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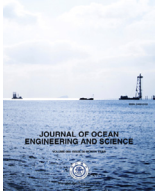

A new computational approach to the fractional-order Liouville equation arising from mechanics of water waves and meteorological forecasts

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ABSTRACT

The current analysis employs the improved F-expansion, modified extended tanh, exponential rational function, and $(g')-$expansion procedures to find a divergent collection of the fractional Liouville equation's exact solutions in the context of beta-derivative. Also, we have given the graphical representations of the obtained results. These plots are useful in describing the dynamic characteristics of the solutions. The investigated equation is very essential in studying the mechanics of water waves and atmospheric predictability. Therefore, our techniques provide a great help in investigating various nonlinear models formulated in the contexts of fractional derivatives arising from oceanography and mathematical physics.

1. Introduction

The mathematical modeling of natural phenomena is crucial to analyzing the world around us. For this reason, we mostly use nonlinear fractional differential equations (NFDEs). These equations arise in numerous subjects of mathematical physics and complex engineering problems, such as viscoelasticity, electroanalytical chemistry, biological diseases, population dynamics, control theory, fluid mechanics, clinical medicine, optics, biological population models, signals processing, and electromagnetics. Many physical laws are defined more precisely in terms of differential equations of arbitrary fractional order. These equations are a generalization of differential equations acquired by the application of fractional operators. The goal of fractional order arose in 1695 with the harmony linking Leibniz and L'Hospital and carries on nowadays. The predominant difference of fractional calculus from classical calculus is that there is no distinguish identification of derivative as in classical analysis. Since there is more than one identification of derivative in the fractional calculus, it supplies the chance to use the most appropriate identification and verify the appropriate solution to the problem. Despite relations between various sorts of fractional operators, they may vary in the physical meaning [1–4].

There are numerous types of fractional derivatives, such as Caputo, conformable derivative, Riemann-Liouville fractional derivatice, Abu-Shady–Kaabar fractional derivative [5], and many other derivatives in [6,7]. The authors worked on retrieving exact solutions of different equations in sense of these derivatives. Various studies have been conducted on investigating fractional differential equations such as fractional hybrid boundary value problems with three-point integral hybrid conditions [8], fractional differential inclusions [9], fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions [10] (see also [11]), coupled Caputo conformable pantograph equations [12], hybrid Caputo fractional thermostat problem with

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hybrid boundary value conditions [13], p-Laplacian nonperiodic nonlinear boundary value problem [14], fractional Schrödinger equation with second-order spatio-temporal dispersion [15], ψ-Caputo fractional differential equation [16], high order fractional integro-differential equation [17], multi-singular pointwise defined fractional q-integro-differential equation [18] (see also [19,20]). Fractional calculus has been applied in several topics from science and engineering such as modeling HIV-1 infection of CD4+ T-cell [21], transient response of the parallel RCL circuit [22], modeling hearing loss due to Mumps virus [23], worm transmission in wireless sensor network [24], and elastic beam equation [25].

Raza et al. considered the electrical microtubule model with local M-derivative to find soliton solutions [26]. Arshed et al. dealt with the Kraenkel-Manna-Merle (KMM) model with beta-derivative to retrieve singular, dark, periodic solutions, and combo solitons [27]. Kaabar et al. worked on the $(3+1)$ Wazwaz-Benjamin-Bona-Mahony equation, formulated in the context of conformable derivative, to find soliton solutions [28]. Hosseini et al. presented a work on a nonlinear water wave equation with Caputo derivative [29]. Kumar and Baleanu used homotopy analysis transform strategy to the fractional Klein-Gordon equation via Mittag-Leffler-type kernel [30]. Aydogan et al. employed the Adomian decomposition method to the Rabies's mathematical model via the Caputo-Fabrizio derivative [31].

An important equation in mathematical physics is called Liouville equation which plays an essential role in the mechanics of water waves [32] and atmospheric predictability (see [33]) where the studied dynamical system's state is governed by this equation via its probability density function's time evolution [34].

Ma and Lee investigated the $3+1$ dimensional Jimbo–Miwa equation's exact solutions via the transformed rational function method [35] where the known methods, F-expansion and $(g\prime)-$expansion, are special cases of the this approach. The transformed rational function method is an efficient approach in finding the traveling wave and explicit solutions to a Riccati equation where the F-expansion and $(g\prime)-$expansion are satisfied according to Eqs. (40)–(42) in [36]. There are some interesting recent studies concerning N-solitons which have been systematically investigated for the nonlocal integrable equations involving the Riemann-Hilbert problems with nonlocal reverse-time NLS hierarchies, the inverse scattering of nonlocal real reverse-spacetime matrix AKNS hierarchies, and reduced nonlocal untegrable mKdV hierarchies [37–39], respectively.

While there are many interesting oceanographic phenomena that have been recently investigated in recent research works such as the Kaup-Boussinesq and generalized Hirota Satsuma coupled KdV systems in the context of fractional calculus arising from water waves and interaction of long waves [40], the investigation of perturbed nonlinear Schrödinger equation's complex soliton solutions in nonlinear optical fibers [41], the study of the $(2+1)$ dimensional Bogoyavlensky-Konopelchenko equation with variable-coefficient in wave propagation in terms of its Lie symmetry analysis and invariant solutions [42], the study of the solitary waves and exact solutions for the fifth-order nonlinear wave equation [43], and the study of generalized Schrödinger-Boussinesq equations for the interaction between complex short wave and real long wave envelope [44], and the investigation of Mikhailov-Novikov-Wang equation [45], our results in this work are unique and novel because according to the best of our knowledge, the exact solutions of fractional Liouville equation, formulated in the sense of beta-derivative, have never been investigated before in other research studies where the investigated equation is very important in studying the mechanics of water waves and atmospheric predictability which are essential in explaining many other related phenomena in the field of ocean science and engineering.

The paper is presented as: We review some fundamental preliminaries. Then, we describe the employed techniques, particularly the improved F-expansion, modified extended tanh-function, exponential rational function, and $(g\prime)$-expansion procedures in Section 3. The extraction of exact solutions for the proposed models and graphical illustrations of the obtained results are provided in Section 4. Finally, the whole research's conclusion is mentioned in Section 5.

2. Fundamental preliminaries of beta derivative

The β derivative with some basic properties are discussed in this section.

Definition. Assume that a given function: $\varsigma(t)$ that is defined for all non-negative t. Then, the β derivative of $f(t)$ of order β is expressed as:

$$T^{\beta}(\varsigma(t)) = \lim_{\varepsilon\to 0}\frac{\varsigma\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-\varsigma(t)}{\varepsilon}, 0<\beta\le 1,$$

and $T^{\beta}(\varsigma(t)) = \frac{d^{\beta}\varsigma(t)}{dt^{\beta}}$.

Some properties are listed in the following theorem for β derivative as:

Theorem. *Assume that* $\varsigma(t)$ *and* $\upsilon(t)$ *be* β *-differentiable functions for all* $t>0$ *and* $\beta\epsilon(0,1]$. *Some basic properties are discussed as follows:*

1. $T^{\beta}(a\varsigma(t)+b\upsilon(t)) = aT^{\beta}(\varsigma(t))+bT^{\beta}(\upsilon(t)), \forall a,b\in R$
2. $T^{\beta}(\varsigma(t)\upsilon(t)) = \upsilon(t)T^{\beta}(\varsigma(t))+\varsigma(t)T^{\beta}(\upsilon(t)),$
3. $T^{\beta}\left(\frac{\varsigma(t)}{\upsilon(t)}\right) = \frac{\upsilon(t)T^{\beta}(\varsigma(t))-\varsigma(t)T^{\beta}(\upsilon(t))}{\upsilon(t)^2},$
4. $T^{\beta}(\varsigma(t)) = \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{d\varsigma(t)}{dt}.$

3. Description of the methodologies

Given a NFDE with beta derivative as:

$$P(u, D_t^{\alpha}u, D_x^{\beta}u, D_t^{\alpha}D_t^{\alpha}u, D_t^{\alpha}D_x^{\beta}u, D_x^{\beta}D_x^{\beta}u, \ldots) = 0, \quad 0<\alpha,\beta<1, \tag{1}$$

where P is a polynomial of $u(x,t)$ and its partial derivatives.

Employing a wave variable

$$u(x,t) = u(\xi), \xi = kx - \frac{c_1}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}, \tag{2}$$

one may rewrite Eq. (1) as a nonlinear ordinary differential equation (ODE)

$$Q(u, u', u'', \ldots) = 0, \tag{3}$$

where the prime represents the differentiation with respect to ξ. One should integrate Eq. (3) possible times with respect to ξ to begin the application of all procedures [46].

3.1. Improved F-expansion procedure

We may consider that the exact solution of Eq. (3) can be written as:

$$u(\xi) = \sum_{i=0}^{N}\alpha_i(\gamma+\mathcal{F}(\xi))^{i} + \sum_{i=1}^{N}\beta_i(\gamma+\mathcal{F}(\xi))^{-i}, \tag{4}$$

where α_N or β_N can be zero at different times but can not be zero at the same time. In Eq. (4) α_i, β_i, γ are arbitrary constants to be determined later, and N is the balancing number. The balancing number can be calculated by balancing the nonlinear terms and the highest order derivatives in Eq. (3).

We know that the Riccati equation takes the form below:

$$\mathcal{F}^{'}(\xi) = \eta + \mathcal{F}^2(\xi), \tag{5}$$

where η is the real parameter. We will give the Riccati equation's solutions as we know.

1: When $\eta < 0$, solutions are given by

$$\mathcal{F}_1 = -\sqrt{-\eta}\tanh\left(\sqrt{-\eta}\xi\right),$$

$$\mathcal{F}_2 = -\sqrt{-\eta}\coth\left(\sqrt{-\eta}\xi\right),$$

2: For $\eta > 0$, solutions are

$$\mathcal{F}_3 = \sqrt{\eta}\tan\left(\sqrt{\eta}\xi\right),$$

$$\mathcal{F}_4 = -\sqrt{\eta}\cot\left(\sqrt{\eta}\xi\right),$$

3: For $\eta = 0$, solution is

$$\mathcal{F}_5 = -\frac{1}{\xi}.$$

Substituting Eq. (4) in Eq. (3) without ignoring Eq. (5), we verify a polynomial in $(\gamma + \mathcal{F}(\xi))^i$ and $(\gamma + \mathcal{F}(\xi))^{-i}$. Next, each coefficient of the resulted polynomial is collected to 0, so we conclude with a set of algebraic equations of α_i, β_i, γ, η, k and c_1.

From the solutions of the valuesfor α_i, β_i, γ, η, k and c_1 are obtained. We substitute the value of α_i, β_i, γ, η, k and c_1 into Eq. (4) with solutions of Riccati equation, we generally verify new exact solutions of Eq. (1) [47].

3.2. *The modified procedure of extended tanh-function*

The desired solution of Eq. (3) is considered as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i(\phi(\xi))^i + \sum_{j=1}^{N} b_j(\phi(\xi))^{-j}, \tag{6}$$

The constants: $a_i (i = 0, 1, \ldots, N)$, b_j $(j = 1, \ldots, N)$ are needed to be computed, and a_N and b_N can not be zero at the same time. In Eq. (6), N is called the balancing number. If the nonlinear terms and the highest order derivatives in Eq. (3) are balanced, then this term can be easily determined. $\phi(\xi)$ is the following auxiliary equation's solution:

$$\phi^{'}(\xi) = \phi(\xi)^2 + \eta, \tag{7}$$

where η is given as constant. Eq. (7) solutions can be presented in 3 cases as below:

I. When $\eta < 0$, the hyperbolic solution is obtained as:

$$\phi(\xi) = -\sqrt{-\eta}\tanh\left(\sqrt{-\eta}\xi\right) \text{ or } \phi(\xi) = -\sqrt{-\eta}\coth\left(\sqrt{-\eta}\xi\right). \tag{8}$$

II. When $\eta > 0$, the trigonometric solution is obtained as:

$$\phi(\xi) = \sqrt{\eta}\tan\left(\sqrt{\eta}\xi\right) \text{ or } \phi(\xi) = \sqrt{-\eta}\cot\left(\sqrt{\eta}\xi\right). \tag{9}$$

III. When $\eta = 0$, the rational solution is obtained as:

$$\phi(\xi) = -\frac{1}{\xi}. \tag{10}$$

If we substitute Eq. (6) into Eq. (3) associated with Eq. (7) then by making all the coefficients of $\phi^i(\xi)$ to be set to 0, we get an algebraic equation system for η, k, c_1, $a_i (i = 0, \ldots, N)$, b_j $(j = 1, \ldots, N)$. If the found determining equation system is computed via MAPLE software, a variety of exact solutions is obtained [48].

3.3. *The procedure of exponential rational function*

The Eq. (3) solution is assumed as [49,50]:

$$u(\xi) = \sum_{i=0}^{N} \frac{a_i}{(1 + e^{\xi})^i}, \tag{11}$$

where a_N $(a_N \neq 0)$ are constants. N here is balancing number as mentioned previously. We substitute Eq. (11) into Eq. (3) and all terms with the same order of $e^{i\xi}$ $(i = 0, 1, 2, 3, \ldots)$ are collected. From the solutions of these system, we find the desired equation's solutions.

3.4. *The $\left(g'\right)$-expansion procedure*

The Eq. (3) is assumed as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i\left(g'\right)^i, \tag{12}$$

where a_i $(i = 0, \ldots, N)$ are constants to be obtained. N is found by the technique of homogeneous balance. Here, g satisfies the following ODE:

$$g'' = a + bg' + cg'^2. \tag{13}$$

The solutions of Eq. (13) are given by:

1: When $\Delta = 4ac - b^2 < 0$, the solution is expressed:

$$g = \frac{1}{2c}\left[\ln\left(\tanh^2\left(\frac{\xi\sqrt{-\Delta}}{2}\right) - 1\right) - b\xi\right],$$

and

$$g' = \frac{1}{2c}\left[\sqrt{-\Delta}\tanh\left(-\frac{\xi\sqrt{-\Delta}}{2}\right) - b\right]. \tag{14}$$

2: When $\Delta = 4ac - b^2 = 0$, the solution is expressed:

$$g = -\frac{1}{c}\left[\ln(\xi) + \frac{\xi b}{2}\right],$$

and

$$g' = -\frac{1}{c}\left(\frac{1}{\xi} + \frac{b}{2}\right). \tag{15}$$

3: When $\Delta = 4ac - b^2 > 0$, the solution is expressed:

$$g = \frac{1}{2c}\left[\ln\left(\tan^2\left(\frac{\xi\sqrt{\Delta}}{2}\right) + 1\right) - b\xi\right],$$

and

$$g' = \frac{1}{2c}\left[\sqrt{\Delta}\tan\left(\frac{\xi\sqrt{\Delta}}{2}\right) - b\right]. \tag{16}$$

If Eq. (12) is substituted into Eq. (3), with (13) and each coefficient of all powers of $\left(g'\right)^i$ is equated to 0, a determining system is found. The found system includes a_i $(i = 0, \ldots, N), a, b, c, k$ and c_1. If we compute the determining system, the values of a_i and c_1 are found. If we substitute the obtained values into (12) along with Eqs. (14),(15) and (16), we get all possible solutions [51–53].

4. An application: The fractional Liouville equation with beta derivative

The considered procedures are separately implemented in this section to the fractional Liouville equation with beta derivative in the following form,

$$D_t^{\gamma}(u_x) + e^u = 0. \tag{17}$$

We firstly use:

$$u(x,t) = U(\xi), \xi = kx - \frac{c_1}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, v = e^u, \tag{18}$$

to the Eq. (17). This carries an ODE

$$-kc_1 v''v + kc_1(v')^2 + v^3 = 0, \tag{19}$$

Here, the balancing number is founded as $N = 2$ by implementing the homogenous balance principle.

4.1. Improved F-expansion procedure

Since the balance number is 2, let us assume the exact solution as follows:

$$v(\xi) = \alpha_0 + \alpha_1(\gamma + \mathcal{F}(\xi)) + \alpha_2(\gamma + \mathcal{F}(\xi))^2 + \frac{\beta_1}{(\gamma + \mathcal{F}(\xi))} + \frac{\beta_2}{(\gamma + \mathcal{F}(\xi))^2}. \tag{20}$$

If Eq. (20) is substituted in Eq. (19) with the aid of Eq. (5), a determining equation system is found. If we solve obtained determining equation system, we obtain the following solution family:

$$\alpha_0 = \alpha_0, \alpha_1 = \alpha_2 = 0, \beta_1 = -2\alpha_0\gamma, \beta_2 = \alpha_0(\gamma^2 + \eta),$$
$$c_1 = \frac{\alpha_0}{2k(\gamma^2 + \eta)}, \gamma = \gamma.$$

1. When $\eta < 0$,

$$u_1(x,t) = \ln\left(-\frac{\alpha_0\eta\left(\tanh\left(\sqrt{-\eta}\left(kx - \frac{\left(\frac{\alpha_0}{2k(\gamma^2+\eta)}\right)}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2 - 1\right)}{\left(\gamma - \sqrt{-\eta}\tanh\left(\sqrt{-\eta}\left(kx - \frac{\left(\frac{\alpha_0}{2k(\gamma^2+\eta)}\right)}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)\right)^2}\right).$$

2. When $\eta > 0$,

$$u_2(x,t) = \ln\left(\frac{\alpha_0\eta\left(1 + \tan\left(\sqrt{\eta}\left(kx - \frac{\left(\frac{\alpha_0}{2k(\gamma^2+\eta)}\right)}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2\right)}{\left(\gamma + \sqrt{\eta}\tan\left(\sqrt{\eta}\left(kx - \frac{\left(\frac{\alpha_0}{2k(\gamma^2+\eta)}\right)}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)\right)^2}\right).$$

3. When $\eta = 0$,

$$u_3(x,t) = \ln\left(\frac{\alpha_0}{\left(\gamma\left(kx - \frac{\left(\frac{\alpha_0}{2k\gamma^2}\right)}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right) - 1\right)^2}\right).$$

$|u_1(x,t)|$ is represented graphically in Fig. 1(a) when $\alpha_0 = 0.5, \eta = -0.1, \beta = 0.2, k = 0.01, \gamma = 2$. $|u_2(x,t)|$ is represented graphically in Fig. 1(b) when $\alpha_0 = 0.2, \eta = 3, \beta = 0.1, k = 1, \gamma = 2$. $u_3(x,t)$ is represented graphically in Fig. 1(c) when $\alpha_0 = 0.01, \beta = 0.1, k = 1, \gamma = 2$.

4.2. The modified procedure of extended tanh-function

We determine the homogeneous balance $N = 2$ from Eq. (19), thus the solution of Eq. (19) is expressed as:

$$v(\xi) = a_0 + a_1\phi(\xi) + a_2(\phi(\xi))^2 + \frac{b_1}{\phi(\xi)} + \frac{b_2}{(\phi(\xi))^2}. \tag{21}$$

Surrogating Eq. (21) associated with Eq. (7) into Eq. (19), then assuming the coefficients of $\phi(\xi)^j$ are 0, a determining equation system is obtained. If we solve the obtained system, the constants' values are obtained.

Solution Family I: $a_0 = \frac{b_2}{\eta}, \quad a_1 = a_2 = b_1 = 0, b_2 = b_2, \quad c_1 = \frac{b_2}{2\eta^2 k}$,

When $\eta < 0$,

$$u_4(x,t) = \ln\left(-\frac{b_2}{\eta\sinh\left(\sqrt{-\eta}\left(kx - \frac{b_2}{2\eta^2 k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2}\right).$$

When $\eta > 0$,

$$u_5(x,t) = \ln\left(\frac{b_2}{\eta\sin\left(\sqrt{\eta}\left(kx - \frac{b_2}{2\eta^2 k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2}\right).$$

If we put $\gamma = 0$ in the obtained solutions of improved F-expansion method, we obtain the same solutions of this family.

Solution Family II: $a_0 = \eta a_2, a_1 = 0, \quad a_2 = a_2, b_1 = b_2 = 0, \quad c_1 = \frac{a_2}{2k}$,

When $\eta < 0$,

$$u_6(x,t) = \ln\left(\frac{\eta a_2}{\cosh\left(\sqrt{-\eta}\left(kx - \frac{a_2}{2k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2}\right).$$

When $\eta > 0$,

$$u_7(x,t) = \ln\left(\frac{\eta a_2}{\cos\left(\sqrt{\eta}\left(kx - \frac{a_2}{2k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2}\right).$$

When $\eta = 0$,

$$u_8(x,t) = \ln\left(\frac{a_2}{\left(kx - \frac{a_2}{2k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)^2}\right).$$

$|u_6(x,t)|$ is represented graphically in Fig. 2(a) when $\eta = -9, \beta = 0.9999, k = 0.3, a_2 = -0.11$. $|u_7(x,t)|$ is represented graphically in Fig. 2(b) when $\eta = 9, \beta = 0.9999, k = 0.3, a_2 = -0.11$. $|u_8(x,t)|$ is represented graphically in Fig. 2(c) when $\beta = 0.9999, k = 0.5, a_2 = 0.5$.

Solution Family III: $a_0 = 2\eta a_2, a_1 = 0, \quad a_2 = a_2, b_1 = 0, b_2 = \eta^2 a_2, \quad c_1 = \frac{a_2}{2k}$,

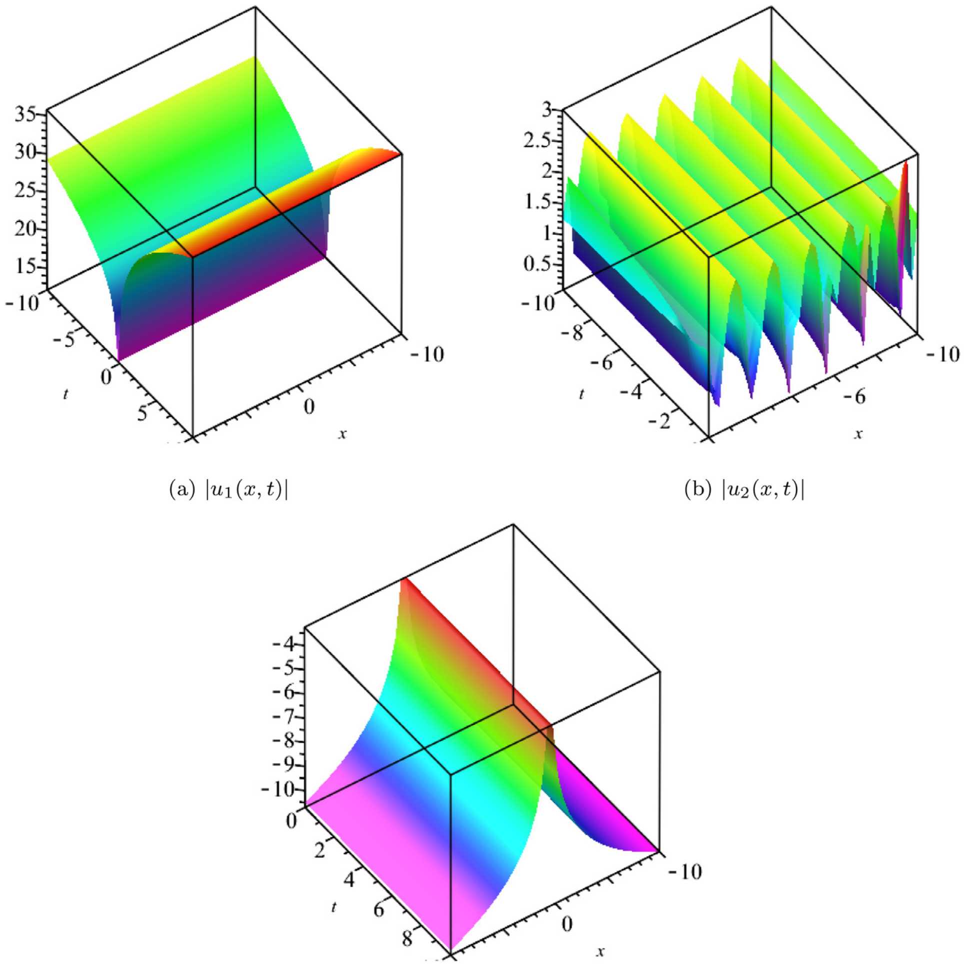
When $\eta < 0$,

$$u_9(x,t) = \ln\left(-\frac{\eta a_2}{\sinh\left(\sqrt{-\eta}\left(kx - \frac{a_2}{2k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2\cosh\left(\sqrt{-\eta}\left(kx - \frac{a_2}{2k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2}\right).$$

When $\eta > 0$,

$$u_{10}(x,t) = \ln\left(\frac{\eta a_2}{\sin\left(\sqrt{\eta}\left(kx - \frac{a_2}{2k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2\cos\left(\sqrt{\eta}\left(kx - \frac{a_2}{2k\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\right)^2}\right).$$

When $\eta = 0$, we obtain the same solution as the 3rd solution in solution family II.



(a) $|u_1(x,t)|$ (b) $|u_2(x,t)|$

(c) $u_3(x,t)$

Fig. 1. Graphical representations of (a) when $\alpha_0 = 0.5, \eta = -0.1, \beta = 0.2, k = 0.01, \gamma = 2$; (b) when $\alpha_0 = 0.2, \eta = 3, \beta = 0.1, k = 1, \gamma = 2$; and (c) when $\alpha_0 = 0.01, \beta = 0.1, k = 1, \gamma = 2$.

4.3. *The procedure of exponential rational function*

Since the balancing number is 2, from the considered procedure, the potential solution is expressed as:

$$v(\xi) = a_0 + \frac{a_1}{(1+e^{\xi})} + \frac{a_2}{(1+e^{\xi})^2}. \tag{22}$$

Then, Eq. (22) is substituted into the reduced equation Eq. (19) and collect all terms with the same order of $e^{i\xi}$ $(n = 0, 1, 2, 3, 4, 5, 6)$. From the solutions of the equation system, we obtain the following results:

$$a_0 = 0, a_1 = a_1, a_2 = -a_1, c_1 = -\frac{a_1}{2k}.$$

and

$$u_{11}(x,t) = \ln\left(\frac{a_1 e^{\left(kx+\frac{a_1}{2k\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}}{\left(1+e^{\left(kx+\frac{a_1}{2k\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)}\right)^2}\right). \tag{23}$$

$u_{11}(x,t)$ is represented graphically in Fig. 3 when $a_1 = 0.08, k = -3, \beta = 0.9$.

4.4. *The* $\left(g'\right)$*-expansion method*

Since the balancing number is 2, from the considered procedure, the given equation's exact solution is expressed as:

$$v(\xi) = a_0 + a_1\left(g'\right) + a_2\left(g'\right)^2, \ (a_2 \neq 0). \tag{24}$$

If we substitute Eq. (24) in Eq. (19) without ignoring Eq. (13) and equate each coefficient of all powers of $\left(g'\right)^i$ to 0, we get a determining equation system. If the obtained system is solved, we obtain:

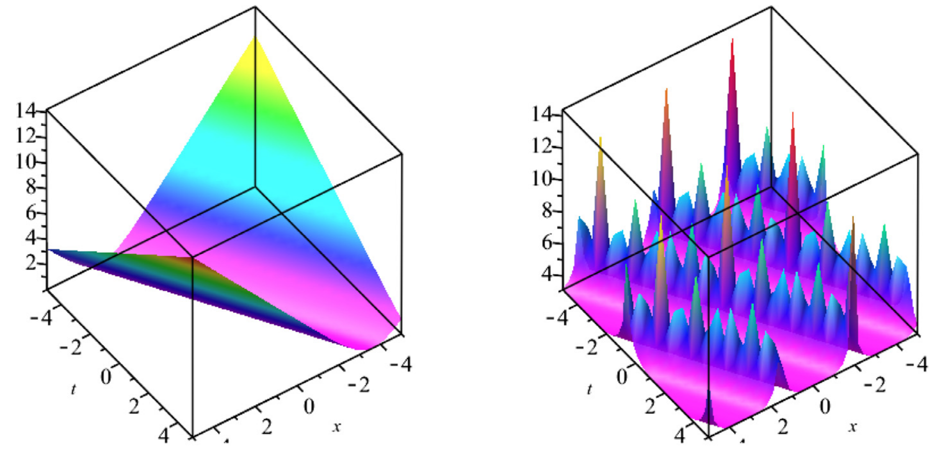
$$a_0 = \frac{aa_2}{c}, a_1 = \frac{a_2 b}{c}, a_2 = a_2, c_1 = \frac{a_2}{2kc^2},$$

here a, b and c are arbitrary constants.

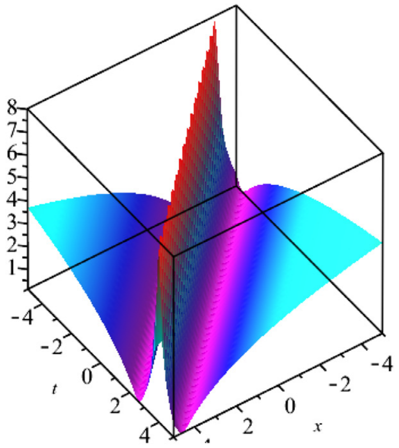
The solutions of Eq. (17) are presented as:

1. If $\Delta = 4ac - b^2 < 0$,

$$u_{12}(x,t) = \ln\left(\frac{a_2\left(4ac-b^2\right)}{4c^2\cosh\left(\frac{\left(kx-\frac{a_2}{2kc^2\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\sqrt{-4ac+b^2}}{2}\right)^2}\right). \tag{25}$$



(a) $|u_6(x,t)|$ (b) $|u_7(x,t)|$



(c) $|u_8(x,t)|$

Fig. 2. Graphical representations of (a) when $\eta = -9, \beta = 0.9999, k = 0.3, a_2 = -0.11$; (b) when $\eta = 9, \beta = 0.9999, k = 0.3, a_2 = -0.11$; and (c) when $\beta = 0.9999, k = 0.5, a_2 = 0.5$.

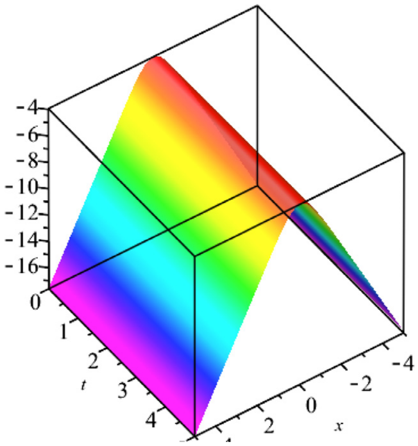


Fig. 3. Graphical representation of $u_{11}(x,t)$ when $a_1 = 0.08, k = -3, \beta = 0.9$.

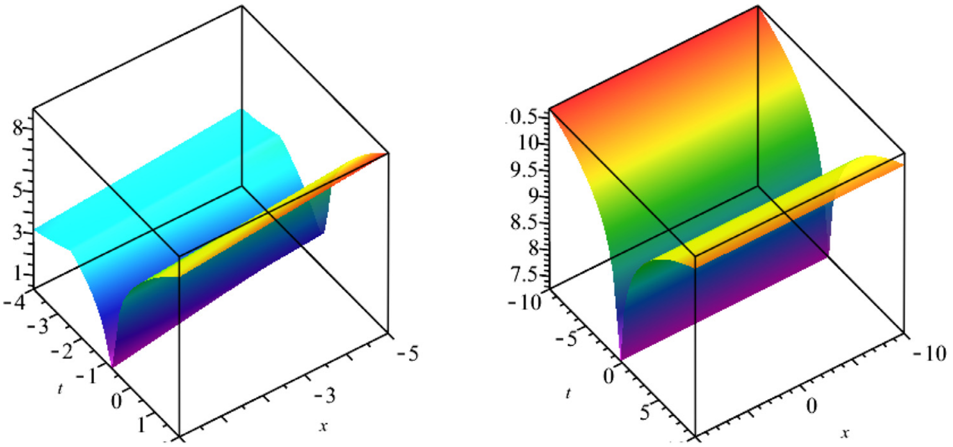
2. If $\Delta = 4ac - b^2 = 0$,

$$u_{13}(x,t) = \ln\left(\frac{a_2}{c^2\left(kx - \frac{a_2}{2kc^2\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)^2}\right). \tag{26}$$

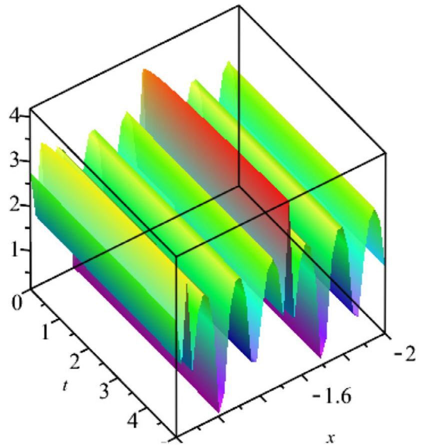
3. If $\Delta = 4ac - b^2 > 0$,

$$u_{14}(x,t) = \ln\left(\frac{a_2\left(ac - \frac{b^2}{4}\right)\left(\tan\left(\frac{\left(kx - \frac{a_2}{2kc^2\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right)\sqrt{4ac - b^2}}{2}\right)^2 + 1\right)}{c^2}\right). \tag{27}$$

$|u_{12}(x,t)|$ is represented graphically in Fig. 4(a) when $a = 1, c = 4, b = -5, \beta = 0.4, k = -0.07, a_2 = 1.5$. $|u_{13}(x,t)|$ is



(a) $|u_{12}(x,t)|$ (b) $|u_{13}(x,t)|$



(c) $|u_{14}(x,t)|$

Fig. 4. Graphical representations of (a) when $a = 1, c = 4, b = -5, \beta = 0.4, k = -0.07, a_2 = 1.5$; (b) when $a = 0.3, c = 0.3, b = 0.6, \beta = 0.4, k = -0.07, a_2 = 1.5$; and (c) when $a = 1, c = 4, b = -1, \beta = 0.05, k = 10, a_2 = 0.2$.

represented graphically in Fig. 4(b) when $a = 0.3, c = 0.3, b = 0.6, \beta = 0.4, k = -0.07, a_2 = 1.5$. $|u_{14}(x,t)|$ is represented graphically in Fig. 4(c) when $a = 1, c = 4, b = -1, \beta = 0.05, k = 10, a_2 = 0.2$.

5. Conclusion

Recently, a great consideration has been paid to find exact solutions to NFDEs. In this work, we have obtained some new traveling wave solutions to the fractional Liouville equation with beta derivative. We have separately applied the improved F-expansion, modified extended tanh, exponential rational function, and G' procedures to find a divergent collection of exact solutions to the fractional Liouville equation with beta-derivative. We have also plotted the graphical representations of the solutions to realize the behavior of the nonlinear model. The obtained results via the current work are up to date. We can mention one of the accomplishments of the current manuscript is that the prospect of applying the procedures considered in the current work can be paved in solving other nonlinear models that are introduced in some branches of ocean engineering, atmospheric sciences, and mathematical physics. We end up very efficient solutions using mathematical software, MAPLE. Undoubtedly, in modeling and recognizing certain physical characteristics of the used equation in distinct areas, these existing solutions may act as a leading role in modeling scenarios.

Data Availability

No data were used in this study.

Declaration of Competing Interest

The authors declare no conflict of interest.

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