Journal of Ocean Engineering and Science 9 (2024) 437–453

Contents lists available at ScienceDirect

Journal of Ocean Engineering and Science

journal homepage: www.elsevier.com/locate/joes

Original Article

A unique computational investigation of the exact traveling wave solutions for the fractional-order Kaup-Boussinesq and generalized Hirota Satsuma coupled KdV systems arising from water waves and interaction of long waves

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ARTICLE INFO

Article history:
Received 20 February 2022
Revised 15 March 2022
Accepted 15 March 2022
Available online 18 March 2022

Keywords:
Symbolic computation
Fractional differential equations
Beta derivative
Auxiliary equation method
Solitary solutions
Nonlinear equations

ABSTRACT

A novel technique, named auxiliary equation method, is applied in this research work for obtaining new traveling wave solutions for two interesting proposed systems: the Kaup-Boussinesq system and generalized Hirota-Satsuma coupled KdV system with beta time fractional derivative. Our solutions were obtained using MAPLE software. This technique shows a great potential to be applied in solving various nonlinear fractional differential equations arising from mathematical physics and ocean engineering. Since a standard equation has not been used as an auxiliary equation for this technique, different and novel solutions are obtained via this technique.

1. Introduction

Nonlinear fractional differential equations have been seen in various modeling phenomena of physics and engineering [1–10]. Many researchers work on investigating exact solutions to these equations due to the importance of these equations in interpreting the physical meaning for some interesting models [11–22]. In particular, various research works have been conducted on studying these equations with beta time derivative due to its applicability in multidisciplinary sciences where investigating their exact solutions can provide us with a better understanding for nonlinear fractional differential equations with beta time derivative by studying their soliton theory and explicit formulas [23,24]. To solve these nonlinear fractional equations exactly, various methods have been proposed for this purpose such as the method of modified simple equation [25], modified Kudryashov method [26] (see also [27]), the method of auxiliary equation [28] (see also [29]), the He's variational approach[30], and the method of $\exp(-\Phi(\xi))$-expansion [31] (see also [32]). Other research works studied the exact solutions of nonlinear partial differential equations arising from various oceanographic phenomena in [33–35]. Some interesting research works systematically investigated the integrable equations' N-soliton solutions to nonlinear dispersive wave equations via the Hirota direct technique for both $(1+1)$-dimensional integrable equations [36] and $(2+1)$-dimensional integrable equations [37] in addition to a novel class of equations in $(2+1)$-dimensions [38,39]. The nonlocal integrable equations involving the Riemann-Hilbert problems with nonlocal reverse-time NLS hierarchies and the inverse scattering of nonlocal real reverse-spacetime matrix AKNS hierarchies have been investigated in [40,41], respectively. In addition, some research studies propose approximate-analytical solutions for nonlinear fractional differential equations formulated in the senses of fractional derivatives such as homotopy method

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https://doi.org/10.1016/j.joes.2022.03.012

[42], generalized form of double Laplace transform [43], and triple Laplace transform [44]. Recently, a generalized definition of fractional derivative, named for simplicity as Abu-Shady–Kaabar fractional derivative, [45] provided a simple tool for obtaining the analytical solutions of various classes of fractional differential equations which can be employed further in investigating various nonlinear partial differential equations arising from scientific phenomena.

While several important research studies concerning phenomena arising from ocean engineering and science have been recently conducted such the well-investigated study of Lie symmetry analysis and invariant solutions of the $(2+1)$ dimensional Bogoyavlensky-Konopelchenko equation with variable-coefficient in wave propagation [46], perturbed nonlinear Schrödinger equation's complex soliton solutions in nonlinear optical fibers [47], generalized Schrödinger-Boussinesq equations for the interaction between complex short wave and real long wave envelope [48], and fifth-order nonlinear wave equation's solitary waves and exact solutions [49], our results provide uniquely a novel tool for obtaining new traveling wave solutions for the Kaup-Boussinesq system and generalized Hirota-Satsuma coupled KdV system in the context of beta time fractional derivative arising from water waves and interaction of long waves. The investigated systems in this study are very important in the field of ocean engineering where this study can provide new direction to provides some explanations to the behavior of various scientific phenomena.

Motivated by the above methods and some other research works, particularly [50–54], the results in our work are new and novel because both of the Kaup-Boussinesq system and generalized Hirota satsuma coupled KdV system with beta time fractional derivative are investigated in this research work via auxiliary equation method (AEqM) to obtain their traveling wave solutions. This research work is organized as follows: In Section 2, some basic properties concerning beta derivative are examined. Section 3 provides a basic step-by-step introduction about auxiliary equation method (AEqM). In section 4, we obtain our traveling wave solutions via AEqM for the studied Kaup-Boussinesq system and generalized Hirota satsuma coupled KdV system with beta time fractional derivative. Our conclusion is given in section 4.

2. Beta derivative and its properties

In this section, we will give the definition of the beta derivative and some basic properties.

Definition 2.1 [55]**.** Let $f(t)$ be a function defined for all non-negative t. Then, the beta derivative of $f(t)$ of order β is given by

$$T^{\beta}(f(t)) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(t)}{\varepsilon}, 0<\beta\le 1$$

and $T^{\beta}(f(t)) = \frac{d^{\beta}f(t)}{dt^{\beta}}$.

Some properties are given for beta derivative in the following theorem:

Theorem 2.1 *[56]. Let $f(t)$ and $g(t)$ be two functions with the derivative of β for all $t>0$ where $\beta\in(0,1]$. Then, we have:*

1. $T^{\beta}(af(t)+bg(t)) = aT^{\beta}(f(t))+bT^{\beta}(g(t)), \forall a,b\in R$
2. $T^{\beta}(f(t)g(t)) = g(t)T^{\beta}(f(t))+f(t)T^{\beta}(g(t))$,
3. $T^{\beta}(\frac{f(t)}{g(t)}) = \frac{g(t)T^{\beta}(f(t))-f(t)T^{\beta}(g(t))}{g(t)^2}$,
4. $T^{\beta}(f(t)) = (t+\frac{1}{\Gamma(\beta)})^{1-\beta}\frac{df(t)}{dt}$.

3. Mathematical preliminaries of AEqM

This section describes step-by-step the basics of AEqM which will be used in obtaining our original results in the next section. AEqM is defined as a direct method for obtaining new multiple traveling wave solutions for fractional order nonlinear fractional differential equations. This technique was first proposed by Sirendaoreji in [57,58], and then Abdoulkary et al. applied this technique for finding the exact traveling wave solutions of the proposed nonlinear Schrödinger equation in [59].

To apply this novel technique, let us state following important steps according to the results in [57–59]:

AEqM STEP I: Let us consider a general nonlinear fractional differential equation of the following type:

$$\Psi(\Omega,\ D_t^{\beta}\Omega, \Omega_x,\ D_{tt}^{\beta}\Omega, \Omega_{xt}, \Omega_{xx}, \ldots) = 0, \tag{1}$$

where the unknown function is denoted by Ω, and Ψ represents a polynomial of Ω and its partial and beta derivatives. Let us now express the traveling wave variable as follows:

$$\Omega(x,t) = \vartheta(\zeta),\quad \zeta = x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{2}$$

transforms Eq. (1) to the following nonlinear ordinary differential equation:

$$\Psi(\vartheta, \vartheta', \vartheta'', \ldots) = 0, \tag{3}$$

where the prime represents the derivation with respect to ζ.

AEqM STEP II: Let us assume that the solution of Eq. (3) can be expressed in the following form:

$$\vartheta(\zeta) = \sum_{j=0}^{2M} a_j\Phi^j(\zeta) \tag{4}$$

where $a_j (j=0,\ldots,2M)$ represents the constants that will be obtained in our results' section, M is a positive integer that can be found via the homogeneous balance method (HBM) between the highest order derivatives and the nonlinear terms that can be seen in Eq. (3). $\Phi(\zeta)$ satisfies the variable separated ODE as follows

$$\Phi'^2(\zeta) = a\Phi^2(\zeta)+b\Phi^4(\zeta)+c\Phi^6(\zeta) \tag{5}$$

where the parameters: a,b,c that are needed to be found.

AEqM STEP III: Let us now substitute the ansatz Eq. (4) along with Eq. (5) into Eq. (3), and equate the coefficients of all powers of $\Phi(\zeta)$ to zero; hence, a set of algebraic equations for the unknowns $a,b,c,d,a_j (j=0,\ldots,2M)$ and λ is obtained. This set of algebraic equations can be solved using Maple software, and our obtained solutions are substituted back into Eq. (3) in order to find the exact traveling wave solutions for Eq. (1).

4. Main results

The method of auxiliary equation is applied in this section to obtain new exact solutions for some fractional order nonlinear fractional differential equations. Both of the Kaup-Boussinesq system and the generalized Hirota-Satsuma coupled KdV system with beta time fractional derivatives will be studied in our work.

4.1. Kaup-Boussinesq system with beta derivative

In this section, we will consider the Kaup-Boussinesq system with beta derivative as follows:

$$D_t^{\beta}\Omega-\varphi_{xxx}-2(\Omega\varphi)_x = 0,\quad D_t^{\beta}\varphi-\Omega_x-\left(\varphi^2\right)_x = 0. \tag{6}$$

where the water surface height above a horizontal bottom is represented by $\Omega(x,t)$, and the horizontal velocity field is represented by $\varphi(x,t)$ [63]. The proliferation scheme for Kaup-Boussinesq system, named dispersive water wave system, was investigated [60]. Li and Wang gave the symmetry reduction this system, and then they used the trial equation method to find exact solutions to describe its dynamical behavior [61]. Singla and Rana have considered this equation system by using the group invariance approach and power series expansion method [62]. In addition, Kilic and Inc have obtained exact solutions to this system by using first integral method [63]. Let us now substitute the traveling wave transformation into Eq. (6) as follows:

$$\Omega(x,t)=\vartheta(\zeta),\varphi(x,t)=\theta(\zeta),\zeta=x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}. \tag{7}$$

Then, the equation system can be found as follows:

$$\begin{aligned}&-\lambda\vartheta'-\theta'''-2\theta\vartheta'-2\vartheta\theta'=0,\\&-\lambda\theta'-\vartheta'-2\theta\theta'=0.\end{aligned} \tag{8}$$

We integrate the second equation of the system once with respect to ζ to find:

$$\vartheta=-\lambda\theta-\theta^2. \tag{9}$$

Then, by substituting Eq. (9) into the first equation of the system and integrating it once with respect to ζ, we find

$$\theta''-\lambda^2\theta-3\lambda\theta^2-2\theta^3=0. \tag{10}$$

According to the HBM, we find the balancing number as 1. Hence, the solution can be expressed as:

$$\theta=a_0+a_1\Phi+a_2\Phi^2. \tag{11}$$

Where the constants: a_0, a_1 and a_2 are needed to be found, while $\Phi(\zeta)$ is an unknown function that are needed to be obtained, too. By substituting this solution into Eq. (10), and collecting all the terms with the same power of $\Phi^j (j=0,1,\ldots,6)$,and then by equating each coefficient to zero, a set of algebraic equations is obtained as follows:

$$\begin{aligned}&\Phi^6: \ -2a_2^3+8a_2\lambda=0,\\&\Phi^5: \ -6a_1a_2^2+3a_1\lambda=0,\\&\Phi^4: \ -6a_0a_2^2-6a_1^2a_2-3a_2^2\lambda+6a_2b=0,\\&\Phi^3: \ -12a_0a_1a_2-2a_1^3-6a_1a_2\lambda+2a_1b=0,\\&\Phi^2: \ -6a_0^2a_2-6a_0a_1^2-6a_0a_2\lambda-3a_1^2\lambda-a_2\lambda^2+4aa_2=0,\\&\Phi^1: \ -6a_0^2a_1-6a_0a_1\lambda-a_1\lambda^2+aa_1=0,\\&\Phi^0: \ -2a_0^3-3\lambda a_0^2-\lambda^2a_0=0.\end{aligned}$$

By using the Maple software, the set of algebraic equations is solved as follows:

$$a=\frac{\lambda^2}{4},b=\frac{a_2\lambda}{2},c=\frac{a_2^2}{4},a_0=0,a_1=0, \tag{12}$$

Case 1: If we substitute Eq. (12) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=-\frac{\sqrt{-\left(-c_1\lambda e^{-\lambda\zeta}+a_2\right)\lambda}}{-c_1\lambda e^{-\lambda\zeta}+a_2}. \tag{13}$$

Therefore, the solitary wave solutions (SWSs) of the studied Kaup-Boussinesq system with beta derivative are expressed as:

$$\Omega_1(x,t)=\frac{a_2\lambda^2}{-c_1\lambda e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}-\left(-\frac{a_2\lambda}{-c_1\lambda e^{-\lambda(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}+a_2}\right)^2 \tag{14}$$

and

$$\varphi_1(x,t)=-\frac{a_2\lambda}{-c_1\lambda e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}, \tag{15}$$

where c_1 is an arbitrary constant. Fig. 1 shows (a) $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 1.5<x<2; -1<t<1$ and (b) $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; -1<x<2; 0<t<1$. Fig. 2 shows (a) $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 1.5<x<2; -1<t<1$ and (b) $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; -1<x<2; 0<t<1$. Fig. 3 shows (a) $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 1.5<x<2; -1<t<1$ and (b) $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; -1<x<2; 0<t<1$. Fig. 4 shows (a) $\varphi_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 0.5<x<1; -2<t<2$, (b) $\varphi_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 0.5<x<1; -2<t<2$, and (c) $\varphi_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 0.5<x<1; -2<t<2$.

Case 2: If we substitute Eq. (12) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=-\frac{\sqrt{-\left(-c_1\lambda e^{\lambda\zeta}+a_2\right)\lambda}}{-c_1\lambda e^{\lambda\zeta}+a_2} \tag{16}$$

The SWSs of the Kaup-Boussinesq system with beta derivative are expressed as:

$$\Omega_2(x,t)=\frac{a_2\lambda^2}{-c_1\lambda e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}-\left(-\frac{a_2\lambda}{-c_1\lambda e^{\lambda(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}+a_2}\right)^2, \tag{17}$$

and

$$\varphi_2(x,t)=-\frac{a_2\lambda}{-c_1\lambda e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}, \tag{18}$$

where c_1 is an arbitrary constant.

With the help of Maple software, the set of algebraic equations is solved as follows:

$$a=\frac{\lambda^2}{4},b=-\frac{a_2\lambda}{2},c=\frac{a_2^2}{4},a_0=-\lambda,a_1=0,a_2=a_2 \tag{19}$$

Case 3: If we substitute Eq. (12) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=-\frac{\sqrt{\left(c_1\lambda e^{\lambda\zeta}+a_2\right)\lambda}}{-c_1\lambda e^{\lambda\zeta}+a_2} \tag{20}$$

The SWSs of the studied Kaup-Boussinesq system with beta derivative are expressed as

$$\Omega_3(x,t)=\frac{\lambda^3c_1e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{c_1\lambda e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}-\left(-\frac{\lambda^2c_1e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{c_1\lambda e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}\right)^2, \tag{21}$$

and

$$\varphi_3(x,t)=-\frac{\lambda^2c_1e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{c_1\lambda e^{\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}, \tag{22}$$

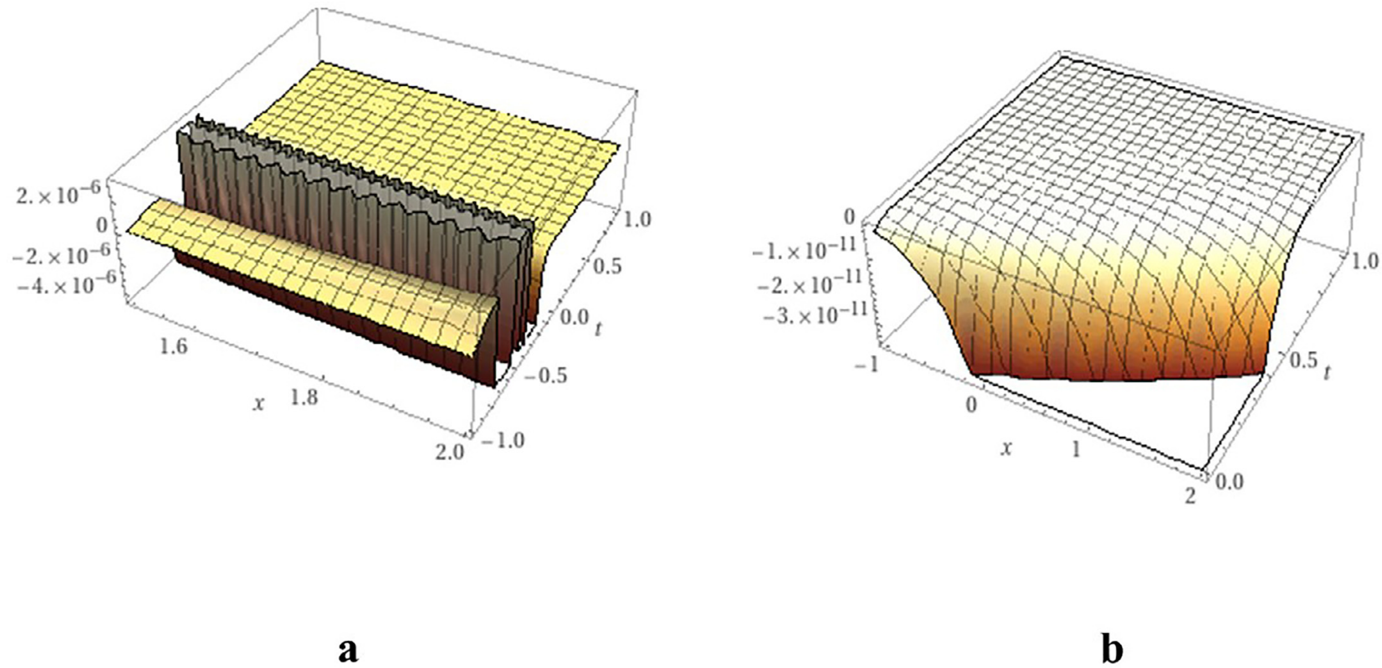


Fig. 1. 3D Plot of $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 1.5<x<2; -1<t<1$ in (a) and $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; -1<x<2; 0<t<1$ in (b).

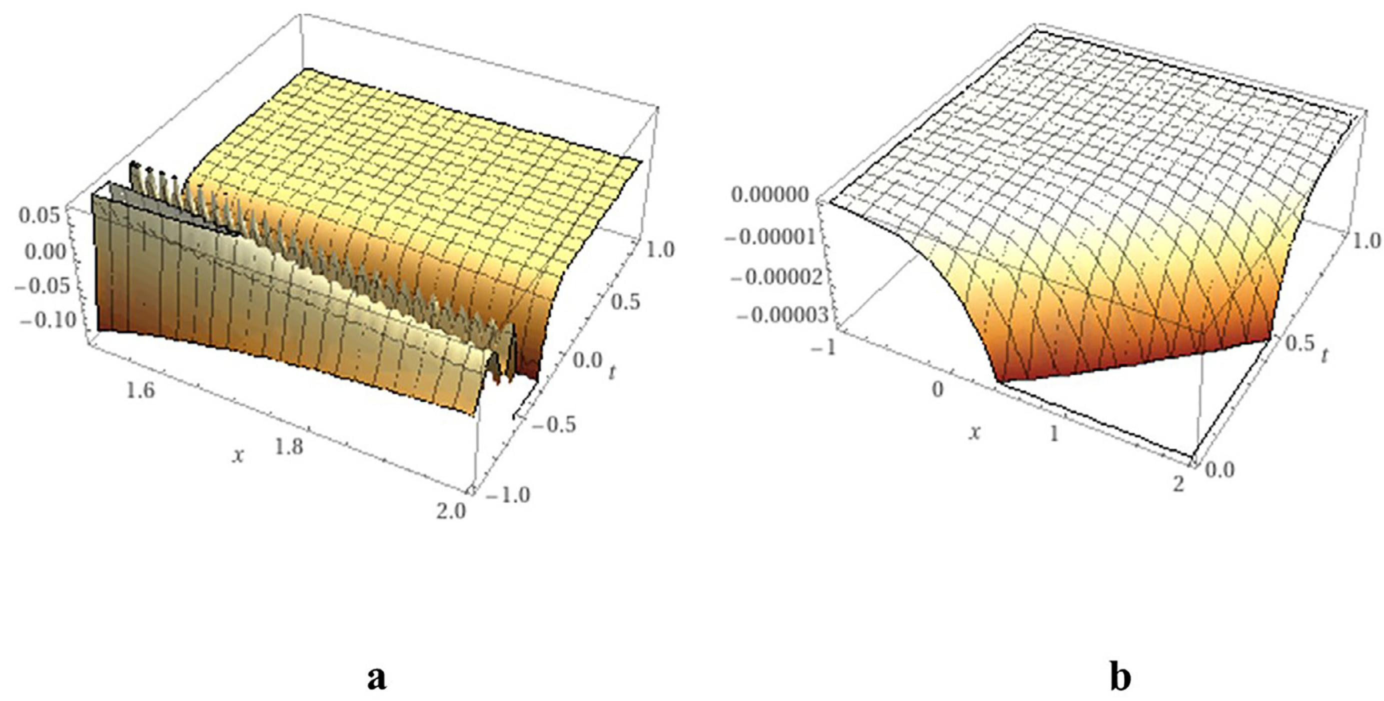
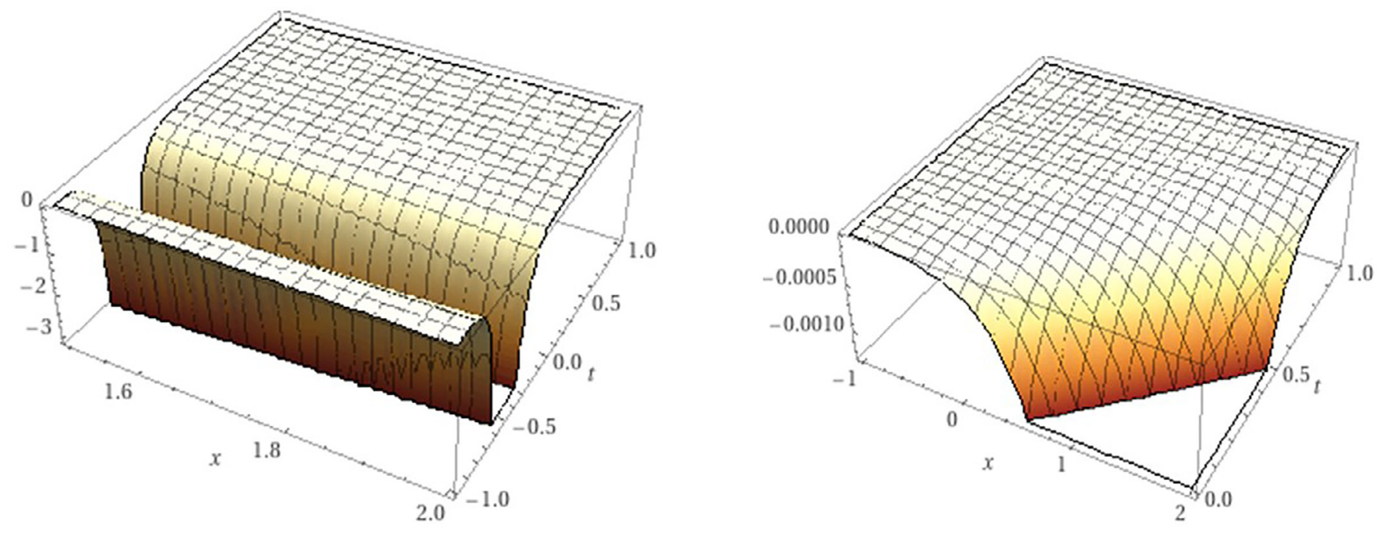


Fig. 2. 3D Plot of $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 1.5<x<2; -1<t<1$ in (a) and $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; -1<x<2; 0<t<1$ in (b).

where c_1 is an arbitrary constant. Fig. 5 shows (a) $\Omega_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 1.5<x<2; -1<t<1$ and (b) $\Omega_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; -1<x<2; 0<t<1$. Fig. 6 shows (a) $\Omega_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 1.5<x<2; -1<t<1$ and (b) $\Omega_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; -1<x<2; 0<t<1$. Fig. 7 shows (a) $\Omega_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 1.5<x<2; -1<t<1$ and (b) $\Omega_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; -1<x<2; 0<t<1$. Fig. 8 shows (a) $\varphi_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 1.5<x<2; -1<t<1$, (b) $\varphi_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 1.5<x<2; -1<t<1$, and (c) $\varphi_3(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 1.5<x<2; -1<t<1$.

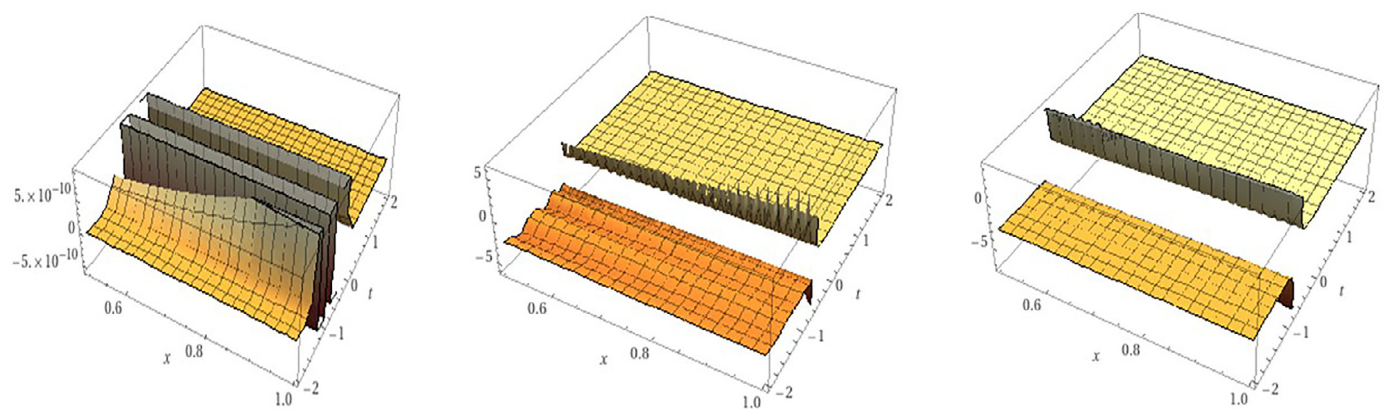
Case 4: If we substitute Eq. (12) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=\frac{\sqrt{\left(c_1\lambda e^{-\lambda\zeta}+a_2\right)\lambda}}{c_1\lambda e^{\lambda\xi}+a_2} \tag{23}$$



a b

Fig. 3. 3D Plot of $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 1.5<x<2; -1<t<1$ in (a) and $\Omega_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; -1<x<2; 0<t<1$ in (b).



a b c

Fig. 4. 3D Plot of (a) $\varphi_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 0.5<x<1; -2<t<2$, (b) $\varphi_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 0.5<x<1; -2<t<2$, and (c) $\varphi_1(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 0.5<x<1; -2<t<2$.

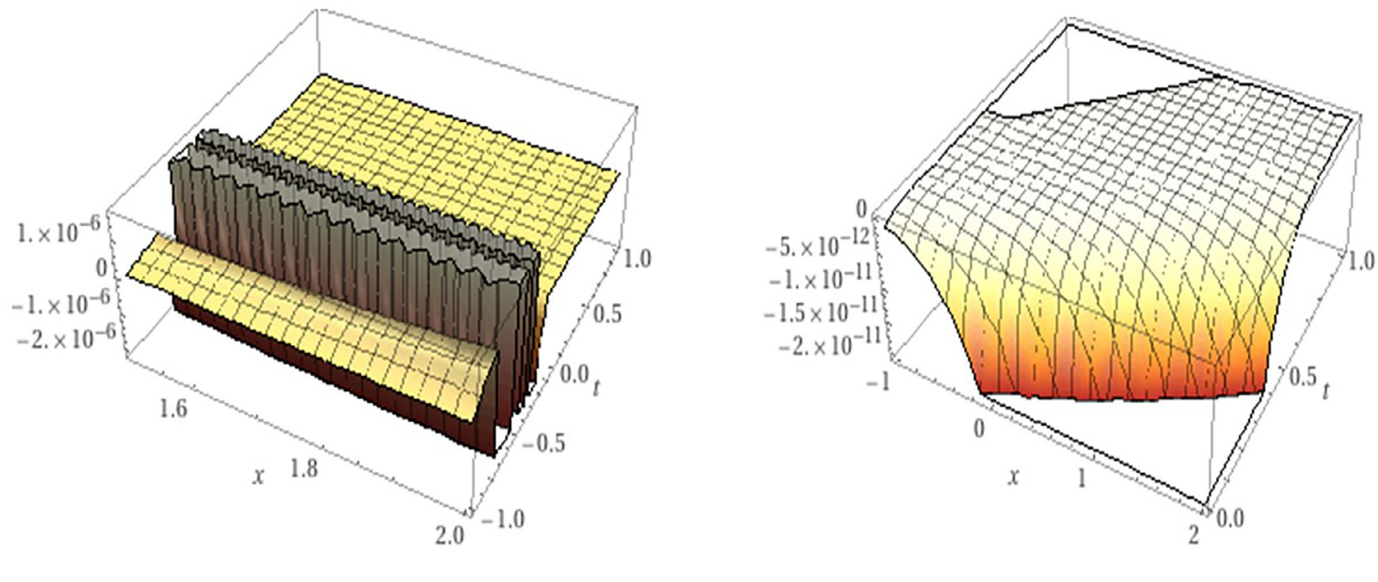
The SWSs of the studied Kaup-Boussinesq system with beta derivative are expressed as:

$$\Omega_4(x,t)=\frac{\lambda^3 c_1 e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{c_1\lambda e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}-\left(-\frac{\lambda^2 c_1 e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{c_1\lambda e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}\right)^2, \tag{24}$$

and

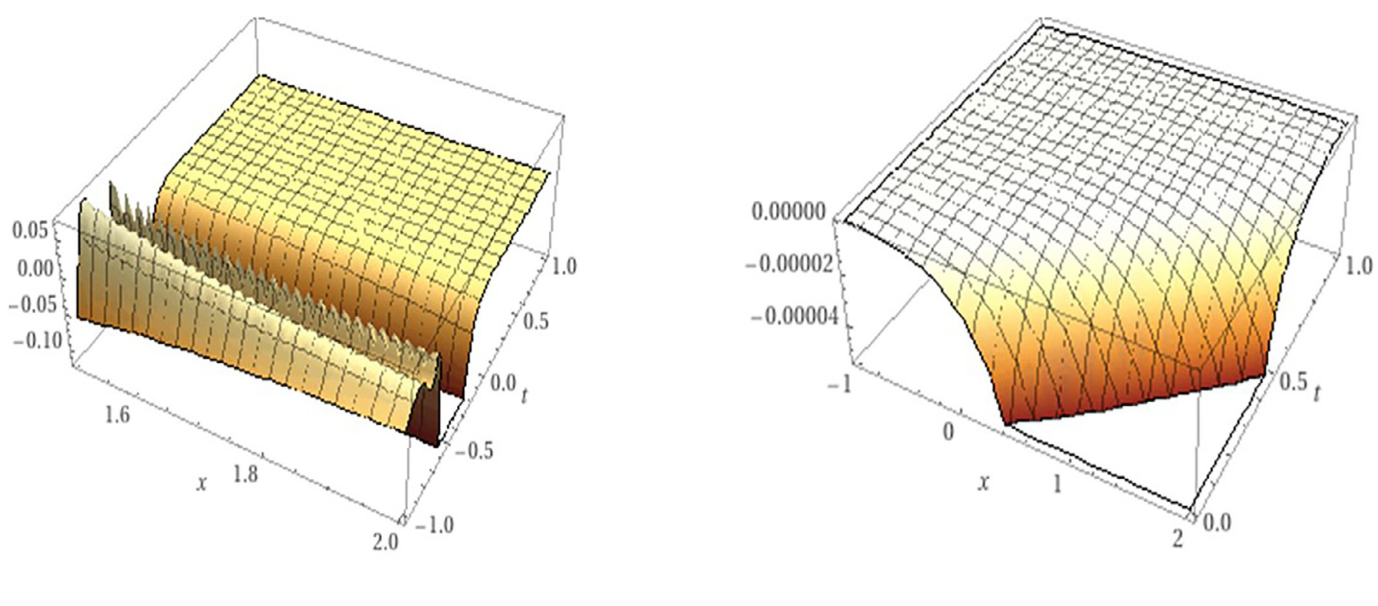
$$\varphi_4(x,t)=-\frac{\lambda^2 c_1 e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{c_1\lambda e^{-\lambda\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}+a_2}, \tag{25}$$

where c_1 is an arbitrary constant. Fig. 9 shows (a) $\varphi_4(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 0.5<x<1; -2<t<2$, (b) $\varphi_4(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 0.5<x<1; -2<t<2$, and (c) $\varphi_4(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 0.5<x<1; -2<t<2$.



a **b**

Fig. 5. 3D Plot of $\Omega_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.25; 1.5 < x < 2; -1 < t < 1$ in (a) and $\Omega_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.25; -1 < x < 2; 0 < t < 1$ in (b).



a **b**

Fig. 6. 3D Plot of $\Omega_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.50; 1.5 < x < 2; -1 < t < 1$ in (a) and $\Omega_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.50; -1 < x < 2; 0 < t < 1$ in (b).

4.2. The generalized Hirota-Satsuma coupled KdV system with beta time fractional derivative

Hirota and Satsuma proposed the Hirota-Satsuma system of equations where two long waves' interaction with different dispersion relations was described[64].

$$D_t^{\beta}\Omega = \frac{1}{4}\Omega_{xxx} + 3\Omega\Omega_x + 3\left(-\varphi^2 + \mu\right)_x$$

$$D_t^{\beta}\varphi = -\frac{1}{2}\varphi_{xxx} - 3\Omega\varphi_x,$$

$$D_t^{\beta}\mu = -\frac{1}{2}\mu_{xxx} - 3\Omega\mu_x \tag{26}$$

If $\mu = 0$, then Eq. (26) is reduced to the Hirota-Satsuma coupled KdV equation. Saberi and Hejazi have dealt this system with Rieman-Liouville sense and they founded lie symmetry analysis, conservation laws and exact solutions [65]. Liu and Chen have

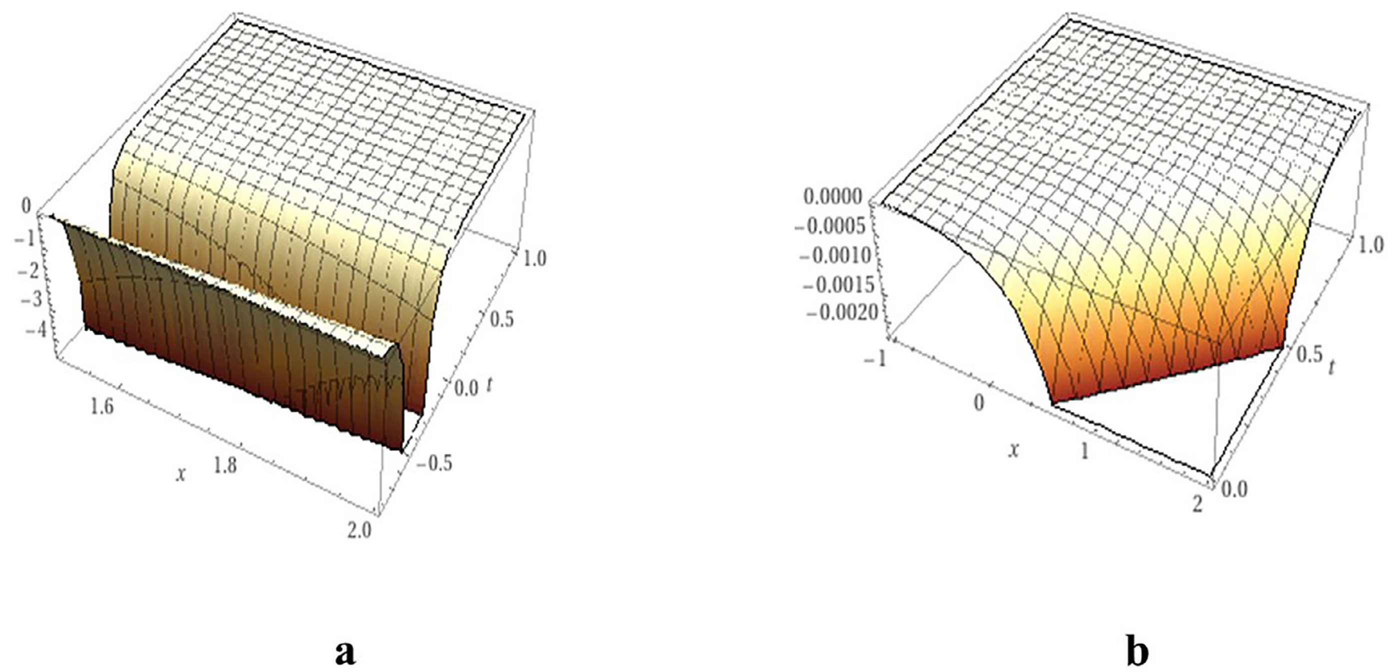


Fig. 7. 3D Plot of $\Omega_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.75; 1.5 < x < 2; -1 < t < 1$ in (a) and $\Omega_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.75; -1 < x < 2; 0 < t < 1$ in (b).

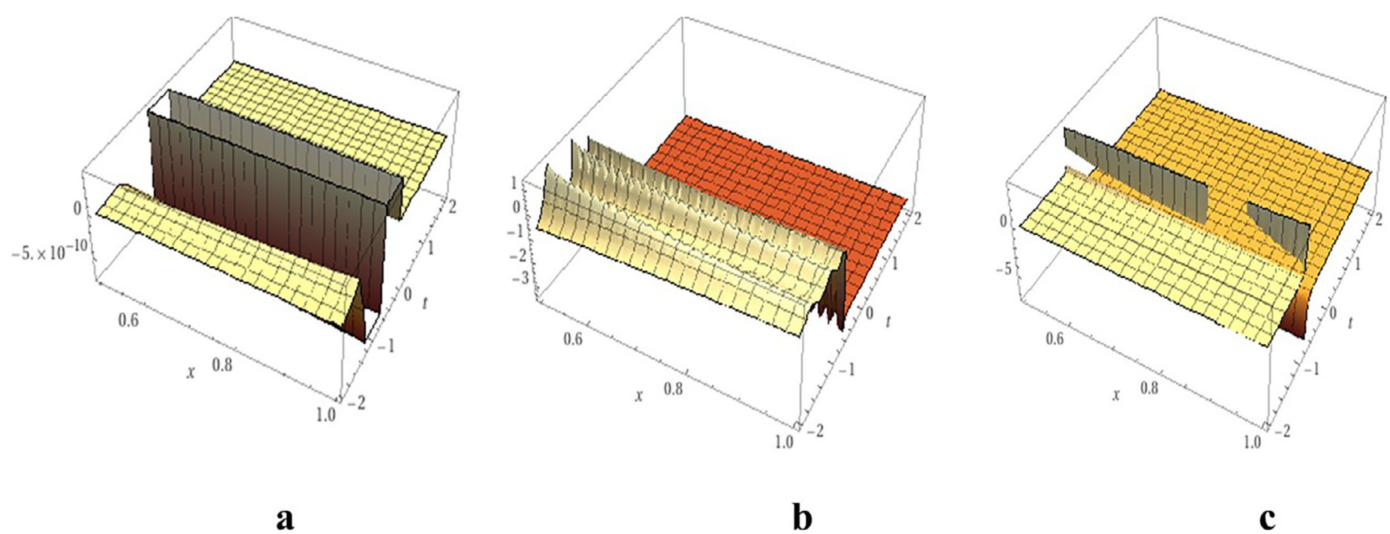


Fig. 8. 3D Plot of (a) $\varphi_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.25; 1.5 < x < 2; -1 < t < 1$, (b) $\varphi_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.50; 1.5 < x < 2; -1 < t < 1$, and (c) $\varphi_3(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.75; 1.5 < x < 2; -1 < t < 1$.

considered this system by the functional variable method [66]. Kurt et al. applied sub-equation method and dummyTXdummy–power series method for this system [67]. Also, H. Yépez-Martínez and J. F. Gómez-Aguilar have considered conformable form of this system to find exact solutions [68].

$$\Omega(x,t) = \frac{1}{\lambda}\vartheta(\zeta)^2$$
$$\varphi(x,t) = -\lambda + \varphi(\zeta)$$
$$\mu(x,t) = 2\lambda^2 - 2\lambda\vartheta(\zeta)$$
$$\zeta = x - \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} \tag{27}$$

where λ is a constant.

$$\lambda\vartheta'' + 2\vartheta^3 - 2\lambda^2\vartheta = 0, \tag{28}$$

where ϑ'' denotes $\frac{d^2\vartheta}{d\zeta^2}$. Let us now balance the linear term of highest order ϑ'' with the highest order nonlinear term ϑ^3 from Eq. (28) yields $M = 1$.

$$\Phi^6(\zeta) : 2a_2^3 + 8\lambda c a_2 = 0,$$
$$\Phi^5(\zeta) : 6a_1a_2^2 + 3\lambda c a_1 = 0,$$
$$\Phi^4(\zeta) : 6a_0a_2^2 + 6a_1^2a_2 + 6\lambda b a_2 = 0,$$
$$\Phi^3(\zeta) : 2\lambda b a_1 + 12a_0a_1a_2 + 2a_1^3 = 0,$$
$$\Phi^2(\zeta) : 4\lambda a a_2 - 2\lambda^2 a_2 + 6a_0^2a_2 + 6a_0a_1^2 = 0,$$
$$\Phi^1(\zeta) : \lambda a a_1 - 2\lambda^2 a_1 + 6a_0^2a_1 = 0,$$
$$\Phi^0(\zeta) : 2a_0^3 - 2\lambda^2 a_0 = 0.$$

With the help of Maple software, the algebraic equation system is solved as follows:

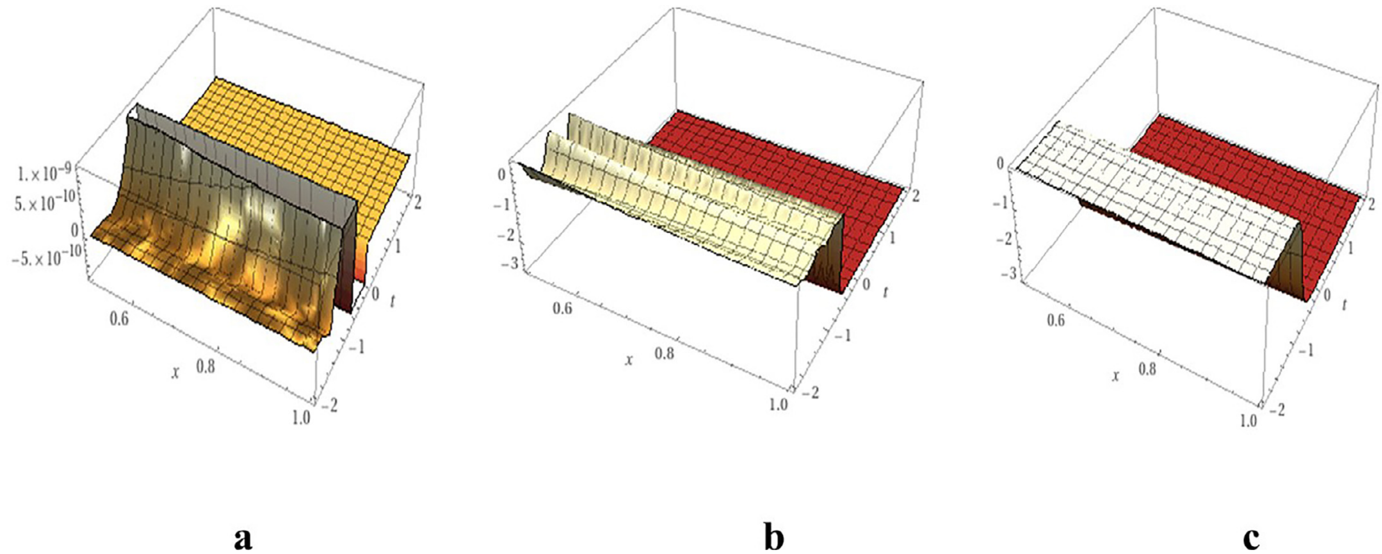


Fig. 9. 3D Plot of (a) $\varphi_4(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.25; 0.5<x<1; -2<t<2$, (b) $\varphi_4(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.50; 0.5<x<1; -2<t<2$, and (c) $\varphi_4(x,t)$ when $\lambda=3; a_2=1; c_1=0.25; \beta=0.75; 0.5<x<1; -2<t<2$.

Case 1.1:

$$a=2\lambda, b=-\frac{a_1^2}{\lambda}, c=0,$$
$$a_0=0, a_2=0 \tag{29}$$

If we substitute Eq. (29) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=\frac{4\lambda e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}(\zeta+c_1)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}\zeta}{\lambda}\right)}} \tag{30}$$

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t)=\frac{4a_1\lambda e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}}, \tag{31}$$

$$\varphi(x,t)=\frac{4a_1\lambda e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}}-\lambda, \tag{32}$$

$$\mu(x,t)=2\lambda^2-2\lambda\frac{4a_1\lambda e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}}, \tag{33}$$

where c_1 is an arbitrary constant.

Case 1.2: If we substitute Eq. (29) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=-\frac{4\lambda e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}(\zeta+c_1)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{2\lambda^3}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}\xi}{\lambda}\right)}} \tag{34}$$

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t)=-\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{2\lambda^3}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}}, \tag{35}$$

$$\varphi(x,t)=-\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{2\lambda^3}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}}-\lambda, \tag{36}$$

$$\mu(x,t)=2\lambda^2+2\lambda\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{2\lambda^3}c_1}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}}, \tag{37}$$

where c_1 is an arbitrary constant.

Case 1.3: If we substitute Eq. (29) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=\frac{4\lambda e^{\left(\frac{\sqrt{2\lambda^3}(\xi+c_1)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}\zeta}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}} \tag{38}$$

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t)=\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}}, \tag{39}$$

$$\varphi(x,t)=\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}}-\lambda, \tag{40}$$

$$\mu(x,t)=2\lambda^2-2\lambda\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}}, \tag{41}$$

where c_1 is an arbitrary constant.

Case 1.4: If we substitute Eq. (29) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=-\frac{4\lambda e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}(\zeta+c_1)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\lambda t)}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}} \tag{42}$$

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t)=-\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}}, \tag{43}$$

$$\varphi(x,t)=-\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}}-\lambda, \tag{44}$$

$$\mu(x,t)=2\lambda^2+2\lambda\frac{4\lambda a_1 e^{\left(\frac{\sqrt{\lambda^3}\sqrt{2}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}+c_1\right)}{\lambda}\right)}\left(-\lambda^2+\sqrt{\lambda^3}\sqrt{\lambda}\right)}{e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})}{\lambda}\right)}+8\lambda^4a_1^2e^{\left(\frac{2\sqrt{\lambda^3}\sqrt{2}c_1}{\lambda}\right)}}, \tag{45}$$

where c_1 is an arbitrary constant.

Case 2.1:

$$a=-\lambda,\ c=-\frac{b^2}{4\lambda},\ a_0=\lambda,\\ a_1=0,\ a_2=-b \tag{46}$$

If we substitute Eq. (46) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta)=e^{\left(\sqrt{-\lambda}(-\zeta+c_1)\right)}\sqrt{-\frac{2\lambda e^{\left(2\sqrt{-\lambda}\zeta\right)}}{e^{\left(2\sqrt{-\lambda}\xi\right)}-be^{\left(2\sqrt{-\lambda}c_1\right)}}} \tag{47}$$

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t)=\frac{\lambda\left(e^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)}+be^{\left(2\sqrt{-\lambda}c_1\right)}\right)}{e^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)}-be^{\left(2\sqrt{-\lambda}c_1\right)}}, \tag{48}$$

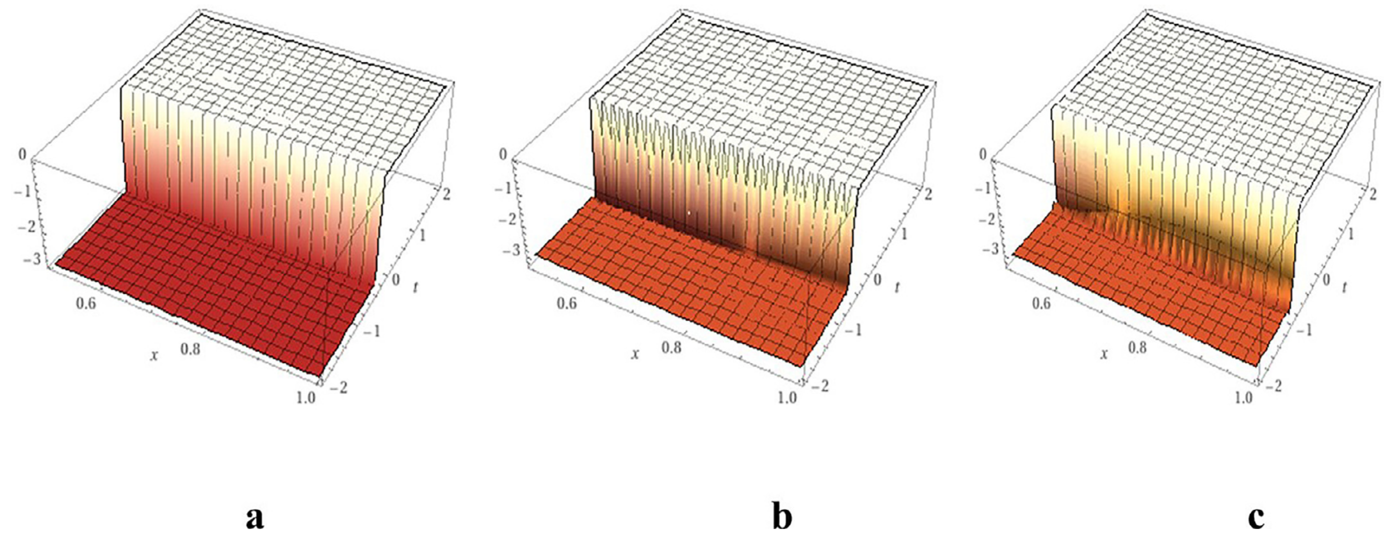


Fig. 10. 3D Plot of (a) $\Omega(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$, (b) $\Omega(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$, and (c) $\Omega(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$.

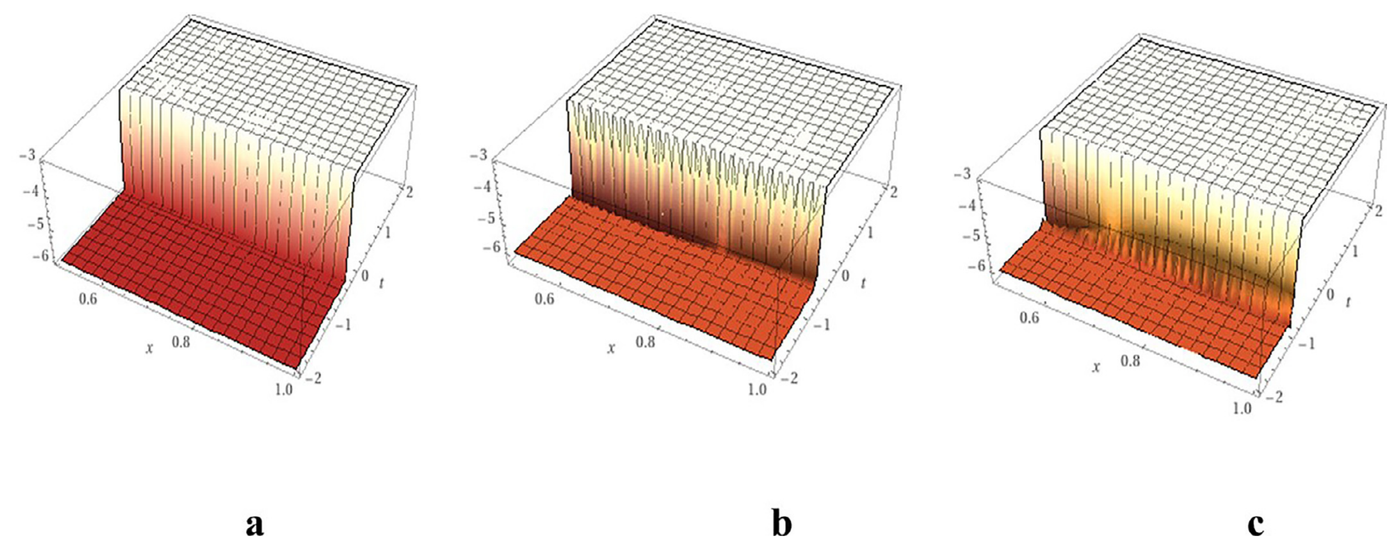


Fig. 11. 3D Plot of (a) $\varphi(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$, (b) $\varphi(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 0.25; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$, and (c) $\varphi(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$.

$$\varphi(x,t) = \frac{\lambda\left(e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)} + be^{\left(2\sqrt{-\lambda}c_1\right)}\right)}{e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)} - be^{\left(2\sqrt{-\lambda}c_1\right)}} - \lambda, \tag{49}$$

$$\mu(x,t) = 2\lambda^2 - 2\lambda\frac{\lambda\left(e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)} + be^{\left(2\sqrt{-\lambda}c_1\right)}\right)}{e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)} - be^{\left(2\sqrt{-\lambda}c_1\right)}}, \tag{50}$$

where c_1 is an arbitrary constant. Fig. 10 shows (a) $\Omega(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$, (b) $\Omega(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$, and (c) $\Omega(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$. Fig. 11 shows (a) $\varphi(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$, (b) $\varphi(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$, and (c) $\varphi(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$. Fig. 12 shows (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$. Fig. 13 shows (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$. Fig. 14 shows (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$.

Case 2.2: If we substitute Eq. (46) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta) = \sqrt{2}\sqrt{\frac{\lambda e^{\left(2\sqrt{-\lambda}\zeta\right)}}{-e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\zeta\right)}}} \tag{51}$$

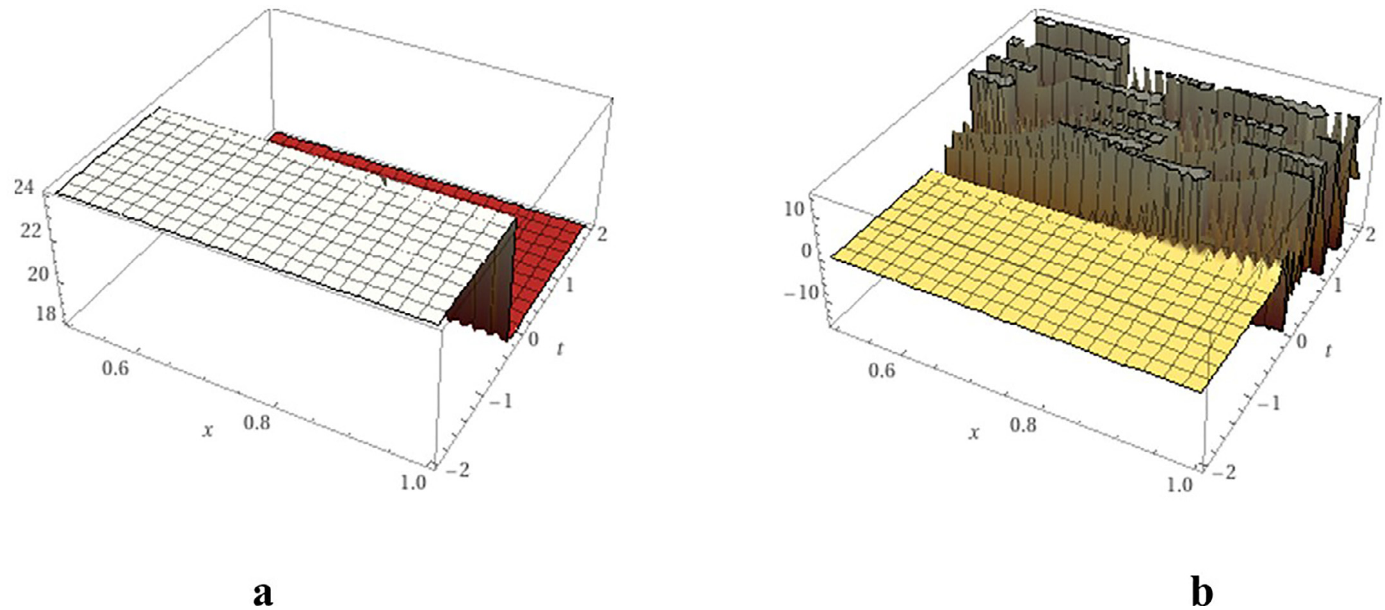


Fig. 12. 3D Plot of (a) the real part of $\mu(x,t)$ when $\lambda=3; b=-1; a_2=1; c_1=1; \beta=0.25; 0.5<x<1; -2<t<2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda=3; b=-1; a_2=1; c_1=1; \beta=0.25; 0.5<x<1; -2<t<2$.

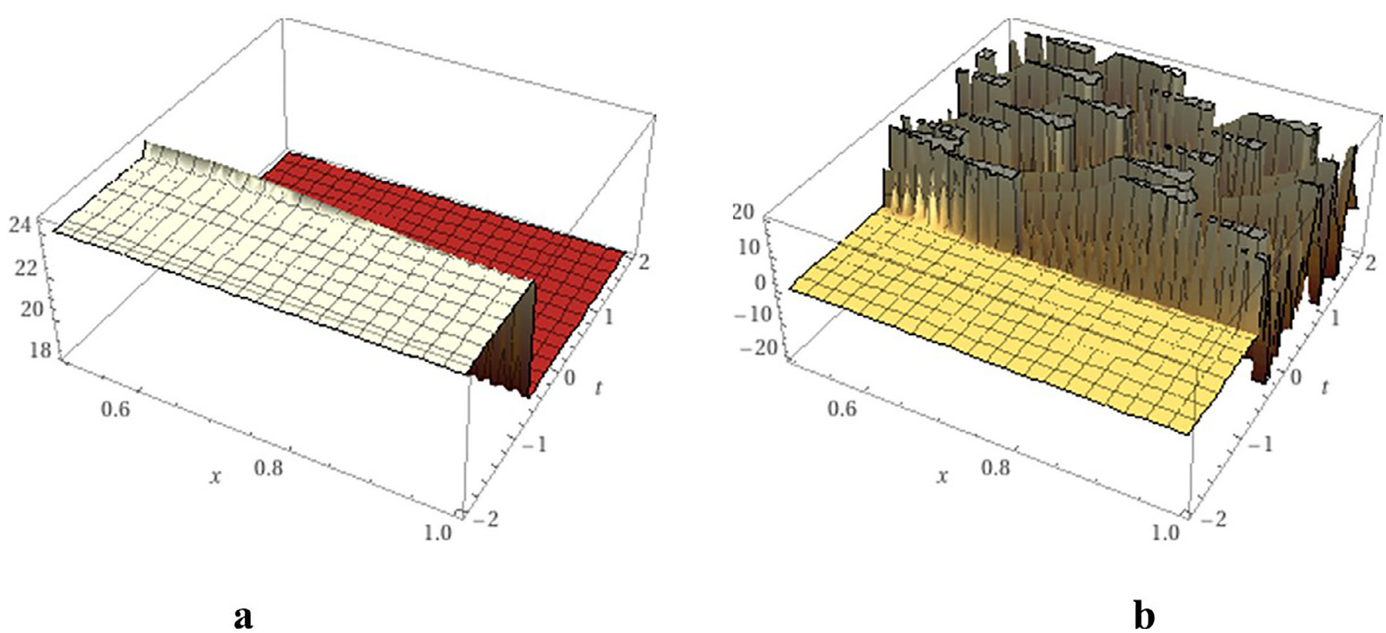


Fig. 13. 3D Plot of (a) the real part of $\mu(x,t)$ when $\lambda=3; b=-1; a_2=1; c_1=1; \beta=0.50; 0.5<x<1; -2<t<2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda=3; a_2=1; c_1=1; \beta=0.50; 0.5<x<1; -2<t<2$.

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t)=-\frac{\lambda(e^{\left(2\sqrt{-\lambda}c_1\right)}+be^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)})}{-e^{\left(2\sqrt{-\lambda}c_1\right)}+be^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)}}, \tag{52}$$

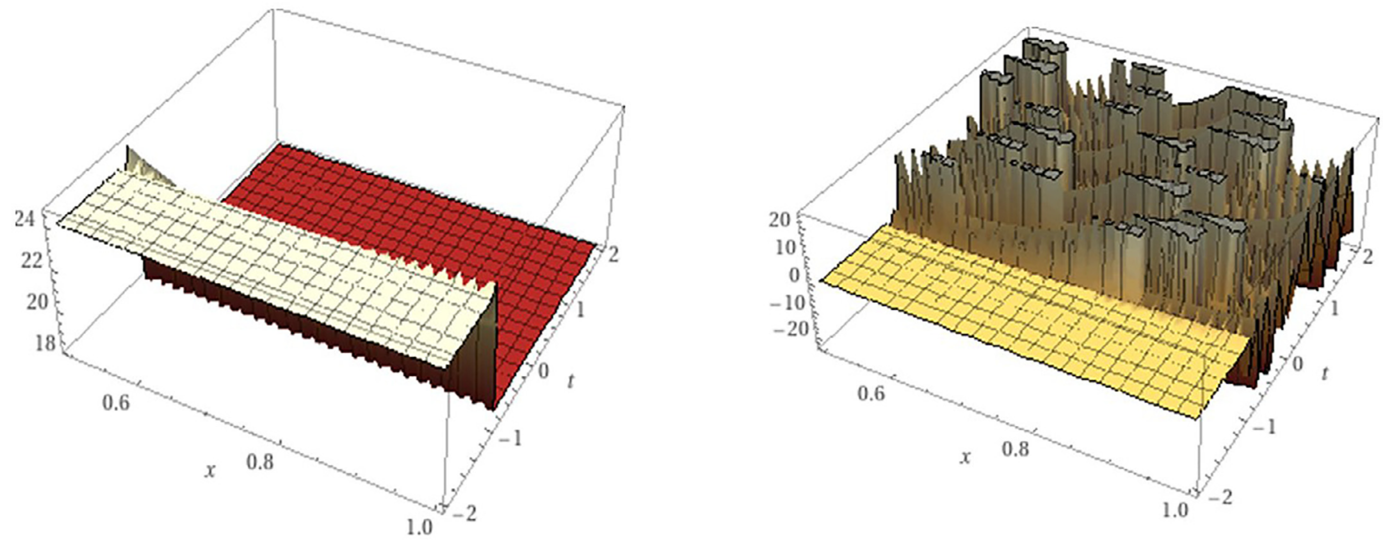
$$\varphi(x,t)=-\frac{\lambda(e^{\left(2\sqrt{-\lambda}c_1\right)}+be^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)})}{-e^{\left(2\sqrt{-\lambda}c_1\right)}+be^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)}}-\lambda \tag{53}$$

$$\mu(x,t)=2\lambda^2+2\lambda\frac{\lambda(e^{\left(2\sqrt{-\lambda}c_1\right)}+be^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)})}{-e^{\left(2\sqrt{-\lambda}c_1\right)}+be^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)}} \tag{54}$$

where c_1 is an arbitrary constant.

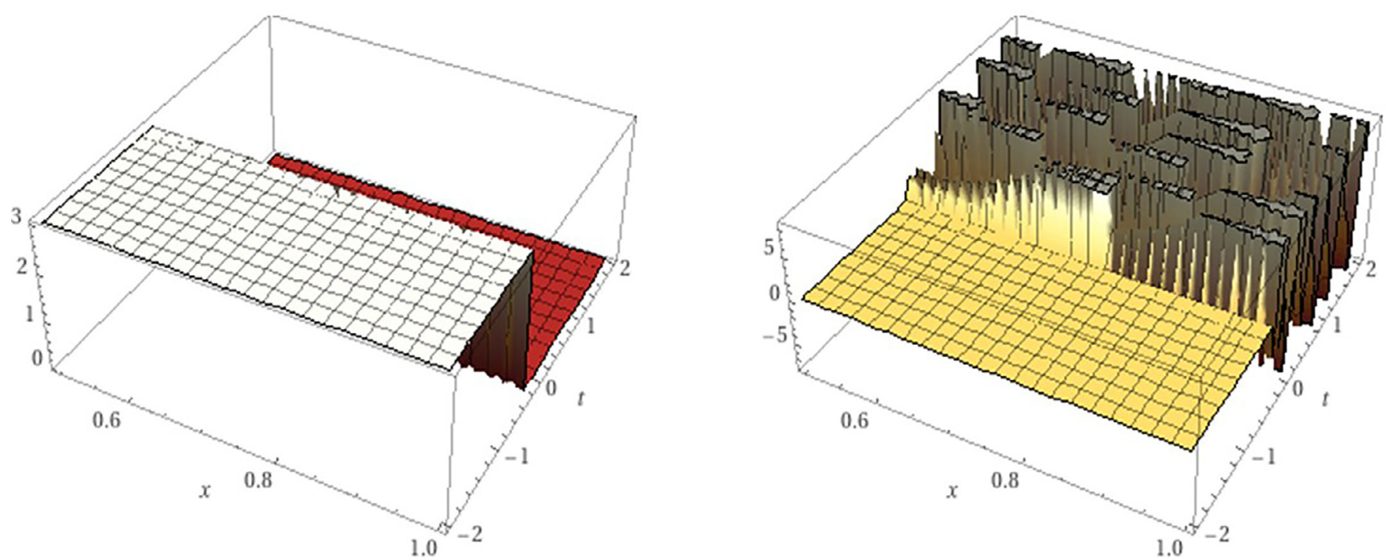
Case 3.1:

$$a=-\lambda, c=-\frac{b^2}{4\lambda}, a_0=-\lambda,$$
$$a_1=0, a_2=b. \tag{55}$$



a **b**

Fig. 14. 3D Plot of (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = -1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$.



a **b**

Fig. 15. 3D Plot of (a) the real part of $\Omega(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\Omega(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$.

If we substitute Eq. (55) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta) = e^{\left(\sqrt{-\lambda}(-\zeta+c_1)\right)}\sqrt{2}\sqrt{\frac{\lambda e^{\left(2\sqrt{-\lambda}\zeta\right)}}{-e^{\left(2\sqrt{-\lambda}\zeta\right)} + be^{\left(2\sqrt{-\lambda}c_1\right)}}} \tag{56}$$

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t) = \frac{\lambda(e^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)} + be^{\left(2\sqrt{-\lambda}c_1\right)})}{-e^{\left(2\sqrt{-\lambda}(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta})\right)} + be^{\left(2\sqrt{-\lambda}c_1\right)}} \tag{57}$$

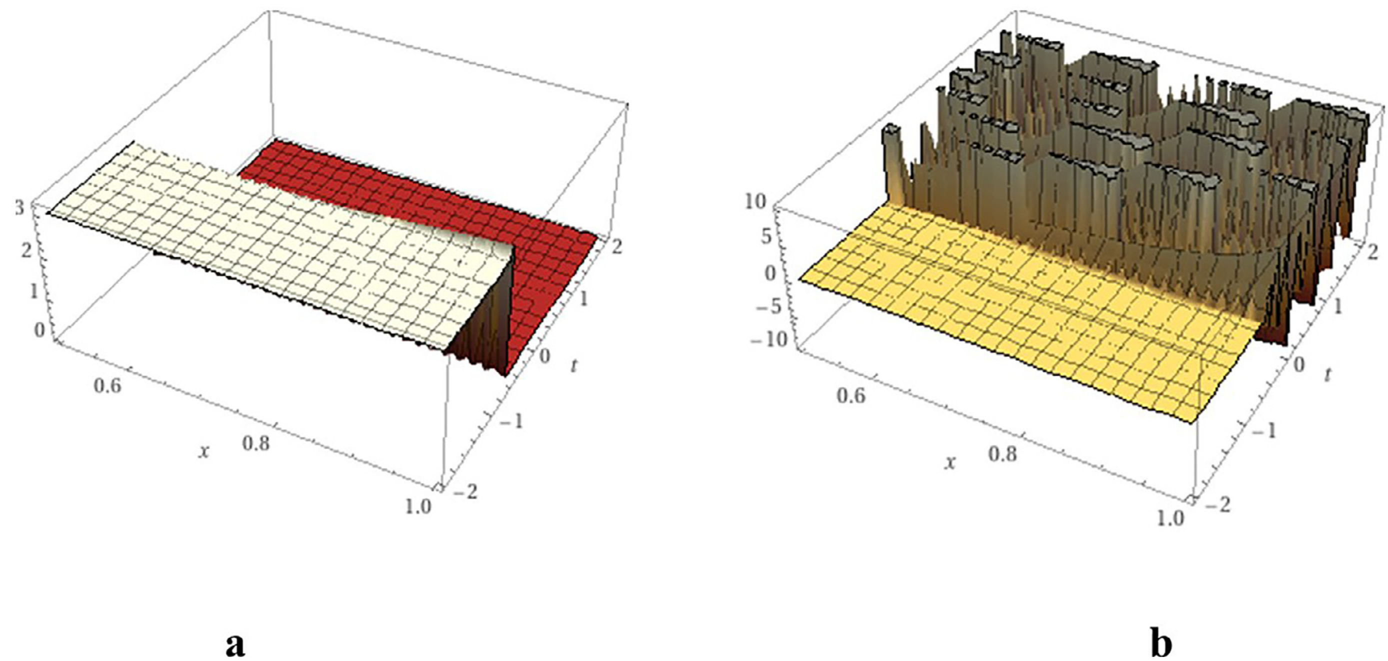


Fig. 16. 3D Plot of (a) the real part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.50; 0.5<x<1; -2<t<2$ and (b) the imaginary part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.50; 0.5<x<1; -2<t<2$.

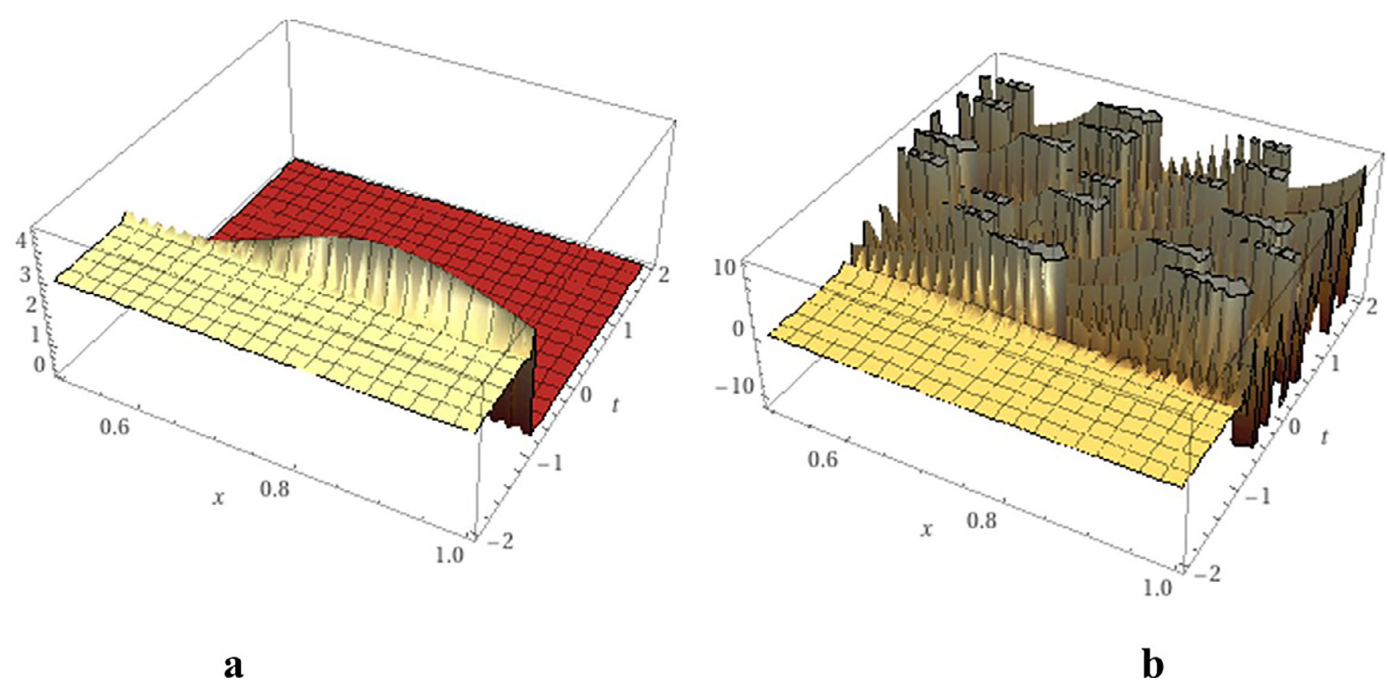


Fig. 17. 3D Plot of (a) the real part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.75; 0.5<x<1; -2<t<2$ and (b) the imaginary part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.75; 0.5<x<1; -2<t<2$.

$$\varphi(x,t)=\frac{\lambda\left(e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}+be^{\left(2\sqrt{-\lambda}c_1\right)}\right)}{-e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}+be^{\left(2\sqrt{-\lambda}c_1\right)}}-\lambda \tag{58}$$

$$\mu(x,t)=2\lambda^2-2\lambda\frac{\lambda\left(e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}+be^{\left(2\sqrt{-\lambda}c_1\right)}\right)}{-e^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}+be^{\left(2\sqrt{-\lambda}c_1\right)}} \tag{59}$$

where c_1 is an arbitrary constant. Fig. 15 shows (a) the real part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.25; 0.5<x<1; -2<t<2$ and (b) the imaginary part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.25; 0.5<x<1; -2<t<2$. Fig. 16 shows (a) the real part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.50; 0.5<x<1; -2<t<2$ and (b) the imaginary part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.50; 0.5<x<1; -2<t<2$. Fig. 17 shows (a) the real part of $\Omega(x,t)$ when $\lambda=3; b=1; a_2=1; c_1=1; \beta=0.75; 0.5<x<1; -2<t<2$

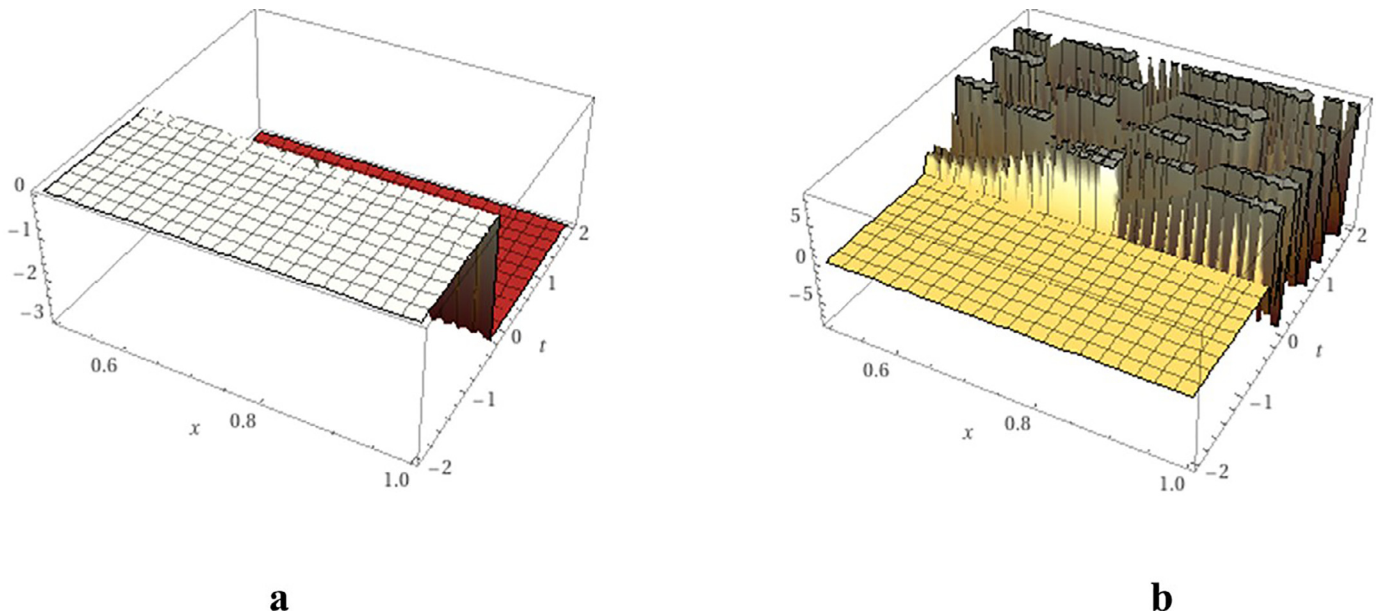


Fig. 18. 3D Plot of (a) the real part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$.

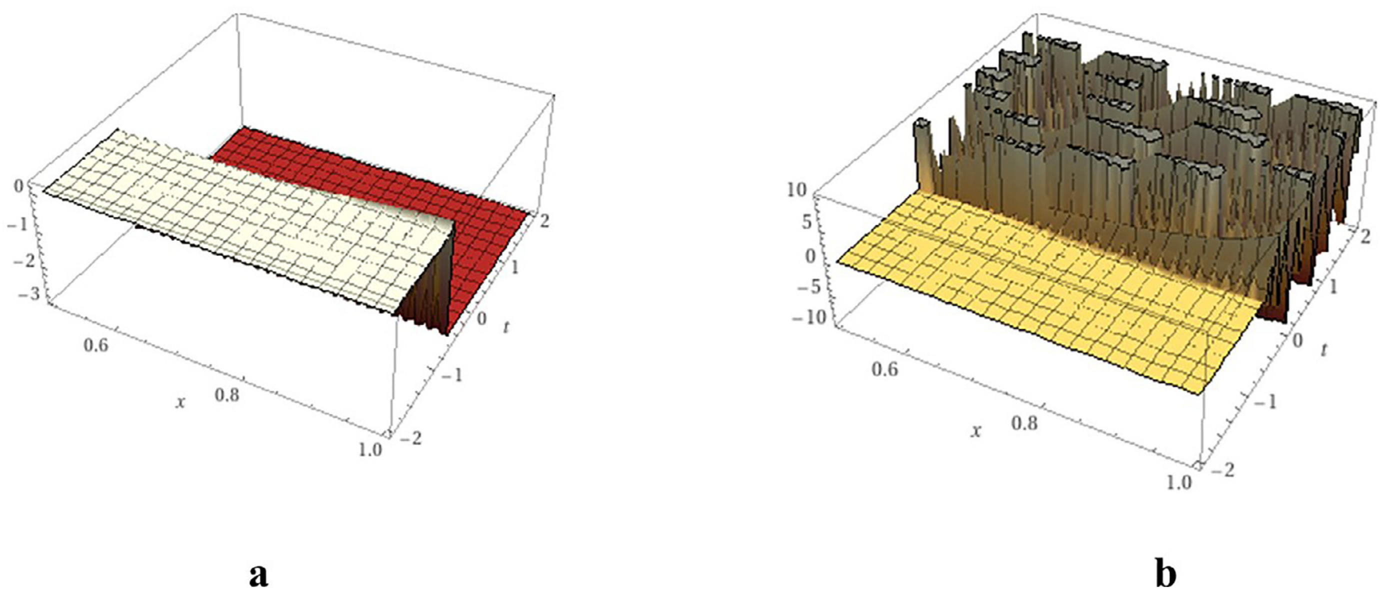
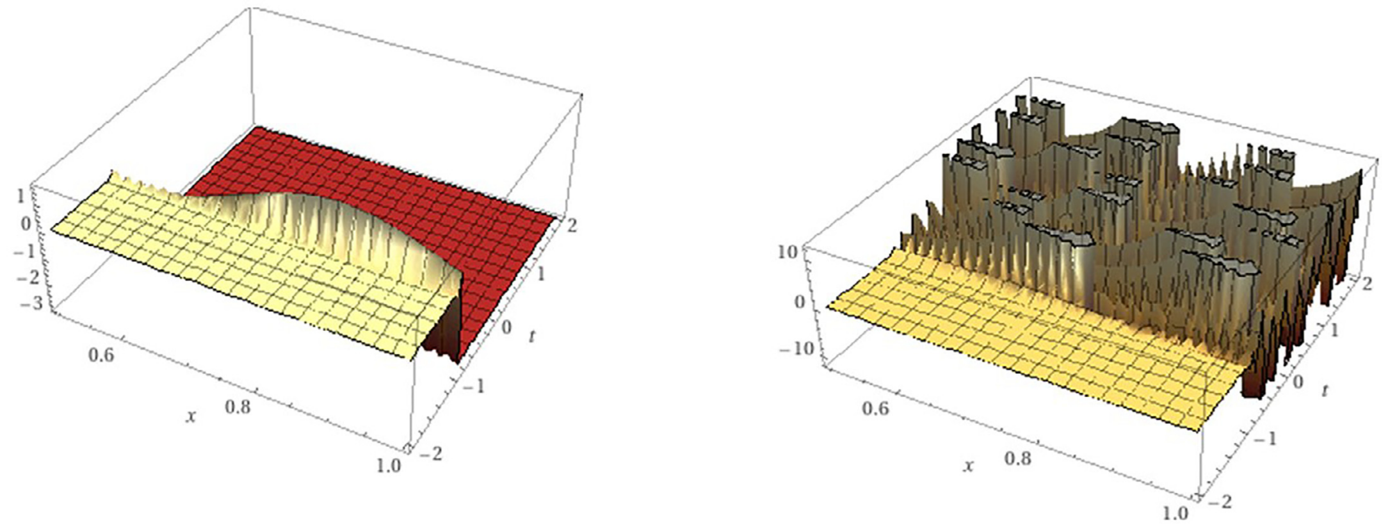


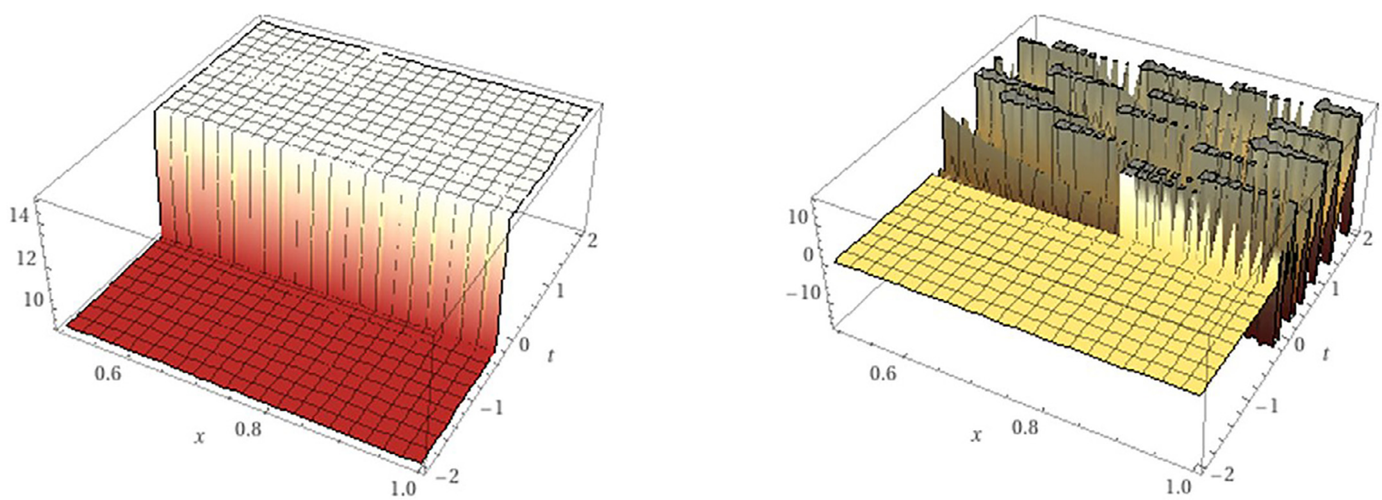
Fig. 19. 3D Plot of (a) the real part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$.

and (b) the imaginary part of $\Omega(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$. Fig. 18 shows (a) the real part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$. Fig. 19 shows (a) the real part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$. Fig. 20 shows (a) the real part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$. Fig. 21 shows (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$. Fig. 22 shows (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$. Fig. 23 shows (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$.



a b

Fig. 20. 3D Plot of (a) the real part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\varphi(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$.



a b

Fig. 21. 3D Plot of (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.25; 0.5 < x < 1; -2 < t < 2$.

Case 3.2: If we substitute Eq. (55) into Eq. (5) and then solve the obtained equation, we find the auxiliary equation as follows:

$$\Phi(\zeta) = \sqrt{\frac{2\lambda e^{\left(2\sqrt{-\lambda}\zeta\right)}}{-e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\zeta\right)}}} \tag{60}$$

The SWSs of the generalized Hirota-Satsuma coupled KdV system can be founded as follows:

$$\Omega(x,t) = \frac{\lambda\left(e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}\right)}{-e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}}, \tag{61}$$

$$\varphi(x,t) = \frac{\lambda\left(e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}\right)}{-e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}} - \lambda, \tag{62}$$

$$\mu(x,t) = 2\lambda^2 - 2\lambda\frac{\lambda\left(e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}\right)}{-e^{\left(2\sqrt{-\lambda}c_1\right)} + be^{\left(2\sqrt{-\lambda}\left(x-\frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)}}, \tag{63}$$

where c_1 is an arbitrary constant.

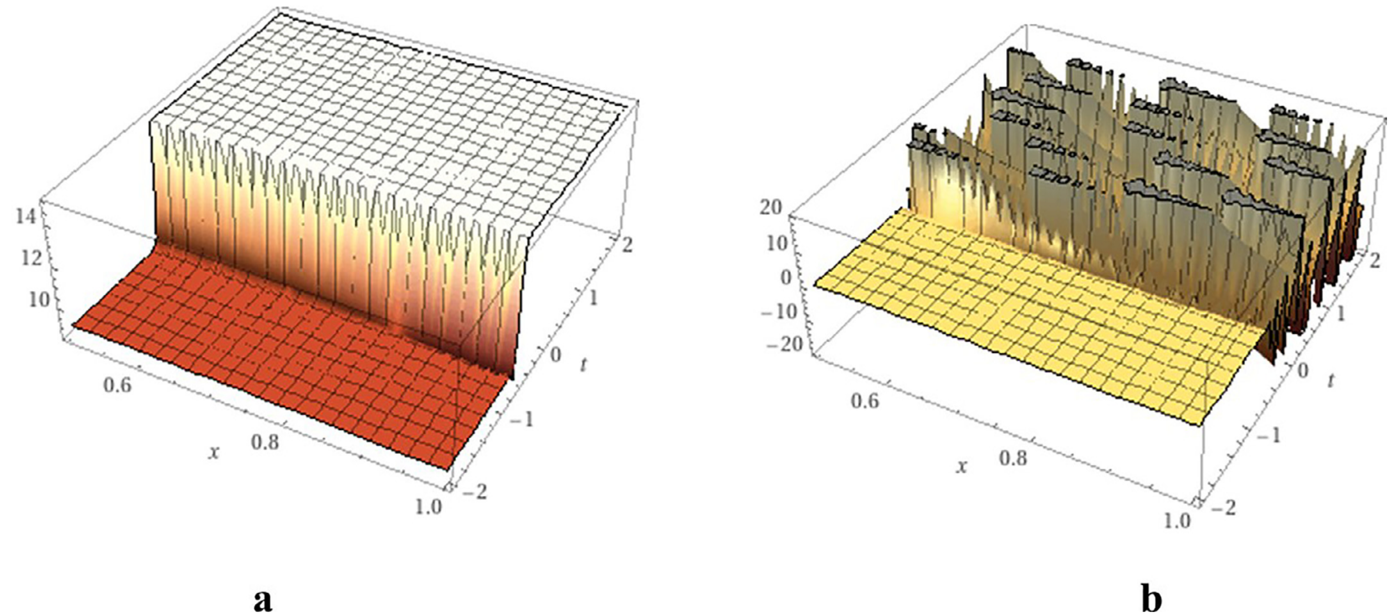


Fig. 22. 3D Plot of (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.50; 0.5 < x < 1; -2 < t < 2$.

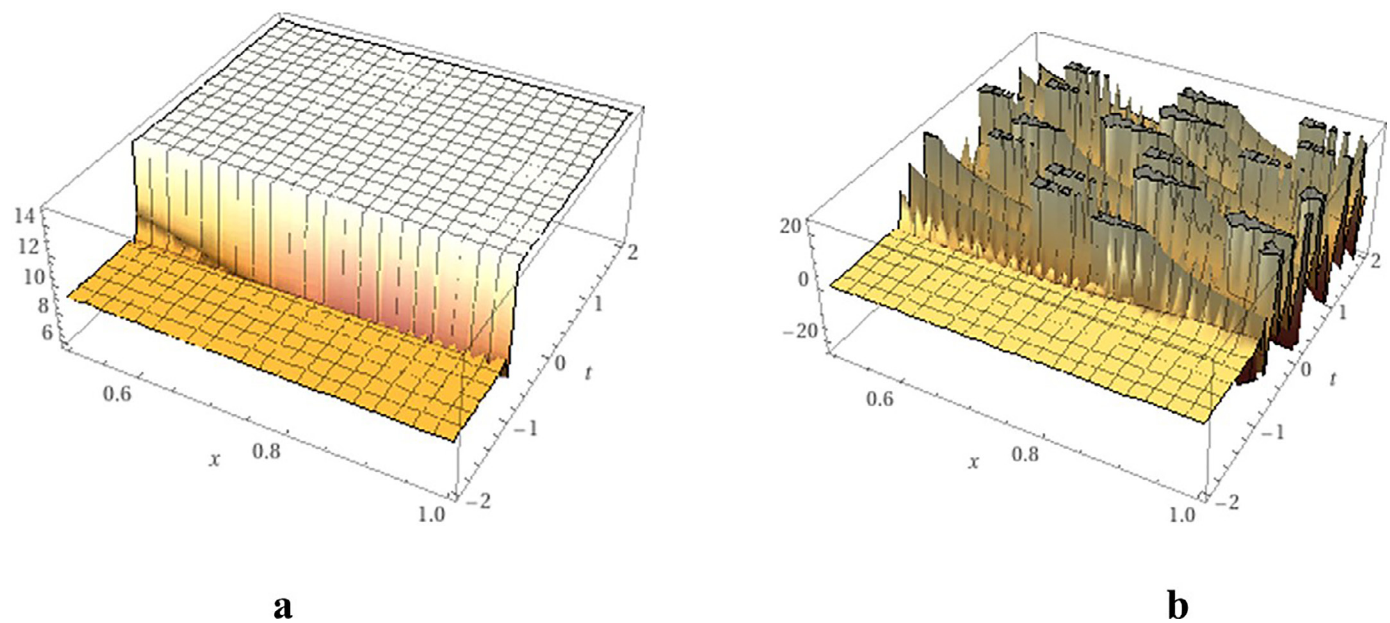


Fig. 23. 3D Plot of (a) the real part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$ and (b) the imaginary part of $\mu(x,t)$ when $\lambda = 3; b = 1; a_2 = 1; c_1 = 1; \beta = 0.75; 0.5 < x < 1; -2 < t < 2$.

5. Conclusion

The auxiliary equation method has been successfully applied to solve two different nonlinear fractional differential equation systems with beta derivative. Therefore, we have showed that this technique can be applicable to the fractional order nonlinear systems. According to all our new results, this technique is considered direct, standard, and computerizable where complicated algebraic calculations can be easily done with the help of this technique. All solutions in this work have been obtained using MAPLE software. This technique can be further extended to solve various nonlinear fractional differential equations that arise from mathematical physics, solitons' theory, and multidisciplinary sciences. Since we have not used a standard equation as an auxiliary equation for this technique, this technique has the potential to give different and fresh solutions.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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