mu+\frac{a_1\lambda^2}{2}-a_3\lambda^4+3a_3l^4-2a_6\alpha_1^2\mu^2+a_1l^2-16a_3\mu^2-\frac{a_6\alpha_1^2\lambda^4}{8}+a_6\alpha_1^2\lambda^2\mu+8a_3\mu\lambda^2+3a_3l^2\lambda^2-12a_3l^2\mu\right)t\right)} \left(\frac{\alpha_1\lambda}{2} + \frac{2\mu\alpha_1}{\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-(2a_1l+8a_3l^3)t+C))-\lambda}\right). \tag{3.21}$$

2.3. When $\lambda \neq 0$, $\mu = 0$, and $\lambda^2 - 4\mu > 0$,

$$q_{2,3}(x,t) = e^{i\left(lx-\left(\frac{a_1\lambda^2}{2}-a_3\lambda^4+3a_3l^4+a_1l^2-\frac{a_6\alpha_1^2\lambda^4}{8}+3a_3l^2\lambda^2\right)t\right)} \left(\frac{\alpha_1\lambda}{2} + \frac{\alpha_1\lambda}{\cosh(\lambda(x-(2a_1l+8a_3l^3)t+C))+\sinh(\lambda(x-(2a_1l+8a_3l^3)t+C))-1}\right). \tag{3.22}$$

2.4. When $\lambda \neq 0$, $\mu \neq 0$, and $\lambda^2 - 4\mu = 0$

$$q_{2,4}(x,t) = e^{i\left(lx-(3a_3l^4+a_1l^2)t\right)} \left(\frac{\alpha_1\lambda}{2} - \frac{\alpha_1(\lambda^2(\xi+C))}{2(\lambda(\xi+C)+2)}\right). \tag{3.23}$$

2.5. When $\lambda = 0$, $\mu = 0$, and $\lambda^2 - 4\mu = 0$,

$$q_{2,5}(x,t) = e^{i\left(lx-(3a_3l^4+a_1l^2)t\right)} \frac{\alpha_1}{x-(2a_1l+8a_3l^3)t+C}. \tag{3.24}$$

Box II.

Substituting Eq. (3.33) along with Eq. (2.13) into Eq. (3.4), then assuming the coefficients of $\phi(\xi)^n$ zero,

$$\begin{aligned}
&24(\ln a)^4\alpha_1a_3 + 6(\ln a)^2\alpha_1^3a_6 + \alpha_1^5a_4 = 0,\\
&-60(\ln a)^4\alpha_1a_3 + 10(\ln a)^2\alpha_0\alpha_1^2a_6 - 10(\ln a)^2\alpha_1^3a_6 + 5\alpha_0\alpha_1^4a_4 = 0,\\
&\quad 50(\ln a)^4\alpha_1a_3 + 4(\ln a)^2\alpha_0^2\alpha_1a_6 - 16(\ln a)^2\alpha_0\alpha_1^2a_6 + 4(\ln a)^2\alpha_1^3a_6\\
&\qquad +12l(\ln a)^2\alpha_1a_3 + 10\alpha_0^2\alpha_1^3a_4 + 2(\ln a)^2\alpha_1a_1 + \alpha_1^3a_5 = 0,\\
&-15(\ln a)^4\alpha_1a_3 - 6(\ln a)^2\alpha_0^2\alpha_1a_6 + 6(\ln a)^2\alpha_0\alpha_1^2a_6 - 18l^2(\ln a)^2\alpha_1a_3\\
&\qquad +10\alpha_0^3\alpha_1^2a_4 - 3(\ln a)^2\alpha_1a_1 + 3\alpha_0\alpha_1^2a_5 = 0,\\
&\quad (\ln a)^4\alpha_1a_3 + 2(\ln a)^2\alpha_0^2\alpha_1a_6 + 6l^2(\ln a)^2\alpha_1a_3 + (\ln a)^2\alpha_1a_1\\
&\qquad +5\alpha_0^4\alpha_1a_4 - 3l^4\alpha_1a_3 - l^2\alpha_1a_1 + 3\alpha_0^2\alpha_1a_5 + \alpha_1\omega = 0,\\
&\alpha_0^5a_4 - 3l^4\alpha_0a_3 - l^2\alpha_0a_1 + \alpha_0^3a_5 + \alpha_0\omega = 0,
\end{aligned} \tag{3.34}$$

is obtained. If we solve the Sys. (3.34), we obtain values of the constants.

Solution 1:

$$\begin{aligned}
&\alpha_0 = 0, \alpha_1 = \alpha_1, a_1 = \frac{\alpha_1^2a_6\left(5(\ln a)^2+6l^2\right)}{6(\ln a)^2}, a_3 = -\frac{\alpha_1^2a_6}{6(\ln a)^2},\\
&a_4 = -\frac{2(\ln a)^2a_6}{\alpha_1^2}, a_5 = \frac{8(\ln a)^2a_6}{3}, \omega = -\frac{\alpha_1^2a_6\left(4(\ln a)^4-5l^2(\ln a)^2-3l^4\right)}{6(\ln a)^2}.
\end{aligned} \tag{3.35}$$

If we use above values, we get following exact solution:

$$q_1(x,t) = \frac{\alpha_1 e^{i\left(lx+\frac{\alpha_1^2a_6\left(4(\ln a)^4-5l^2(\ln a)^2-3l^4\right)t}{6(\ln a)^2}\right)}}{1+\delta a^{x-\frac{\alpha_1^2a_6l\left(5(\ln a)^2+2l^2\right)t}{3(\ln a)^2}}}. \tag{3.36}$$

Solution 2:

$$\begin{aligned}
&\alpha_0 = -\frac{\alpha_1}{2}, \alpha_1 = \alpha_1,\\
&a_1 = \frac{\alpha_1^2\left(-a_6(\ln a)^4+\left(-\frac{5a_4\alpha_1^2}{6}+6l^2a_6-2a_5\right)(\ln a)^2+\alpha_1^2a_4l^2\right)}{4(\ln a)^4},\\
&a_3 = -\frac{\alpha_1^2\left(\alpha_1^2a_4+6(\ln a)^2a_6\right)}{24(\ln a)^4},\\
&\omega = \frac{\alpha_1^2\left(\left(-\frac{a_4\alpha_1^2}{2}-2l^2a_6-2a_5\right)(\ln a)^4+\left(6l^4a_6+l^2\left(-\frac{5a_4\alpha_1^2}{3}-4a_5\right)\right)(\ln a)^2+l^4\alpha_1^2a_4\right)}{8(\ln a)^4}.
\end{aligned} \tag{3.37}$$

According to above values, we get:

$$q_2(x,t) = \left(-\frac{\alpha_1}{2} + \frac{\alpha_1}{1+\delta a^{x-\frac{l\alpha_1^2\left(-3a_6(\ln a)^4+\left(-\frac{5a_4\alpha_1^2}{2}+6l^2a_6-6a_5\right)(\ln a)^2+l^2\alpha_1^2a_4\right)t}{6(\ln a)^4}}}\right) e^{i\left(lx-\frac{\alpha_1^2\left(\left(-\frac{a_4\alpha_1^2}{2}-2l^2a_6-2a_5\right)(\ln a)^4+\left(6l^4a_6+l^2\left(-\frac{5a_4\alpha_1^2}{3}-4a_5\right)\right)(\ln a)^2+l^4\alpha_1^2a_4\right)t}{8(\ln a)^4}\right)}. \tag{3.38}$$

Solution 3:

$$\begin{aligned}
&\alpha_0 = -\alpha_1, \alpha_1 = \alpha_1, a_1 = \frac{\alpha_1^2a_6\left(5(\ln a)^2+6l^2\right)}{6(\ln a)^2}, a_3 = -\frac{\alpha_1^2a_6}{6(\ln a)^2},\\
&a_4 = -\frac{2a_6(\ln a)^2}{\alpha_1^2},\\
&a_5 = \frac{8(\ln a)^2a_6}{3}\omega = -\frac{\alpha_1^2a_6\left(4(\ln a)^4-5l^2(\ln a)^2-3l^4\right)}{6(\ln a)^2},
\end{aligned} \tag{3.39}$$

then we obtain following solution

$$q_3(x,t) = \left(-\alpha_1 + \frac{\alpha_1}{1 + \delta a^{x - \frac{\alpha_1^2 a_6 l \left(5(\ln a)^2 + 2l^2\right) t}{3(\ln a)^2}}} \right) e^{i\left(lx + \frac{\alpha_1^2 a_6 \left(4(\ln a)^4 - 5l^2 (\ln a)^2 - 3l^4\right) t}{6(\ln a)^2} \right)} \tag{3.40}$$

4. Conclusion

Generally speaking, we found some new traveling solutions of a model involving an integrable equation for wave packet envelope via the $\exp(-\Phi(\xi))$-expansion and modified Kudryashov procedures. And we have plotted the 3D and 2D graphics of some of the obtained solutions at some certain values to analyze their dynamical behaviors. The obtained results are including dark, breather, bright, and periodic soliton solutions (Figs. 1–7). The considered techniques are productive to obtain different types of solutions. For the future works, we may apply new techniques to the considered equation and find new different type of solutions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Burger S, Bongs K, Detter S, Ertmer W, Sengstock K. Dark solitons in Bose–Einstein condensates. *Phys Rev Lett*. 1999;83:5198–5201.
2. Javid A, Raza N. Chiral solitons of the (1+2)-dimensional nonlinear Schrodinger's equation. *Mod Phys Lett B*. 2019;33(32):1950401.
3. Ma WX. Linear superposition of Wronskian rational solutions to the KdV equation. *Commun Theor Phys*. 2021;73(6):065001.
4. Kumar D, Hosseini K, Kaabar MKA, Kaplan M, Salahshour S. On some novel solution solutions to the generalized Schrödinger-Boussinesq equations for the interaction between complex short wave and real long wave envelope. *J Ocean Eng Sci*. 2021. http://dx.doi.org/10.1016/j.joes.2021.09.008.
5. Yao SW, Akinyemi L, Mirzazadeh M, Inc M, Hosseini K, Senol M. Dynamics of optical solitons in higher-order Sasa-Satsuma equation. *Results Phys*. 2021;30:104825.
6. Ye R, Zhang Y, Ma WX. Darboux transformation and dark vector soliton solutions for complex mKdV systems. *Partial Differ Equ Appl Math*. 2021;4:100161.
7. Kaplan M, Ozer MN. Auto-bäcklund transformations and solitary wave solutions for the nonlinear evolution equation. *Opt Quantum Electron*. 2018;50(1):33.
8. Akbulut A, Kaplan M, Tascan F. Conservation laws and exact solutions of phi-four (phi-4) equation via the (G'/G, 1/G)-expansion method. *Z Nat forsch A*. 2016;71(5):439–446.
9. Liu JG, Osman MS. Nonlinear dynamics for different nonautonomous wave structures solutions of a 3D variable-coefficient generalized shallow water wave equation. *Chin J Phys*. 2021. http://dx.doi.org/10.1016/j.cjph.2021.10.026.
10. Wang F, Ma WX. Lump solutions to nonlinear PDEs involving Hirota derivative $D_t^2 D_x D_y$. *Mod Phys Lett B*. 2020;34(18):2050197.
11. Zada L, Nawaz R, Nisar KS, Tahir M, et al. New approximate-analytical solutions to partial differential equations via auxiliary function method. *Partial Diff Equ Appl Math*. 2021;4:100045.
12. Kaabar M. Novel methods for solving the conformable wave equation. *J New Theory*. 2020;31:56–85.
13. Az-Zo'bi EA, Jhangeer A, Rezazadeh H, Ali MN, Kaabar MKA. New soliton solutions for the higher-dimensional non-local Ito equation. *Nonlinear Eng*. 2021;10(1):374–384.
14. Bhanotar SA, Kaabar MKA. Analytical solutions for the nonlinear partial differential equations using the conformable triple Laplace transform decomposition method. *Int J Differ Equ Appl*. 2021:2021.
15. Abu-Shady M, Kaabar MKA. A generalized definition of the fractional derivative with applications. *Math Probl Eng*. 2021:2021.
16. Kaplan M, Butt AR, Thabet H, Akbulut A, Raza N, Kumar D. An effective computational approach and sensitivity analysis to pseudo-parabolic-type equations. *Waves Random Complex Media*. 2021. http://dx.doi.org/10.1080/17455030.2021.1989081.
17. Younis M, Seadawy AR, Baber MZ, Husain S, et al. Analytical optical soliton solutions of the Schrö dinger-Poisson dynamical system. *Results Phys*. 2021;27:104369.
18. Tang Y, Liang Z, Zhou M. New exact travelling solutions of the generalized Hirota equation. *Partial Diff Equ Appl Math*. 2021;4:100112.
19. Manukure S, Zhou Y, Ma WX. Lump solutions to a (2+1)-dimensional extended KP equation. *Comput Math Appl*. 2018;75(7):2414–2419.
20. Zayed EME, Alurrfi KAE. Extended auxiliary equation method and its applications for finding the exact solutions for a class of nonlinear Schrödinger-type equations. *Appl Math Comput*. 2016;289:111–131.
21. Zayed EME, Al-Nowehy AG. Solitons and other exact solutions for a class of nonlinear Schrödinger-type equations. *Optik*. 2017;130:1295–1311.
22. Barashenkov IV, Makhankov VG. Soliton-like bubbles in the system of interacting bosons. *Phys Lett A*. 1988;128:52–56.
23. Kartavenko VG. Soliton-like solutions in nuclear hydrodynamics. *Sov J Nucl Phys*. 1984;40:240–246.
24. Kumar A, Sarkar SN, Ghatak AK. Effect of fifth-order non-linearity in refractive index on Gaussian pulse propagation in lossy optical fibers. *Opt Lett*. 1986;11:321–323.
25. Kudryashov NA. Optical solitons of model with integrable equation for wave packet envelope. *Chaos, Solitons Fractals*. 2020;141:110325.
26. Roshid HO, Alam MD, Akbar MA, Islam R. Traveling wave solutions of the simplified MCH equation via $\exp(-\Phi(\xi))$-expansion method. *Br J Math Comput Sci*. 2015;5(5):595–605.
27. Khan K, Akbar MA. The $\exp(-\Phi(\xi))$-expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation. *Int J Dyn Syst Diff Equ*. 2015;5(1):72–83.
28. Kabir MM, Khajeh A, Aghdam EA, Koma AY. Modified Kudryashov method for finding exact solitary wave solutions of higher-order nonlinear equations. *Math Methods App Sci*. 2011;34(2):213–219.
29. Ege SM, Misirli E. The modified kudryashov method for solving some evolution equations. *AIP Conf Proc*. 2012;1470:244.
30. Hereman W, Nuseir A. Symbolic methods to construct exact solutions of nonlinear partial differential equations. *Math Comput Simulation*. 1997;43(1):13–27.
31. Kaplan M. Two different systematic techniques to find analytical solutions of the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. *Chin J Phys*. 2018;56(5):2523–2530.
32. Hosseini K, Ansari R. New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method. *Waves Random Complex Media*. 2017;27(4):628–636.
33. Zayed EME, Alurrfi KAE. The modified kudryashov method for solving some seventh order nonlinear PDEs in mathematical physics. *World J Model Simul*. 2015;11(4):308–319.
34. Hosseini K, Akbulut A, Baleanu D, Salahshour S. The Sharma-Tasso-Olver-Burgers equation: Its conservation laws and kink solitons. *Commun Theor Phys*. 2021. http://dx.doi.org/10.1088/1572-9494/ac4411.