

On a new technique for solving the nonlinear conformable time-fractional differential equations

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Abstract Real-world phenomena often are modelled by the nonlinear fractional differential equations. In this work, a novel technique called the $\exp(-\phi(\varepsilon))$ method is employed to find the exact solutions of nonlinear FDEs. Some well-known time-fractional differential equations in the context of conformable derivative, viz. the time-fractional modified Benjamin–Bona–Mahony (BBM) equation and the time-fractional Cahn–Hilliard (CH) equation are considered to test the usefulness of the method. The utility of the $\exp(-\phi(\varepsilon))$ method in solving nonlinear FDEs is proved.

Keywords Time-fractional modified BBM equation · Time-fractional CH equation · Conformable derivative · $\exp(-\phi(\varepsilon))$ method · Exact solutions

1 Introduction

FDEs are the generalizations of classical differential equations with integer orders which are used for modelling a lot of phenomena in the areas of fluid dynamics, mathematical biology, and so on. Several practical methods, including exp-function method (Dehghan et al. 2011; Dehghan and Manafian 2011; Jahani and Manafian 2016; Manafian and Lakestani 2015; Manafian 2015), modified Kudryashov method (Saha Ray and Sahoo 2016; Hosseini et al. 2017a, b), modified trial equation method (Bulut and Pandir 2013; Tandogan and Bildik 2016; Bulut et al. 2013), (G'/G) -expansion method (Yasar and Giresunlu 2016; Manafian et al. 2017; Teymuri Sindi and Manafian 2017a; Younis 2013),

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and improved $\tan(\phi(\xi)/2)$ -expansion method (Manafian 2016; Manafian and Lakestani 2016; Teymuri Sindi and Manafian 2017b) have been exerted to seek the exact solutions of nonlinear differential equations. Among other methods, the $\exp(-\phi(\varepsilon))$ method has achieved significant attention due to its capability in establishing the exact solutions of nonlinear differential equations. For instance, Taşcan and Akbulut (2017) applied the $\exp(-\phi(\varepsilon))$ method to the Bogoyavlenskii and modified KdV–Zakharov–Kuznetsev equations and derived new exact solutions of these equations. Hosseini et al. (2017c) employed the $\exp(-\phi(\varepsilon))$ method to the nonlinear time-fractional Klein–Gordon equations in conformable context and obtained the exact solutions of these equations. Hosseini et al. (2017d) also exerted the $\exp(-\phi(\varepsilon))$ method to the nonlinear conformable time-fractional Boussinesq equations and constructed various exact solutions of these equations. More works are found in Roshid et al. (2014), Islam et al. (2015), Kaplan and Bekir (2016), Hafez et al. (2015), Korkmaz and Hosseini (2017), Hosseini and Gholamin (2015), Mirzazadeh (2016), Sahoo and Saha Ray (2015), Hosseini et al. (2017e), Korkmaz (2017, 2018), Demiray and Bulut (2017), Eslami (2016) and Guner et al. (2017).

In this article, the $\exp(-\phi(\varepsilon))$ method is recommended to search the exact solutions of the time-fractional conformable modified BBM equation (Topsakal et al. 2016) and the time-fractional conformable CH equation (Baleanu et al. 2015), i.e.

$$D_t^\alpha u + u_x - vu^2 u_x + u_{xxx} = 0, \quad t > 0, \quad 0 < \alpha \leq 1,$$

and

$$D_t^\alpha u - u_x - 6u(u_x)^2 - (3u^2 - 1)u_{xx} + u_{xxx} = 0, \quad t > 0, \quad 0 < \alpha \leq 1.$$

Different schemes have been implemented to solve above equations, such as (G'/G) and $(G'/G, 1/G)$ -expansion methods (Topsakal et al. 2016), improved (G'/G) -expansion method (Baleanu et al. 2015), modified Kudryashov method (Ege and Misirli 2014), and generalized tanh-coth method (Manafian and Lakestani 2017).

The remainder of this study is as below: In Sect. 2, the conformable derivative and some useful characteristics of it are reviewed. In Sect. 3, the $\exp(-\phi(\varepsilon))$ method is described briefly. In Sect. 4, the exact solutions of aforementioned equations are obtained. Paper ends with a short note about the conclusions, in Sect. 5.

2 Conformable fractional derivative

For the function $f : (0, \infty) \rightarrow R$, the α th order of conformable fractional derivative is given by Khalil et al. (2014)

$$T_\alpha(f)(t) = \lim_{\tau \rightarrow 0} \frac{f(t + \tau t^{1-\alpha}) - f(t)}{\tau},$$

in which $\alpha \in (0, 1]$.

A number of useful properties of the conformable derivative are as follows (Khalil et al. 2014; Eslami and Rezaadeh 2016)

1.

$$T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \quad \forall a, b \in R.$$

2.

$$T_x(t^\mu) = \mu t^{\mu-\alpha}, \quad \forall \mu \in \mathbb{R}.$$

3.

$$T_x(fg) = fT_x(g) + gT_x(f).$$

4.

$$T_x\left(\frac{f}{g}\right) = \frac{gT_x(f) - fT_x(g)}{g^2}.$$

Furthermore, the chain rule of conformable derivative is as follows

$$T_x(fog)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

3 Exp(-φ(ε)) method

Consider the following nonlinear conformable time-fractional differential equation

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0. \tag{1}$$

With the use of transformation $u(x, t) = f(\varepsilon)$ where $\varepsilon = kx - l(t^\alpha/\alpha)$, Eq. (1) is changed into a nonlinear ODE as

$$G(f, f', f'', \dots) = 0. \tag{2}$$

We search a solution for Eq. (2) in the form

$$f(\varepsilon) = a_0 + a_1 \exp(-\phi(\varepsilon)) + \dots + a_N \exp(-N\phi(\varepsilon)), \tag{3}$$

where N is calculated using the homogeneous balance principle (HBP) and $\phi(\varepsilon)$ is a function that satisfies a first-order equation as

$$\phi'(\varepsilon) = \exp(-\phi(\varepsilon)) + \mu \exp(\phi(\varepsilon)) + \lambda.$$

Now, several cases can be taken:

Case 1 If $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, then

$$\phi_1(\varepsilon) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\varepsilon + C)\right) - \lambda}{2\mu}\right).$$

Case 2 If $\lambda^2 - 4\mu > 0$, $\mu = 0$, and $\lambda \neq 0$, then

$$\phi_2(\varepsilon) = -\ln\left(\frac{\lambda}{\cosh(\lambda(\varepsilon + C)) + \sinh(\lambda(\varepsilon + C)) - 1}\right).$$

Case 3 If $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, then

$$\phi_3(\varepsilon) = \ln \left(\frac{\sqrt{4\mu - \lambda^2} \tan \left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\varepsilon + C) \right) - \lambda}{2\mu} \right).$$

Case 4 If $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, and $\lambda \neq 0$, then

$$\phi_4(\varepsilon) = \ln \left(-\frac{2\lambda(\varepsilon + C) + 4}{\lambda^2(\varepsilon + C)} \right).$$

Case 5 If $\lambda^2 - 4\mu = 0$, $\mu = 0$, and $\lambda = 0$, then

$$\phi_5(\varepsilon) = \ln((\varepsilon + C)).$$

Now, a polynomial in $\exp(-\phi(\varepsilon))$ is secured by substituting Eq. (3) into left hand side of Eq. (2). By setting each coefficient of this polynomial to zero, we will acquire a nonlinear system whose solution gives a series of exact solutions for the Eq. (1).

4 Applications

In the current section, the exact solutions of the time-fractional modified BBM equation and the time-fractional CH equation are gained using the $\exp(-\phi(\varepsilon))$ method.

4.1 Time-fractional Benjamin–Bona–Mahony equation

With the use of transformation $u(x, t) = f(\varepsilon)$ where $\varepsilon = kx - l(t^\alpha/\alpha)$, the conformable time-fractional BBM equation is reduced to a nonlinear ODE as follows

$$(k - l)f' - vkf'f^2 + k^3f''' = 0.$$

Now, by integrating above equation with respect to ε , we find

$$k^3f'' + (k - l)f - \frac{1}{3}kvf^3 = 0, \tag{4}$$

where the integration constant is assumed to be zero.

4.1.1 Applying the $\exp(-\phi(\varepsilon))$ method

By means of the HBP, we acquire $N = 1$. As a result, Eq. (4) has a formal solution as

$$f(\varepsilon) = a_0 + a_1 \exp(-\phi(\varepsilon)). \tag{5}$$

By inserting Eqs. (5) in (4) and setting the coefficient of each power of $\exp(-\phi(\varepsilon))$ to zero, we reach a nonlinear system as follows

$$(k - l)a_0 + k^3\lambda\mu a_1 - \frac{1}{3}kva_0^3 = 0,$$

$$\begin{aligned}
 (k - l + k^3 \lambda^2 + 2k^3 \mu) a_1 - k v a_0^2 a_1 &= 0, \\
 3k^3 \lambda a_1 - k v a_0 a_1^2 &= 0, \\
 2k^3 a_1 - \frac{1}{3} k v a_1^3 &= 0.
 \end{aligned}$$

By using the *Maple* software, we derive two solutions as follows:

Case 1

$$a_0 = \frac{\sqrt{6} \lambda k}{2\sqrt{v}}, \quad a_1 = \frac{\sqrt{6} k}{\sqrt{v}}, \quad l = -\frac{1}{2} (k^2 \lambda^2 - 4k^2 \mu - 2) k.$$

In this case, we obtain the following exact solutions for the conformable time-fractional BBM equation

$$\begin{aligned}
 u_1(x, t) &= \frac{\sqrt{6} \lambda k}{2\sqrt{v}} - \frac{2\sqrt{6} k \mu}{\sqrt{v} \left(\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (kx + \frac{1}{2} (k^2 \lambda^2 - 4k^2 \mu - 2) k \frac{t^\alpha}{\alpha} + C) \right) + \lambda \right)}, \\
 u_2(x, t) &= \frac{\sqrt{6} \lambda k}{2\sqrt{v}} + \frac{\sqrt{6} k \lambda}{\sqrt{v} (\cosh(\lambda (kx + \frac{1}{2} (k^2 \lambda^2 - 2) k \frac{t^\alpha}{\alpha} + C)) + \sinh(\lambda (kx + \frac{1}{2} (k^2 \lambda^2 - 2) k \frac{t^\alpha}{\alpha} + C)) - 1)}, \\
 u_3(x, t) &= \frac{\sqrt{6} \lambda k}{2\sqrt{v}} - \frac{2\sqrt{6} k \mu}{\sqrt{v} \left(\sqrt{4\mu - \lambda^2} \tan \left(\frac{\sqrt{4\mu - \lambda^2}}{2} (kx + \frac{1}{2} (k^2 \lambda^2 - 4k^2 \mu - 2) k \frac{t^\alpha}{\alpha} + C) \right) + \lambda \right)}, \\
 u_4(x, t) &= \frac{\sqrt{6} \lambda k}{2\sqrt{v}} - \frac{\sqrt{6} k \lambda^2 (kx - k \frac{t^\alpha}{\alpha} + C)}{\sqrt{v} (2\lambda (kx - k \frac{t^\alpha}{\alpha} + C) + 4)}, \\
 u_5(x, t) &= \frac{\sqrt{6} k}{\sqrt{v} (kx - k \frac{t^\alpha}{\alpha} + C)}.
 \end{aligned}$$

Case 2

$$a_0 = -\frac{\sqrt{6} \lambda k}{2\sqrt{v}}, \quad a_1 = -\frac{\sqrt{6} k}{\sqrt{v}}, \quad l = -\frac{1}{2} (k^2 \lambda^2 - 4k^2 \mu - 2) k.$$

In this case, we derive the following exact solutions for the conformable time-fractional BBM equation

$$\begin{aligned}
 u_6(x, t) &= -\frac{\sqrt{6}\lambda k}{2\sqrt{v}} \\
 &\quad + \frac{2\sqrt{6}k\mu}{\sqrt{v}\left(\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(kx + \frac{1}{2}(k^2\lambda^2 - 4k^2\mu - 2)k\frac{t^\alpha}{\alpha} + C\right)\right) + \lambda\right)}, \\
 u_7(x, t) &= -\frac{\sqrt{6}\lambda k}{2\sqrt{v}} \\
 &\quad - \frac{\sqrt{6}k\lambda}{\sqrt{v}\left(\cosh\left(\lambda\left(kx + \frac{1}{2}(k^2\lambda^2 - 2)k\frac{t^\alpha}{\alpha} + C\right)\right) + \sinh\left(\lambda\left(kx + \frac{1}{2}(k^2\lambda^2 - 2)k\frac{t^\alpha}{\alpha} + C\right)\right) - 1\right)}, \\
 u_8(x, t) &= -\frac{\sqrt{6}\lambda k}{2\sqrt{v}} \\
 &\quad + \frac{2\sqrt{6}k\mu}{\sqrt{v}\left(\sqrt{4\mu - \lambda^2} \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\left(kx + \frac{1}{2}(k^2\lambda^2 - 4k^2\mu - 2)k\frac{t^\alpha}{\alpha} + C\right)\right) + \lambda\right)}, \\
 u_9(x, t) &= -\frac{\sqrt{6}\lambda k}{2\sqrt{v}} + \frac{\sqrt{6}k\lambda^2\left(kx - k\frac{t^\alpha}{\alpha} + C\right)}{\sqrt{v}\left(2\lambda\left(kx - k\frac{t^\alpha}{\alpha} + C\right) + 4\right)}, \\
 u_{10}(x, t) &= -\frac{\sqrt{6}k}{\sqrt{v}\left(kx - k\frac{t^\alpha}{\alpha} + C\right)}.
 \end{aligned}$$

Figure 1 demonstrates the graph of $u_1(x, t)$ for the values $\alpha = 0.5$ and $\alpha = 1$, when $\lambda = 3, k = 1, v = 4, \mu = 1$, and $C = 0$.

4.2 Time-fractional Cahn–Hilliard equation

By applying the transformation $u(x, t) = f(\varepsilon)$ where $\varepsilon = kx - l(t^\alpha/\alpha)$, the conformable time-fractional CH equation is changed into a nonlinear ODE as below

$$-(l + k)f' - 6f(kf')^2 - k^2(3f^2 - 1)f'' + k^4f^{(4)} = 0.$$

Now, by integrating above equation with respect to ε , we obtain

$$-(l + k)f - 3k^2f'f^2 + k^2f' + k^4f''' = 0, \tag{6}$$

where the integration constant is supposed to be zero.

4.2.1 Applying the $\exp(-\phi(\varepsilon))$ method

With the use of HBP, we gain $N = 1$. As a consequence, Eq. (6) has a formal solution as

$$f(\varepsilon) = a_0 + a_1 \exp(-\phi(\varepsilon)). \tag{7}$$

By inserting Eq. (7) in Eq. (6) and setting the coefficient of each power of $\exp(-\phi(\varepsilon))$ to zero, we get a nonlinear system as

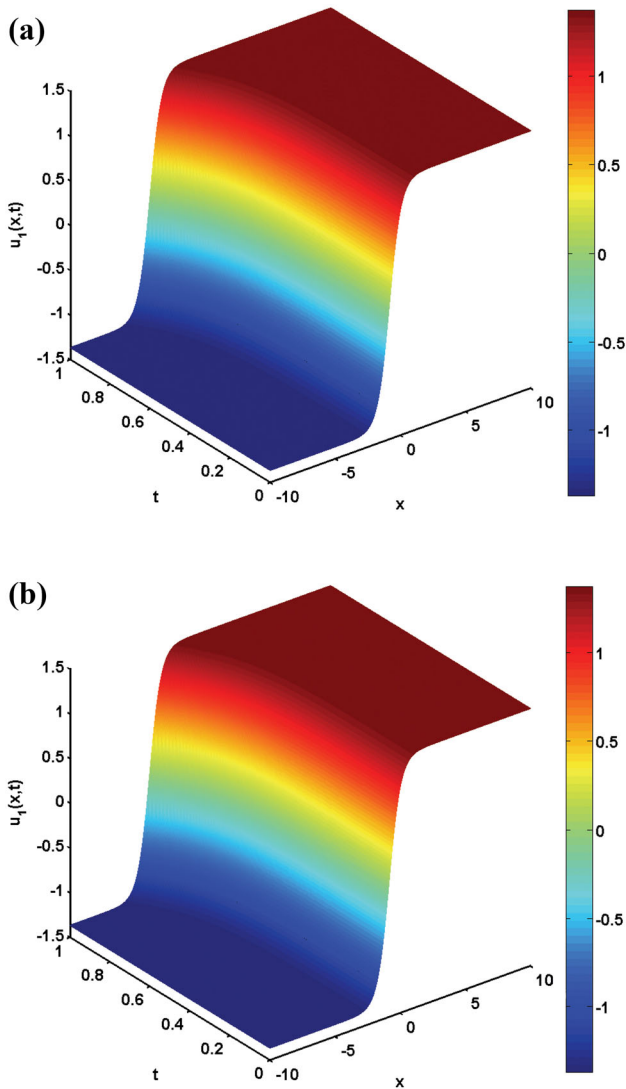


Fig. 1 **a** Graph of $u_1(x, t)$ for $\lambda = 3, k = 1, v = 4, \mu = 1, C = 0,$ and $\alpha = 0.5$. **b** Graph of $u_1(x, t)$ for $\lambda = 3, k = 1, v = 4, \mu = 1, C = 0,$ and $\alpha = 1$

$$\begin{aligned}
 & -(k + l)a_0 - (k^4\lambda^2\mu + 2k^4\mu^2 + k^2\mu)a_1 + 3k^2\mu a_0^2a_1 = 0, \\
 & -(k + l + k^4\lambda^3 + 8k^4\lambda\mu + k^2\lambda)a_1 + 3k^2\lambda a_0^2a_1 + 6k^2\mu a_0a_1^2 = 0, \\
 & -(k^2 + 7k^4\lambda^2 + 8k^4\mu)a_1 + 3k^2a_0^2a_1 + 3k^2\mu a_1^3 + 6k^2\lambda a_0a_1^2 = 0, \\
 & -12k^4\lambda a_1 + 3k^2\lambda a_1^3 + 6k^2a_0a_1^2 = 0, \\
 & -6k^4a_1 + 3k^2a_1^3 = 0.
 \end{aligned}$$

By using the *Maple* software, we secure two solutions as follows:

Case 1

$$a_0 = \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}}, \quad a_1 = \frac{2}{\sqrt{\lambda^2 - 4\mu}}, \quad l = -\frac{2}{\sqrt{2(\lambda^2 - 4\mu)}}, \quad k = \frac{2}{\sqrt{2(\lambda^2 - 4\mu)}}.$$

In this case, we gain the following exact solutions for the conformable time-fractional CH equation

$$u_1(x, t) = \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \left(\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} x + \frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} t^\alpha + C \right) \right) + \lambda \right)},$$

$$u_2(x, t) = \frac{\lambda}{\sqrt{\lambda^2}} + \frac{2\lambda}{\sqrt{\lambda^2} \left(\cosh \left(\lambda \left(\frac{\sqrt{2}}{\sqrt{\lambda^2}} x + \frac{\sqrt{2}}{\sqrt{\lambda^2}} t^\alpha + C \right) \right) + \sinh \left(\lambda \left(\frac{\sqrt{2}}{\sqrt{\lambda^2}} x + \frac{\sqrt{2}}{\sqrt{\lambda^2}} t^\alpha + C \right) \right) - 1 \right)},$$

$$u_3(x, t) = \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \left(\sqrt{4\mu - \lambda^2} \tan \left(\frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} x + \frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} t^\alpha + C \right) \right) + \lambda \right)}.$$

Case 2

$$a_0 = -\frac{\lambda}{\sqrt{\lambda^2 - 4\mu}}, \quad a_1 = -\frac{2}{\sqrt{\lambda^2 - 4\mu}}, \quad l = \frac{2}{\sqrt{2(\lambda^2 - 4\mu)}}, \quad k = -\frac{2}{\sqrt{2(\lambda^2 - 4\mu)}}.$$

In this case, we acquire the following exact solutions for the conformable time-fractional CH equation

$$u_4(x, t) = -\frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} + \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \left(\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(-\frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} x - \frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} t^\alpha + C \right) \right) + \lambda \right)},$$

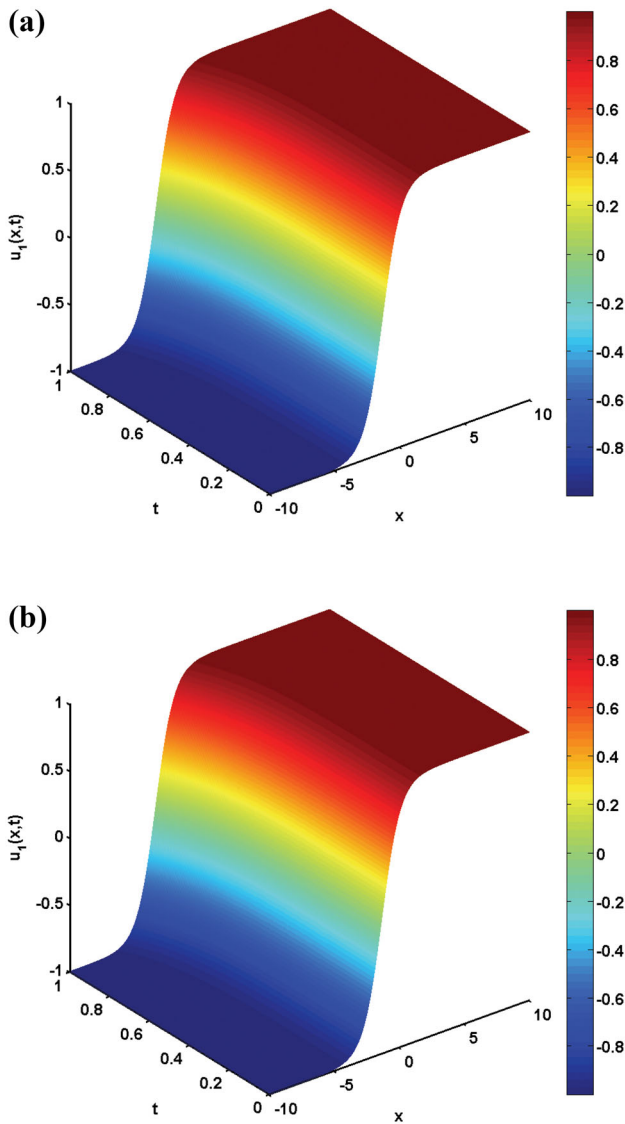


Fig. 2 **a** Graph of $u_1(x, t)$ for $\lambda = 3, \mu = 1, C = 0,$ and $\alpha = 0.5.$ **b** Graph of $u_1(x, t)$ for $\lambda = 3, \mu = 1, C = 0,$ and $\alpha = 1$

$$u_5(x, t) = -\frac{\lambda}{\sqrt{\lambda^2}} \frac{2\lambda}{\sqrt{\lambda^2} \left(\cosh \left(\lambda \left(-\frac{\sqrt{2}}{\sqrt{\lambda^2}} x - \frac{\sqrt{2}}{\sqrt{\lambda^2}} t^\alpha + C \right) \right) + \sinh \left(\lambda \left(-\frac{\sqrt{2}}{\sqrt{\lambda^2}} x - \frac{\sqrt{2}}{\sqrt{\lambda^2}} t^\alpha + C \right) \right) - 1 \right)},$$

$$u_6(x,t) = -\frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} + \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \left(\sqrt{4\mu - \lambda^2} \tan \left(\frac{\sqrt{4\mu - \lambda^2}}{2} \left(-\frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} x - \frac{2}{\sqrt{2(\lambda^2 - 4\mu)}} \frac{t^\alpha}{\alpha} + C \right) \right) + \lambda \right)}.$$

Figure 2 illustrates the graph of $u_1(x, t)$ for the values $\alpha = 0.5$ and $\alpha = 1$, when $\lambda = 3$, $\mu = 1$, and $C = 0$.

5 Conclusion

Some nonlinear time-fractional differential equations with the conformable derivative, viz. the time-fractional modified Benjamin–Bona–Mahony equation and the time-fractional Cahn–Hilliard equation were formally studied, in the present work. A novel method called the $\exp(-\phi(\varepsilon))$ method was adopted successfully to secure the exact solutions of aforementioned equations. As a result, the helpfulness of the $\exp(-\phi(\varepsilon))$ method in solving nonlinear FDEs was proved.

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