



Abundant soliton-type solutions to the new generalized KdV equation via auto-Bäcklund transformations and extended transformed rational function technique

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Abstract

This study investigates new soliton-type solutions to the new generalized KdV (ngKdV) equation. For this purpose, the homogeneous balance method is used to create Auto-Bäcklund transformations of the regarded equation and with the help of the transformations, abundant exact and explicit solutions have been found. We found complexiton solutions to the dealt equation by using the extended transformed rational function technique. We have also given the 3D graphics of the obtained solutions.

Keywords Auto-Bäcklund transformations · Soliton · Hirota direct method · Complexiton solutions · Symbolic computation

1 Introduction

Nonlinear partial differential equations (NPDEs) have a vital role in many aspects of engineering fields. Several specialists recently acquired analytical travelingwave solutions of NPDEs, which play a key role in uncovering knowledge about rheological features or phenomena. These styles are created in a variety of disciplines, including solid-state physics, fiber optics, economics, fluid dynamics, population ecology, plasma physics, neural networks and countless other scientific and technical domains (Arshed and Raza 2020). In this way, in recent years, the finding of soliton solutions to previously described wonders has been exciting and unfathomable the topic of the study, and the associated issue is the construction of soliton wave solutions for a wide range of nonlinear evolution equations (NLEEs) (Raza et al. 2021).

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There are many appealing techniques to explore soliton-type solutions of NPDEs (Ismael et al. 2021; Yagmurlu et al. 2019; Yokus et al. 2018). Solitons are self-trapped wave packets with extraordinary qualities. They are resistant to impediments and retain their original identity during propagation. Some of the techniques are the Hirota’s bilinear method (Hirota 1971; Arshed et al. 2021), Darboux transformation (Matveev and Salle 1991), the inverse scattering method (Ablowitz and Clarkson 1991), the finite difference method (Yokus and Bulut 2018), the Wronskian technique (Ma and You 2004), the similarity reduction method (Zhang and Hu 2018), the first integral technique (Javeed et al. 2019), the extended trial function technique (Ekici et al. 2019), the Exp-function technique (Ganji et al. 2009), extended sinh-Gordon equation expansion method (Chen and Yan 2005) and so on.

It is a fact that the Hirota bilinear method is the most dynamic technique for finding soliton solutions for integrable and non-integrable nonlinear equations by perturbation. The Bäcklund transformation is a powerful method for finding soliton-type solutions to NPDEs from a trivial “seed” and is related to infinite conservation laws and the inverse scattering method.

The ngKdV equation will be discussed in this work. The Hirota conditions, D’Alembert wave, and soliton molecule of this equation have been established in Ma et al. (2022). Using an extension of the homogeneous balancing technique and Maple, the Auto-Bäcklund transformations for this problem are discovered. A large number of travelling-wave and periodic wave solutions have been generated using the transformations described in this work. The extended transformed rational function approach will be used to obtain complexiton solutions, depending on the bilinear equation.

This paper is arranged as follows. The Auto-Bäcklund transformation of the ngKdv equation is found in Sect. 2. In Sect. 3, we find a large number of clear and accurate solutions to the problem at hand. The key phases of the extended transformed rational function approach are described in Sect. 4. The extended transformed rational function approach is then used for the ngKdv problem in Sect. 5. Finally, some conclusions are made.

2 Auto-Bäcklund transformation for the ngKdV equation

A novel version of ngKdV equation is taken into consideration,

$$P(x, y, t) = ax^4 + bx^3t + cx^2 + dy^2 + ext,$$

where the constants a, b, c, d, e are arbitrary. The Hirota bilinear equation is given by the even polynomial in three variables $(P(x, y, t))$ with no constant term,

$$P(D_x, D_y, D_t)f \cdot f = 0.$$

The relevant (2+1)-dimensional bilinear ngKdV equation comes to be

$$\begin{aligned} B(f) : &= (aD_x^4 + bD_x^3D_t + cD_x^2 + dD_y^2 + eD_xD_t)f \cdot f \\ &= 2[a(f_{4,xt}f - 4f_{3x}f_x + 3f_{xx}^2) + b(f_{3x,t}f - 3f_{xt}f_x + 3f_{xt}f_{xx} - f_t f_{3x}) \\ &\quad + c(f_{xx}f - f_x^2) + d(f_{yy}f - f_y^2) + e(f_{xt}f - f_x f_t)] = 0, \end{aligned} \tag{1}$$

here “ f ” is an arbitrary function in x, y and t and $\}D''$ is a known Hirota bilinear derivative operator.

$$D_x^l D_y^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x, y, t) \\ \text{cdot} g(x', y', t')|_{x'=x, y'=y, t'=t}$$

Equation (1) is converted into the following ngKdV equation:

$$N(u) := a(6u_x u_{xx} + u_{xxxx}) + b(3(u_x u_t)_x + u_{3xt}) + cu_{xx} + du_{yy} + eu_{xt} = 0, \tag{2}$$

under a logarithmic transformation (Ma et al. 2022)

$$u = 2(\ln f)_x. \tag{3}$$

When we set $b=c=0$, we get Kadomtsev-Petviashvili equation, by setting $d=0$, we get a class of ngKdV equation. We can get the traditional ngKdV equation connected with $b=c=0$ and the Hirota-Satsuma equation linked with $a=0$ in particular (Ma et al. 2022).

The following solution using the extended homogeneous balance technique is explored (Shang et al. 2011)

$$u(x, y, t) = f'(w)w_x + u_0(x, y, t). \tag{4}$$

Two functions “ f ” and “ w ” are to be determined later, and $u_0(x, y, t)$ is a seed solution of Eq. (2) and is practical to find several other solutions. Inserting (4) into (2), we find:

$$\begin{aligned} & [(6af''f''' + af^{(5)})w_x^5 + (bf^{(5)} + 3bf''f''')w_t w_x^4] + [(10aw_x^3 w_{xx} + 4bw_x^3 w_{xt} \\ & + 6bw_t w_x^2 w_{xx})f^{(4)} + (3bw_x^3 w_{xt} + 3bw_{xx} w_t w_x^2 + 6aw_{xx} w_x^3)f'f'' + (18aw_x^3 w_{xx} \\ & + 6bw_x^3 w w_{xt} + 3bw_t w_{xx} w_x^2 + 9bw_x^2 w_t w_{xx})(f'')^2] + [(6aw_x^3 u_{0x} + 3bw_t w_x^2 u_{0x} \\ & + 3bw_x^3 u_{0t} + 15aw_x w_{xx}^2 + 10aw_x^2 w_{xxx} + 12bw_x w_{xx} w_{xt} + 6bw_x^2 w_{xxt} + 3bw_t w_{xx}^2 \\ & + 4bw_t w_x w_{xxx} + cw_x^3 + dw_y^2 w_x + ew_t w_x^2 - 3bw_t w_x w_{xxx} - 9bw_x w_{xt} w_{xxx} - 6bw_x w_{xx} w_{xt})f'''' \\ & + (6aw_x^2 w_{xxx} + 18aw_x w_{xx}^2 + 3bw_x^2 w_{xxt} - 3bw_t w_{xx}^2)f'f'' + [(6aw_x^2 u_{0xx} + 18aw_x w_{xx} u_{0x} \\ & + 3bw_x^2 u_{0xt} + 6bw_x w_{xt} u_{0x} + 3bw_t w_{xx} u_{0x} + 9bw_x w_{xx} u_{0t} + 3bw_t w_x u_{0xx} + 10aw_{xx} w_{xxx} \\ & + 5aw_x w_{xxxx} + 6bw_{xx} w_{xxt} + 4bw_{xt} w_{xxx} + 4bw_x w_{xxt} + bw_t w_{xxx} + 3cw_x w_{xx} + dw_{yy} w_x \\ & + 2dw_y w_{xy} + 2ew_x w_{xt} + ew_t w_{xx})f'' + (3bw_{xxx} w_{xt} + 3bw_{xx} w_{xxt} + 6aw_{xx} w_{xxx})(f')^2] \\ & + [(6aw_{xx} u_{0xx} + 6aw_{xxx} u_{0x} + 3bw_{xx} u_{0xt} + 3bw_{xxx} u_{0t} + 3bw_{xxt} u_{0x} + 3bw_{xt} u_{0xx} \\ & + aw_{xxxx} + bw_{xxxxt} + cw_{xxx} + dw_{xyy} + ew_{xxt})f'] + [6au_{0x} u_{0xx} + 3bu_{0x} u_{0xt} + 3bu_{0xx} u_{0t} \\ & + au_{0xxx} + bu_{0xxx} + cu_{0xx} + du_{0yy} + eu_{0xt}] = 0. \end{aligned} \tag{5}$$

Then, we set (Kaplan and Ozer 2018)

$$f = g \ln w, \tag{6}$$

where g is an arbitrary constant. From this equation:

$$\begin{aligned}
 f''f''' &= -\frac{g}{12}f^{(5)}, & f'f'' &= -\frac{g}{3}f''' \\
 f'f''' &= -\frac{g}{3}f^{(4)}, & (f'')^2 &= -\frac{g}{6}f^{(4)} \\
 (f')^2 &= -gf'' .
 \end{aligned}
 \tag{7}$$

If we use the equalities (7), we can rewrite Eq. (5) in terms of $f^{(5)}, f^{(4)}, f''', f'', f'$ and $f^{(0)}$. Then by equating their coefficients to zero, we find:

$$\begin{aligned}
 f^{(5)} &: 1 - \frac{g}{2} = 0, \\
 f^{(4)} &: 10aw_x^3w_{xx} + 4bw_x^3w_{xt} + 6bw_t w_x^2w_{xx} + (3bw_x^3w_{xx} + 3bw_{xx}w_t w_x^2 \\
 &\quad + 6aw_{xx}w_x^3)\left(-\frac{g}{3}\right) + (18aw_x^3w_{xx} + 6bw_x^3w_{xt} + 3bw_x^2w_t w_{xx} \\
 &\quad + 9bw_x^2w_t w_{xx})\left(-\frac{g}{6}\right) = 0, \\
 f''' &: 6aw_x^3u_{0x} + 3bw_t w_x^2u_{0x} + 3bw_x^3u_{0t} + 15aw_{xx}^2w_x + 10aw_x^2w_{xxx} + 12bw_x w_{xx}w_{xt} \\
 &\quad + 6bw_x^2w_{xxt} + 3bw_t w_{xx}^2 + 4bw_t w_x w_{xxx} + cw_x^3 + dw_y^2w_x + ew_t w_x^2 \\
 &\quad - 3bw_{xx}w_t w_x - 9bw_{xxx}w_x w_{xt} - 6bw_{xx}w_x w_{xt} + (6aw_x^2w_{xxx} + 18aw_{xx}^2w_x \\
 &\quad + 3bw_x^2w_{xxt} - 3bw_{xx}^2w_t)\left(-\frac{g}{2}\right), \\
 f'' &: 6aw_x^2u_{0xx} + 18aw_x w_{xx}u_{0x} + 3bw_x^2u_{0xt} + 6bw_x w_{xt}u_{0x} + 3bw_t w_{xx}u_{0x} \\
 &\quad + 9bw_x w_{xx}u_{0t} + 3bw_t w_x u_{0xx} + 10aw_{xx}w_{xxx} + 5aw_x w_{xxx} + 6bw_{xx}w_{xxt} + 4bw_{xt}w_{xxx} \\
 &\quad + 4bw_x w_{xxt} + bw_t w_{xxx} + 3cw_x w_{xx} + dw_{yy}w_x + 2dw_y w_{xy} + 2ew_x w_{xt} + ew_t w_{xx} \\
 &\quad + (3bw_{xxx}w_{xt} + 3bw_{xx}w_{xxt} + 6aw_{xx}w_{xxx})(-g) = 0, \\
 f' &: 6aw_{xx}u_{0xx} + 6aw_{xxx}u_{0x} + 3bw_{xx}u_{0xt} + 3bw_{xxx}u_{0t} + 3bw_{xxt}u_{0x} + 3bw_{xt}u_{0xx} \\
 &\quad + aw_{xxxx} + bw_{xxxxt} + cw_{xxx} + dw_{yy} + ew_{xxt} = 0, \\
 f^{(0)} &: 6au_{0xx}u_{0xx} + 3bu_{0x}u_{0xt} + 3bu_{0xx}u_{0t} + au_{0xxx} + bu_{0xxx} + cu_{0xx} + du_{0yy} + eu_{0xt} = 0.
 \end{aligned}
 \tag{8}$$

The first equation of the system (8) gives:

$$g = 2,
 \tag{9}$$

If we use the equalities Eq. (7), we can rewrite the equation system Eq. (8) as

$$\begin{aligned}
 & [w_x(6aw_x^2u_{0x} + 3bw_t w_x u_{0x} + 3bw_x^2u_{0t} - 3aw_{xx}^2 + 4aw_x w_{xxx} - 3bw_{xx} w_{xt} \\
 & + 3bw_x w_{xtt} + 4bw_t w_{xxx} + cw_x^2 + dw_y^2 + ew_t w_x - 3bw_t w_{xxx})]f''' \\
 & + [6aw_x^2u_{0xx} + 18aw_x w_{xx} u_{0x} + 3bw_x^2u_{0xt} + 6bw_x w_{xt} u_{0x} + 3bw_t w_{xx} u_{0x} \\
 & + 9bw_x w_{xx} u_{0t} + 3bw_t w_x u_{0xx} - 2aw_{xxx} w_{xx} + 5aw_x w_{xxxx} - 2bw_{xt} w_{xxx} \\
 & + 4bw_x w_{xxx} + bw_t w_{xxx} + 3cw_x w_{xx} + dw_{yy} w_x + 2dw_y w_{xy} + 2ew_x w_{xt} \\
 & + ew_t w_{xt}]f'' + [6aw_{xx} u_{0xx} + 6aw_{xxx} u_{0x} + 3bw_{xx} u_{0xt} + 3bw_{xxx} u_{0t} \\
 & + 3bw_{xxt} u_{0x} + 3bw_{xt} u_{0xx} + aw_{xxxx} + bw_{xxx} + cw_{xxx} + dw_{xyy} + ew_{xxt}]f' \\
 & + [6au_{0x} u_{0xx} + 3bu_{0x} u_{0xt} + 3bu_{0xx} u_{0t} + au_{0xxx} + bu_{0xxt} + cu_{0xx} + du_{0yy} \\
 & + eu_{0xt}] = 0.
 \end{aligned} \tag{10}$$

Then, to create the differential equations below, we substitute zero for the coefficients of f''' , f'' , f' , f^0 respectively.

$$\begin{aligned}
 & 6aw_x^2u_{0x} + 3bw_t w_x u_{0x} + 3bw_x^2u_{0t} - 3aw_{xx}^2 + 4aw_x w_{xxx} - 3bw_{xx} w_{xt} + 3bw_x w_{xtt} \\
 & + 4bw_t w_{xxx} + cw_x^2 + dw_y^2 + ew_t w_x - 3bw_t w_{xxx} = 0
 \end{aligned} \tag{11}$$

$$\begin{aligned}
 & 6aw_x^2u_{0xx} + 18aw_x w_{xx} u_{0x} + 3bw_x^2u_{0xt} + 6bw_x w_{xt} u_{0x} + 3bw_t w_{xx} u_{0x} + 9bw_x w_{xx} u_{0t} \\
 & + 3bw_t w_x u_{0xx} - 2aw_{xxx} w_{xx} + 5aw_x w_{xxxx} - 2bw_{xt} w_{xxx} + 4bw_x w_{xxx} + bw_t w_{xxx} \\
 & + 3cw_x w_{xx} + dw_{yy} w_x + 2dw_y w_{xy} + 2ew_x w_{xt} + ew_t w_{xt} = 0
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 & 6aw_{xx} u_{0xx} + 6aw_{xxx} u_{0x} + 3bw_{xx} u_{0xt} + 3bw_{xxx} u_{0t} + 3bw_{xxt} u_{0x} + 3bw_{xt} u_{0xx} \\
 & + aw_{xxxx} + bw_{xxx} + cw_{xxx} + dw_{xyy} + ew_{xxt} = 0
 \end{aligned} \tag{13}$$

$$6au_{0x} u_{0xx} + 3bu_{0x} u_{0xt} + 3bu_{0xx} u_{0t} + au_{0xxx} + bu_{0xxt} + cu_{0xx} + du_{0yy} + eu_{0xt} = 0 \tag{14}$$

Differentiating Eq. (11) and integrating Eq. (13) w.r.t x , we can discover the connection between Eqs.(11)–(14). That is to say,

$$\begin{aligned}
 & \frac{\partial}{\partial x} (6aw_x^2u_{0x} + 3bw_t w_x u_{0x} + 3bw_x^2u_{0t} - 3aw_{xx}^2 + 4aw_x w_{xxx} \\
 & - 3bw_{xx} w_{xt} + 3bw_x w_{xtt} + 4bw_t w_{xxx} + cw_x^2 + dw_y^2 + ew_t w_x - 3bw_t w_{xxx}) \\
 & + w_x(6aw_{xx} u_{0xx} + 3bw_{xxx} u_{0t} + bw_{xxx} + cw_{xxx} + dw_{xyy} + ew_{xxt}) \\
 & = 6aw_x^2u_{0xx} + 18aw_x w_{xx} u_{0x} + 3bw_x^2u_{0xt} + 6bw_x w_{xt} u_{0x} + 3bw_t w_{xx} u_{0x} + 9bw_x w_{xx} u_{0t} \\
 & + 3bw_t w_x u_{0xx} - 2aw_{xxx} w_{xx} + 5aw_x w_{xxxx} - 2bw_{xt} w_{xxx} + 4bw_x w_{xxx} + bw_t w_{xxx} \\
 & + 3cw_x w_{xx} + dw_{yy} w_x + 2dw_y w_{xy} + 2ew_x w_{xt} + ew_t w_{xt}.
 \end{aligned} \tag{15}$$

If the following equations are satisfied, the Eq. (15) is satisfied

$$6aw_{xx} u_{0x} + 3bw_{xx} u_{0t} + bw_{xxx} + cw_{xx} + dw_{yy} + ew_{xt} = 0, \tag{16}$$

$$\begin{aligned}
 &6aw_x^2u_{0x} + 3bw_t w_x u_{0x} + 3bw_x^2u_{0t} - 3aw_{xx}^2 + 4aw_x w_{xxx} \\
 &\quad - 3bw_{xx} w_{xt} + 3bw_x w_{xxt} + 4bw_t w_{xxx} + cw_x^2 + dw_y^2 + ew_t w_x \\
 &\quad - 3bw_t w_{xxx} = 0,
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 &6au_{0x}u_{0xx} + 3bu_{0x}u_{0xt} + 3bu_{0xx}u_{0t} + au_{0xxx} + bu_{0xxt} + cu_{0xx} \\
 &\quad + du_{0yy} + eu_{0xt} = 0.
 \end{aligned} \tag{18}$$

Thus, an Auto-Bäcklund transformation for the ngKdV equation is as:

$$u(x, y, t) = 2 \frac{w_x}{w} + u_0, \tag{19}$$

where $u_0(x, y, t)$ and $w(x, y, t)$ satisfy Eqs. (16 and 17). After that, we may have several new solutions to Eq. 2 from Eq. (19) if we solve Eqs. (16) and (17) for any given solution u_0 of Eq. (2).

1. If we set $u_0 = k$ in Eq. (19), we can verify

$$u(x, y, t) = 2 \frac{w_x}{w} + k, \tag{20}$$

where $w(x, y, t)$ is satisfied by the following equations below

$$aw_{xxxx} + bw_{xxt} + cw_{xx} + dw_{yy} + ew_{xt} = 0, \tag{21}$$

$$\begin{aligned}
 &4aw - xw_{xxx} 3aw_{xx}^2 - 3bw_{xx} w_{xt} + 3bw_x w_{xxt} + 4bw_t w_{xxx} + cw_x^2 + dw_y^2 \\
 &\quad + ew_t w_x - 3bw_{xxx} w_t = 0.
 \end{aligned} \tag{22}$$

2. If we set $u_0 = 0$ in (19), we may find Cole-Hopf transformation

$$u(x, y, t) = 2 \frac{w_x}{w}, \tag{23}$$

where w is satisfied by Eqs. (21) and (22).

2.1 Extraction of the solitary wave solutions for ngKdV equation

Utilizing Auto-Bäcklund transformation and maple provided in the prior section, we will search for alternative solitary wave solutions of Eq. (2) in this section (Shang 2007). To do so the Auto-Bäcklund transformation (20) and the result u_0 of Eq. (2) is utilized as a starting point.

1. Take $w(x, y, t)$ be of the type

$$w(x, y, t) = q \cosh(kx + ly + rt + \xi_0) + s \sinh(kx + ly + rt + \xi_0) + p, \tag{24}$$

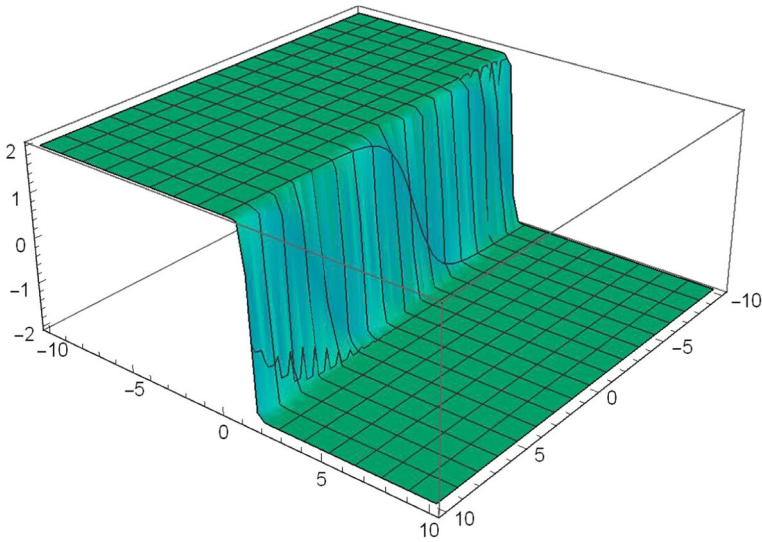


Fig. 1 The solution for $u(x, y, t)$ in Case 1, when $k = 1, \xi_0 = 1, l = 3, a = 1, b=2, c=2, d=3, e=1, p=1, q=2, s = 2, j = 0, y = 0$

where p, k, q, l, r and s are constants to be calculated and ξ_0 is an arbitrary. If we insert Eq. (24) into Eqs. (21) and (22), we find an algebraic equation system (Kaplan et al. 2017).

The by solving the system of algebraic equations, we obtain the following cases.

Case 1

$$k = k, \quad l = l, \quad q = s, \quad r = -\frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}, \quad s = s, \quad p = p, \tag{25}$$

where k, s, p, l are arbitrary. Thus, we find (Fig. 1)

$$\begin{aligned} w_1(x, y, t) = & s \cosh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) \\ & - s \sinh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) + p. \end{aligned} \tag{26}$$

Solitary wave solution for the ngKdV equation Eq. (2) is obtained by plugging Eq. (26) into Eq. (20), as follows

$$\begin{aligned} u_1(x, y, t) = & 2sk \frac{\cosh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) - \sinh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right)}{s \left(\cosh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) - \sinh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) \right) + p} + k \end{aligned} \tag{27}$$

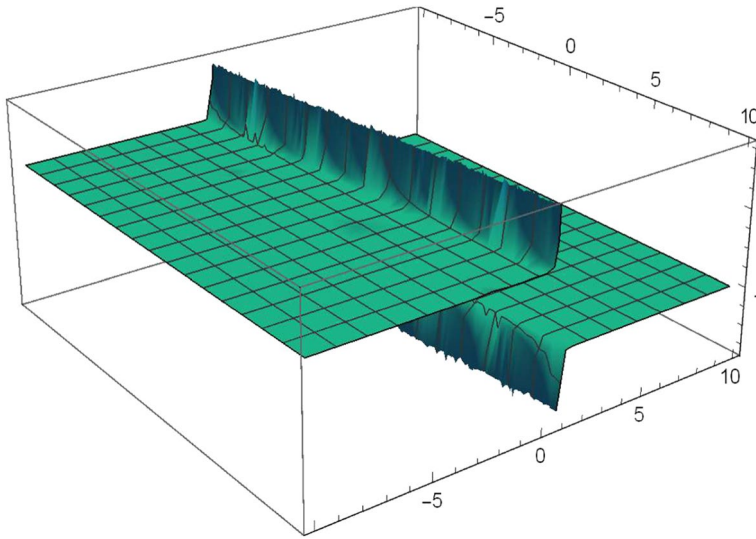


Fig. 2 The periodic solution to ngKdV equation for Case 2, when $k = 1, \xi_0 = 1, a = 1, p = 1, c = 2, s = 2, e = 1, b = 2, q = 2, d = 3, j = 0, y = 0$

Case 2

$$k = k, \quad l = l, \quad q = -s, \quad r = -\frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}, \quad s = s, \quad p = p, \tag{28}$$

where k, s, l, p are non-zero values. Thus, we acquire (Fig. 2)

$$w_2(x, y, t) = -s \cosh\left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0\right) - s \sinh\left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0\right) + p. \tag{29}$$

By putting Eq. (29) into Eq. (20), we find solitary wave solution of Eq. (2) as:

$$u_2(x, y, t) = 2sk \frac{\cosh\left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0\right) + \sinh\left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0\right)}{-s\left(\cosh\left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0\right) - \sinh\left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0\right)\right) + p} + k. \tag{30}$$

2. Take $w(x, y, t)$ of the type

$$w(x, y, t) = q \cos(kx + ly + rt + \xi_0) + s \sin(kx + ly + rt + \xi_0) + p, \tag{31}$$

where p, q, k, r, l and s are constants which are to be determined and ξ_0 is arbitrary.

Inserting Eq. (31) into Eqs. (21) and (22) an algebraic equations. After solving the system, we find:

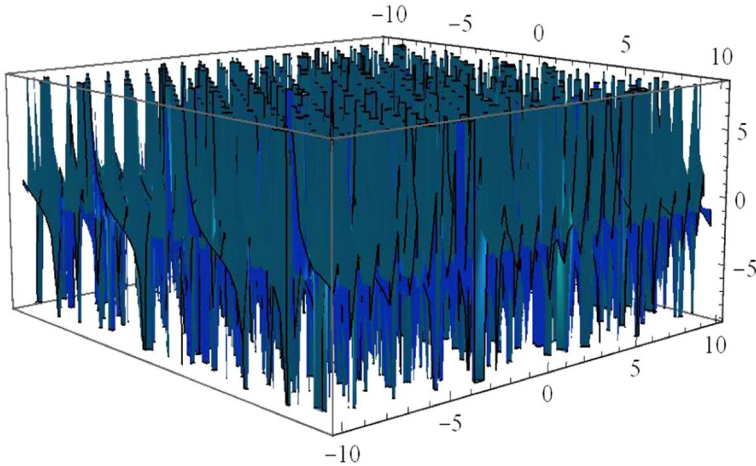


Fig. 3 The solution $u(x, y, t)$ when $k = 1, \xi_0 = 1, l = 3, a = 1, b = 2, c = 2, d = 3, e = 1, p = 1, q = 2, s = 2, j = 0, y = 0$

$$q = st, \quad s = s, \quad k = k, \quad p = p, \quad l = l, \quad r = -\frac{ak^4 - ck^2 - dl^2}{k(bk^2 - e)}, \tag{32}$$

where s, k, p and l are non-zero values. Then, we obtain

$$\begin{aligned} w_3(x, y, t) = & st \cos \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) \\ & - s \sin \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) + p. \end{aligned} \tag{33}$$

Periodic wave solution of Eq. (2) is obtained after putting Eq. (33) into Eq. (20), as shown in Fig. 3

$$\begin{aligned} u_3(x, y, t) &= 2sk \frac{\cosh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) - \sinh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right)}{s \left(\cosh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) - \sinh \left(-kx - ly + \frac{ak^4 + ck^2 + dl^2}{k(k^2b + e)}t - \xi_0 \right) \right)} + k \end{aligned} \tag{34}$$

3. We now investigate the solution of Eq. (2) as below

$$w(x, y, t) = q \exp(k_1x + l_1y + r_1t + \xi_1) + s \exp(k_2x + l_2y + r_1t + \xi_2) + p, \tag{35}$$

where $q, s, p, k_1, l_1, r_1, k_2, l_2, r_2$ are constants which are to be determined later. By putting Eq. (33) into Eqs. (19) and (20), yields

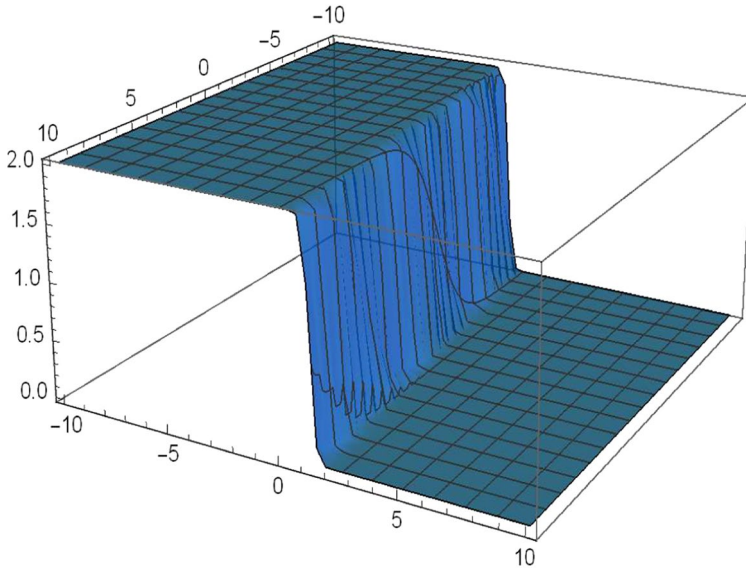


Fig. 4 The travelling wave solution for $u(x, y, t)$, when $k_1 = 1, l_1 = 3, \xi_1 = 1, k_2 = 1, l_2 = 3, \xi_2 = 1, r_2 = 2, a=1, b = 2, c = 2, d = 3, e = 1, p = 1, q = 2, j = 0, y = 0$.

$$\begin{aligned}
 k_1 &= k_1, \quad l_1 = l_1, \quad k_2 = k_2, \quad l_2 = l_2, \quad q = q, \quad p = p, \\
 s &= 0, \quad r_1 = -\frac{(ak_1^4 + ck_1^2 + dl_1^2)}{k_1(bk_1^2 + e)}, \quad r_2 = r_2,
 \end{aligned}
 \tag{36}$$

where k_1, k_2, l_1, l_2, q and r_2 are non-zero values and ξ_1, ξ_2 are constants. Thus, we obtain

$$w_4(x, y, t) = q \exp(k_1x + l_1y - \frac{(ak_1^4 + ck_1^2 + dl_1^2)}{k_1(bk_1^2 + e)}t + \xi_1) + p,
 \tag{37}$$

Afterwards, inserting Eq. (37) into Eq. (20), the exponential solution to the ngKdV equation is acquired. Eq. (2) as follows (Fig. 4)

$$u_4(x, y, t) = 2qk_1 \frac{\exp\left(k_1x + l_1y - \frac{(ak_1^4 + ck_1^2 + dl_1^2)}{k_1(bk_1^2 + e)}t + \xi_1\right)}{q \exp\left(k_1x + l_1y - \frac{(ak_1^4 + ck_1^2 + dl_1^2)}{k_1(bk_1^2 + e)}t + \xi_1\right) + p} + j
 \tag{38}$$

We can simply check the precise explicit solutions of the ngKdV equation for Cole-Hopf transformation given by Eq. (23) by inserting zero for the constant $\}j''$ in the founded solutions.

3 Description of the extended transformed rational function technique

In this section, we will give the main steps of the extended transformed rational function technique (Zhang and Ma 2014). We consider a Hirota bilinear NPDE with independent variables ρ and ν and dependent variable r , given by:

$$Q(\Delta_\rho, \Delta_\nu, \dots)r \cdot r = 0, \tag{39}$$

where $\Delta_\rho, \Delta_\nu, \dots$, are Hirota’s differential operators defined as

$$\Delta_\mu^l r(\mu) \cdot w(\mu) = (\partial_\mu - \partial_{\mu'})^l r(y)w(\mu') \Big|_{\mu'=\mu} = \partial_{\mu'}^l r(\mu + \mu')g(\mu - \mu') \Big|_{\mu'=0}, \quad p \geq 1. \tag{40}$$

After that, we will set the solution of Eq. (39) the following form:

$$r = \frac{l(\eta_1, \eta_2)}{m(\eta_1, \eta_2)} \tag{41}$$

where $l(\eta_1, \eta_2)$ and $m(\eta_1, \eta_2)$ are polynomials and η_1 and η_2 admit the following differential equations ,

$$\eta_1'' = \frac{d^2 \eta_1}{d\xi_1^2} = -\eta_1, \tag{42}$$

$$\eta_2'' = \frac{d^2 \eta_2}{d\xi_2^2} = \eta_2, \tag{43}$$

where $\xi_1 = k_1x + \omega_1t + c_1$ $\xi_2 = k_2x + \omega_2t + c_2$, k_1, k_2, ω_1 and ω_2 can be determined later and c_1 and c_2 are arbitrary constants. Here one may easily get the solutions of Eqs. (42) and (43) as follows:

$$\begin{aligned} \eta_1 &= \pm \sin \xi_1 \text{ or } \eta_1 = \pm \cos \xi_1, \\ \eta_2 &= \pm \sinh \xi_2 \text{ or } \eta_2 = \pm \cosh \xi_2, \end{aligned} \tag{44}$$

respectively. We may find the following equalities from Eq. (44)

$$\eta_1'^2 = 1 - \eta_1^2 \text{ and } \eta_2'^2 = 1 + \eta_2^2. \tag{45}$$

Then, by choosing suitable $p(\eta_1, \eta_2)$ and $q(\eta_1, \eta_2)$, Eq. (39) may be transformed into an algebraic equation contains k_i and ω_i . We then use the symbolic computation to find exact complexiton solutions to Eq. (37).

4 Extended transformed rational function method

By using the transformation mentioned in Eq. (3) the relevant bilinear equation is given by Eq. (1) Assume that

$$f = A\eta_1 + B\eta_2 \tag{46}$$

As Eq. 1 is a (2+1) dimensional equation, then we choose

$$\begin{aligned} \xi_1 &= k_1x + pk_1y + w_1t + g \\ \xi_2 &= k_2x + pk_2y + w_2t + h \end{aligned} \tag{47}$$

where A, B, k_i and w_i are constant values. Inserting Eq. (46) into Eq. (1) and using the following relations

$$\begin{aligned} \eta_1'' &= \frac{d^2\eta_1}{d\xi_1^2} = -\eta_1, & \eta_2'' &= \frac{d^2\eta_2}{d\xi_2^2} = \eta_2, \\ \eta_1'^2 &= 1 - \eta_1^2, & \eta_2'^2 &= 1 + \eta_2^2 \end{aligned} \tag{48}$$

The resulting equation can be expressed in the polynomial form in terms of $\eta_1^2, \eta_2^2, \eta_1\eta_2, \eta_1'\eta_2'$. Accumulating the coefficients of $\eta_1^2, \eta_2^2, \eta_1\eta_2, \eta_1'\eta_2'$, then we obtain the following algebraic equations by equating them equal to zero

$$\begin{aligned} &-12ABak_1^2k_2^2 + 2ABbk_1^3w_1 + 2ABbk_2^3w_2 + 2ABdp^2k_2^2 - 2ABek_1w_1 + 2ABek_2w_2 \\ &-2ABdp^2k_1^2 - 6ABbk_1k_2^2w_1 + 2ABak_1^4 + 2ABak_2^4 - 2ABck_1^2 + 2ABck_2^2 \\ &-6ABbk_1^2k_2w_2 = 0 \\ &8ABak_1^3k_2 - 8ABak_1k_2^3 + 2ABbk_1^3w_2 - 4ABck_1k_2 - 2ABek_1w_2 \\ &-2ABbk_2^3w_1 - 2ABek_2w_1 + 6ABbk_1^2k_2w_1 - 6ABbk_1k_2^2w_2 - 4ABdp^2k_1k_2 = 0 \\ &-8B^2ak_2^4 - 2A^2dp^2k_1^2 - 2B^2dp^2k_2^2 - 2A^2ek_1w_1 - 2B^2ek_2w_2 \\ &+8A^2ak_1^4 - 2A^2ck_1^2 - 2B^2ck_2^2 + 8A^2bk_1^3w_1 - 8B^2bk_2^3w_2 = 0. \end{aligned} \tag{49}$$

By evaluating the above system of simultaneous equations Eq. (49), we get the following result

$$A = \frac{k_2}{k_1} \sqrt{\frac{bk_1^2 - 3bk_2^2 - 3e}{3bk_1^2 - bk_2^2 - 3e}} B, \tag{50}$$

$$w_1 = \frac{-k_1(abk_1^4 + 2abk_1^2k_2^2 + abk_2^4 - bdp^2k_1^2 - bdp^2k_2^2 - aek_1^2 + 3aek_2^2 - bck_1^2 - bck_2^2 + edp^2 + ce)}{b^2k_1^4 + 2b^2k_1^2k_2^2 + b^2k_2^4 - 2bek_1^2 + 2bek_2^2 + e^2}, \tag{51}$$

$$w_2 = \frac{-k_2(abk_1^4 + 2abk_1^2k_2^2 + abk_2^4 + bdp^2k_1^2 + bdp^2k_2^2 + aek_2^2 - 3aek_1^2 + bck_1^2 + bck_2^2 + edp^2 + ce)}{b^2k_1^4 + 2b^2k_1^2k_2^2 + b^2k_2^4 - 2bek_1^2 + 2bek_2^2 + e^2}. \tag{52}$$

By Eq. (43) we can write the solutions of (1) as below:

$$f(x, y, t) = A \left[\sin(k_1x + pk_1y + w_1t + g) + \frac{k_1}{k_2} \sqrt{\frac{3bk_1^2 - bk_2^2 - 3e}{bk_1^2 - 3bk_2^2 - 3e}} \sinh(k_2x + pk_2y + w_2t + h) \right] \tag{53}$$

or

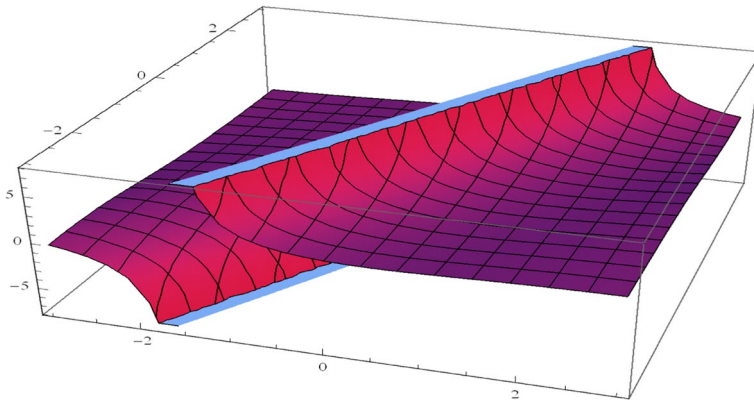


Fig. 5 The travelling solution for $u(x, y, t)$, when $A = 1, g = 1, h = 1, a = 1, b = 2, c = 1, d = 1, e = 0, k_1 = 1, k_2 = 1, p = 1, y = 0$

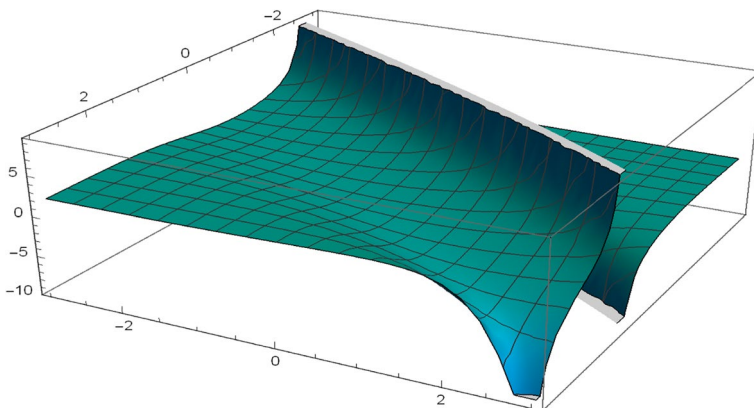


Fig. 6 The travelling solution for $u(x, y, t)$, when $A = 1, g = 1, h = 1, a = 1, b = 2, c = 1, d = 1, e = 0, k_1 = 1, k_2 = 1, p = 1, y = 0$

$$f(x, y, t) = A \left[\cos(k_1x + pk_1y + w_1t + g) + \frac{k_1}{k_2} \sqrt{\frac{3bk_1^2 - bk_2^2 - 3e}{bk_1^2 - 3bk_2^2 - 3e}} \sinh(k_2x + pk_1y + w_2t + h) \right] \tag{54}$$

where w_1 and w_2 are given in Eqs. (51) and (52). A, b, g, h, k_1, k_2 are arbitrary. Putting Eqs. (53) and (54) into Eq. (2) we find u which is shown in Figs. (5) and (6)

5 Conclusion

In the present work, we have used the extended homogeneous balance technique to find Auto-Bäcklund transformations of the new generalized KdV equation. Then, different exact explicit solutions to this equation via these transformations and symbolic computation

have been discovered. Graphically, periodic and travelling wave solutions are found. Then complexiton solutions of the considered equation are constructed by using the extended transformed rational function technique thanks to Hirota bilinear forms. We also gave a graphical representation of the obtained solutions. The considered techniques are powerful and useful. We have used the Maple packet program for the calculations and verifying the correctness of the solutions.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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