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Auto-Bäcklund transformations and solitary wave solutions for the nonlinear evolution equation

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Abstract In the present work, according to the concept of extended homogeneous balance method and with help of Maple, we get auto-Bäcklund transformations for a (2 + 1)-dimensional nonlinear evolution equation. Subsequently, by using these auto-Bäcklund transformation, exact explicit solutions of this equation are obtained.

Keywords Auto-Bäcklund transformation · The extended homogenouous balance method · Periodic wave solutions · Solitary wave solutions

1 Introduction

Nonlinear evolution equations (NLEEs) have been used as the models to describe some complex phenomena in biology, chemistry, communications and especially in almost all branches of physics: fluid mechanics, plasma waves, condensed matter physics, chemical physics, solid state physics, nonlinear optics etc (Wang et al. 2011; Chai et al. 2017; Liu et al. 2017; Inan et al. 2016; Yokus et al. 2017). To better comprehend the mechanisms of those physical phenomena and provide some possible applications, it is necessary to explore their solutions and properties (Qua et al. 2012; Mirzazadeh et al. 2015). For that purpose, some methods have been developed by sciences, such as Hirota method (Hirota 1971; Wazwaz and El-Tantawy 2016; Wazwaz 2009, 2015) Darboux transformation (Xue et al. 2012), inverse scattering method (Ablowitz and Clarkson 1991), homogeneous balance method (Moussa and Shikh 2009; Eslami et al. 2014) similarity reduction method (Fan 2002), linear superposition principle (Ma et al. 2012), Cole–Hopf transformation method (Salas and Gomez 2010), Bäcklund transformation (El-Kalaawy 2014; Gao

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2015, 2016, 2017) Lax pair (Yin et al. 2017), generalized Miura transformation (Miura 1978), modified simple equation method (Jawad et al. 2010; Zayed 2011; Mirzazadeh 2014) extended trial equation method (Gurefe et al. 2013; Ekici et al. 2017; Zhou and Mirzazadeh 2016) first integral method (Mirzazadeh 2016), exp-function method (Fard et al. 2015), (G′/G)-expansion method (Mirzazadeh et al. 2017b; Jawad et al. 2015) sine-Gordon expansion method (Bulut et al. 2016, 2017a, b; Baskonus et al. 2017a, b, c) improved modified extended tanh-function method (Mirzazadeh et al. 2017a), Bäcklund transformation of generalized Riccati equation (Zayed and Al-Nowehy 2017a, b; Zayed et al. 2017a, b; Lu 2012) and so on.

Integrable systems are frequently thought to indicate regular temporal and spatial evolution. It is admitted that the theme of complete integrability of NLEEs is not yet well defined, and there exists no unique definition. Recently, to explore complete integrability of nonlinear partial and ordinary differential equations both, a systematic effort has been spent, and a large number of effective tecniques have been developed in Sahadevan (2001) and Biswas (2009).

In all fields of nonlinear science, Painleve analysis or Painleve test is a frequently used technique for testing the integrability of these NLEEs. In the last decades, the studies on Painleve property were presented in Conte (1999) and Wazwaz and Xu (2016). In Bekir (2007), the Painleve test has been applied for the following (2 + 1)-dimensional nonlinear evolution equation:

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} = 0. \tag{1}$$

Then it is concluded that the equation has the Painleve property and also it is expected to be integrable. Generally speaking, we call a model is Painleve integrable if the model possesses the Painleve property which means that the solutions of the model are single valued about an arbitrary singularity manifold. Equation (1) is a nonlinear evolution equation by nature. Such equations are of great importance in mathematical physics, nonlinear theory and physical applications.

Also, multiple-soliton solutions of this equation have been founded by using the simplified Hirota's method and different types of analytical solutions for the (2 + 1)-dimensional nonlinear evolution equation have been obtained by using the transformed rational function method (Kaplan and Ozer 2018). The aim of this paper is to get different types of explicit solutions of the (2 + 1)-dimensional nonlinear evolution equation by using the Auto-Bäcklund transformations which are obtained with the help of extended homogenous balance method.

The paper is organized as follows. In Sect. 2, we would like to use the extended homogenous balance method to shed light on the Auto-Bäcklund transformations of Eq. (1). In Sect. 3, we use these transformations to get exact explicit solutions with the help of Maple, which include hyperbolic, trigonometric and rational function solutions. And finally, some conclusions are mentioned.

2 Auto-Bäcklund transformations for the (2 + 1)-dimensional nonlinear equation

Based the extended homogeneous balance method (Shang 2007; Shang et al. 2011) we can take the solution of Eq. (1) as

$$u(x,y,t) = f'(w)w_x + u_0(x,y,t), \tag{2}$$

where f, w are two functions are going to be determined, and $u_0(x,y,t)$ is a seed solution of Eq. (1). This seed solution is a simple one and is actually useful for constructing many other solutions. Substituting Eq. (2) into Eq. (1) turns into

$$\begin{aligned}
&\left[f^{(5)} + 6f''f'''\right]w_y w_x^4 + \Big[\left(4w_x^3 w_{xy} + 6w_y w_x^2 w_{xx}\right)f^{(4)}\\
&+\left(6w_x^3 w_{xy} + 12w_y w_x^2 w_{xx}\right)(f'')^2 + \left(3w_{xy}w_x^3 + 3w_{xx}w_y w_x^2\right)f'f'''\Big]\\
&+\Big[\Big(6w_x^2 w_{xxy} + 3w_y(w_{xx})^2 + 3u_{0y}w_x^3 + 2w_t w_y w_x + 4w_y w_x w_{xxx}\\
&+12w_x w_{xx} w_{xy} + 3u_{0x} w_y w_x^2\Big)f''' + \Big(3w_y w_x w_{xxx} + 15w_{xy} w_x w_{xx}\\
&+3w_x^2 w_{xxy} + 3w_{xx}^2 w_y\Big)f''f'\Big] + \Big[\left(3w_{xy}w_{xxx} + 3w_{xx}w_{xxy}\right)(f')^2\\
&+\Big(6w_{xx}w_{xxy} + 4w_{xy}w_{xxx} + 4w_x w_{xxxy} + w_y w_{xxxx} + 2w_{yt}w_x\\
&+2w_y w_{xt} + 2w_t w_{xy} + 3w_x^2 u_{0xy}\\
&+3w_y w_x u_{0xx} + 9u_{0y} w_x w_{xx} + 6u_{0x} w_x w_{xy} + 3u_{0x} w_y w_{xx}\Big)f''\Big]\\
&+\left[w_{xxxxy} + 2w_{xyt} + 3w_{xy}u_{0xx} + 3u_{0y}w_{xxx} + 3w_{xx}u_{0xy} + 3u_{0x}w_{xxy}\right]f'\\
&+\left[3u_{0y}u_{0xx} + 3u_{0x}u_{0xy} + u_{0xxxy} + 2u_{0yt}\right] = 0.
\end{aligned} \tag{3}$$

To simplify this expression, we further assume that

$$f = c\ln w, \tag{4}$$

where c is an arbitrary constant. Then we get

$$\begin{aligned}
&(f')^2 = -cf'', \quad (f'')^2 = -\frac{c}{6}f^{(4)},\\
&f'f''' = -\frac{c}{3}f^{(4)}, \quad f'f'' = -\frac{c}{2}f''',\\
&f''f''' = -\frac{c}{12}f^{(5)}.
\end{aligned} \tag{5}$$

By using the equalities (5), Eq. (3) can be rewritten as the sum of some terms of $f^{(0)}f', f'', f'', f^{(4)}$ and $f^{(5)}$, setting their coefficients to zero yields to

$$
\begin{aligned}
f^{(5)} &: 1-\frac{c}{2}=0,\\
f^{(4)} &: 4w_x^3w_{xy}+6w_yw_x^2w_{xx}-cw_x^3w_{xy}-2cw_yw_x^2w_{xx}-cw_{xy}w_x^3\\
&\quad -cw_{xx}w_yw_x^2=0,\\
f''' &: 6w_x^2w_{xxy}+3w_y(w_{xx})^2+3u_{0y}w_x^3+2w_tw_yw_x+4w_yw_xw_{xxx}\\
&\quad +12w_xw_{xx}w_{xy}+3u_{0x}w_yw_x^2\\
&\quad +\left(3w_yw_xw_{xxx}+15w_{xy}w_xw_{xx}+3w_x^2w_{xxy}+3w_{xx}^2w_y\right)\left(-\frac{c}{2}\right)=0\\
f'' &: -3cw_{xy}w_{xxx}-3cw_{xx}w_{xxy}+6w_{xx}w_{xxy}+4w_{xy}w_{xxx}+4w_xw_{xxxy}\\
&\quad +w_yw_{xxxx}+2w_{yt}w_x+2w_yw_{xt}+2w_tw_{xy}+3w_x^2u_{0xy}+3w_yw_xu_{0xx}\\
&\quad +9u_{0y}w_xw_{xx}+6u_{0x}w_xw_{xy}+3u_{0x}w_yw_{xx}=0,\\
f' &: w_{xxxxy}+2w_{xyt}+3w_{xy}u_{0xx}+3u_{0y}w_{xxx}+3w_{xx}u_{0xy}+3u_{0x}w_{xxy}=0,\\
f^{(0)} &: u_{0xxxy}+3u_{0y}u_{0xx}+3u_{0x}u_{0xy}+2u_{0yt}=0.
\end{aligned}
\tag{6}
$$

From the first equation of the system (6), we find

$$c=2. \tag{7}$$

By using the equalities (5), the equation system (6) can be rewritten as

$$
\begin{aligned}
&\left[w_x\left(3w_xw_{xxy}-3w_{xy}w_{xx}+3u_{0y}w_x^2+2w_tw_y+4w_yw_{xxx}+3u_{0x}w_yw_x\right)\right]f'''\\
&\quad+\left[w_yw_{xxxx}-2w_{xy}w_{xxx}+4w_xw_{xxxy}+2w_{yt}w_x+2w_yw_{xt}+2w_tw_{xy}\right.\\
&\quad\left.+3w_x^2u_{0xy}+3w_yw_xu_{0xx}+9u_{0y}w_xw_{xx}+6u_{0x}w_xw_{xy}+3u_{0x}w_yw_{xx}\right]f''\\
&\quad\left[w_{xxxxy}+2w_{xyt}+3w_{xy}u_{0xx}+3u_{0y}w_{xxx}+3w_{xx}u_{0xy}+3u_{0x}w_{xxy}\right]f'\\
&\quad+\left[u_{0xxxy}+3u_{0y}u_{0xx}+3u_{0x}u_{0xy}+2u_{0yt}\right]=0
\end{aligned}
\tag{8}
$$

Substituting zero instead of coefficients of f''', f'', f', f^0 respectively, we find the following differential equations:

$$3w_xw_{xxy}-3w_{xy}w_{xx}+3u_{0y}w_x^2+2w_tw_y+w_yw_{xxx}+3u_{0x}w_yw_x=0 \tag{9}$$

$$
\begin{aligned}
&w_yw_{xxxx}-2w_{xy}w_{xxx}+4w_xw_{xxxy}+2w_{yt}w_x+2w_yw_{xt}+2w_tw_{xy}\\
&\quad+3w_x^2u_{0xy}+3w_yw_xu_{0xx}+9u_{0y}w_xw_{xx}+6u_{0x}w_xw_{xy}+3u_{0x}w_yw_{xx}=0
\end{aligned}
\tag{10}
$$

$$w_{xxxxy}+2w_{xyt}+3w_{xy}u_{0xx}+3u_{0y}w_{xxx}+3w_{xx}u_{0xy}+3u_{0x}w_{xxy}=0 \tag{11}$$

$$u_{0xxxy}+3u_{0y}u_{0xx}+3u_{0x}u_{0xy}+2u_{0yt}=0 \tag{12}$$

If we differentiate Eq. (9) with respect to x, we can obtain the relation between Eqs. (9)–(11). Namely,

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(3w_x w_{xxy} - 3w_{xy}w_{xx} + 3u_{0y}w_x^2 + 2w_t w_y + w_y w_{xxx} + 3u_{0x}w_y w_x\right)\\
&\quad + w_x \frac{\partial}{\partial x}\left(w_{xxxy} + 3w_{xx}u_{0y} + 3w_{xy}u_{0x} + 2w_{yt}\right)\\
&\quad = w_y w_{xxxx} - 2w_{xy}w_{xxx} + 4w_x w_{xxxy} + 2w_{yt}w_x + 2w_y w_{xt} + 2w_t w_{xy}\\
&\qquad + 3w_x^2 u_{0xy} + 3w_y w_x u_{0xx} + 9u_{0y}w_x w_{xx} + 6u_{0x}w_x w_{xy} + 3u_{0x}w_y w_{xx}
\end{aligned} \tag{13}
$$

Then Eq. (13) can be satisfied, if the following differential equations are satisfied:

$$w_{xxxy} + 3w_{xx}u_{0y} + 3w_{xy}u_{0x} + 2w_{yt} = 0, \tag{14}$$

$$3w_x w_{xxy} - 3w_{xy}w_{xx} + 3u_{0y}w_x^2 + 2w_t w_y + w_y w_{xxx} + 3u_{0x}w_y w_x = 0, \tag{15}$$

$$u_{0xxxy} + 3u_{0y}u_{0xx} + 3u_{0x}u_{0xy} + 2u_{0yt} = 0. \tag{16}$$

Thus, we find an Auto-Bäcklund transformation for (2 + 1)-dimensional nonlinear evolution equation

$$u(x, y, t) = 2\frac{w_x}{w} + u_{0,} \tag{17}$$

where $w(x, y, t)$ and $u_0(x, y, t)$ satisfy Eqs. (14)–(16).

Then we can find the multiple new solutions to Eq. (1) from Eq. (17) by solving Eqs. (14) and (15) for any given solution u_0 of Eq. (1).

1. Setting $u_0 = c$ in Eq. (17), we obtain

$$u(x, y, t) = 2\frac{w_x}{w} + c, \tag{18}$$

   where $w(x, y, t)$ satisfies the following equations

$$w_{xxxy} + 2w_{yt} = 0, \tag{19}$$

$$3w_x w_{xxy} - 3w_{xy}w_{xx} + 2w_t w_y + w_y w_{xxx} = 0. \tag{20}$$

2. Setting $u_0 = 0$ in Eq. (17), we get Cole–Hopf transformation

$$u(x, y, t) = 2\frac{w_x}{w}, \tag{21}$$

   where $w(x, y, t)$ satisfies Eqs. (19) and (20).

3 Abundant solitary wave solutions of (2 + 1)-dimensional nonlinear evolution equation

In this section, we are going to obtain various solitary wave solutions of Eq. (1) by using Maple and the Auto-Bäcklund transformation which is founded in the previous section. To get our goal, let us begin with the Auto-Bäcklund transformation (18) and the constant solution u_0 of Eq. (1).

1. Let $w(x, y, t)$ be of the following form

$$w(x,y,t) = a\cosh(kx+ly+rt+\xi_0) + b\sinh(kx+ly+rt+\xi_0) + p, \tag{22}$$

where a, b, k, l, r and p are constants be determined later and ξ_0 is an arbitrary constant. Substituting (22) into (19) and (20) an algebraic equation system is verified. From the solutions of the system, we obtain the following two cases.

Case 1

$$a = b, \quad b = b, \quad k = k, \quad l = l, \quad p = p, \quad r = -\frac{k^3}{2}, \tag{23}$$

where b, k, l and p are arbitrary non-zero parameters. Therefore, we obtain (Fig. 1)

$$w_1(x,y,t) = b\cosh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) - b\sinh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) + p. \tag{24}$$

Then by substituting Eq. (24) into Eq. (18), we find solitary wave solution for (2 + 1)-dimensional nonlinear evolution Eq. (1) as follows (Fig. 2):

$$u_1(x,y,t) = 2bk\frac{\cosh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) - \sinh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right)}{b\left(\cosh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) - \sinh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right)\right) + p} + c. \tag{25}$$

Case 2

$$a = -b, \quad b = b, \quad k = k, \quad l = l, \quad p = p, \quad r = -\frac{k^3}{2}, \tag{26}$$

where b, k, l and p are arbitrary non-zero parameters. Then, we obtain

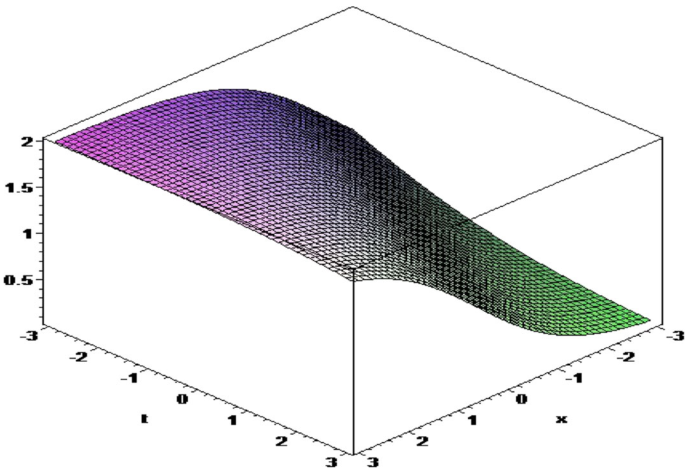

Fig. 1 The traveling solution for $u(x, y, t)$ obtained in Case 1, when $k = 1$, $l = 3$, $\xi_0 = 1$, $a = 2$, $b = 2$, $c = 0$, $p = 2$, $y = 0$, $y = 0$

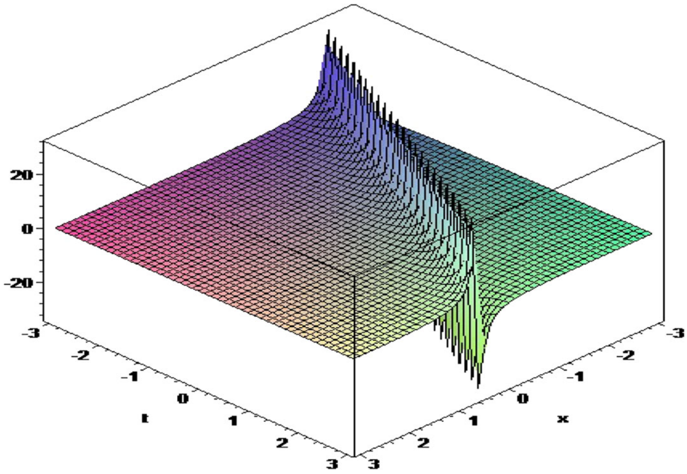

Fig. 2 The traveling solution for $u(x, y, t)$ obtained in Case 2, when $k = 1$, $l = 3$, $\xi_0 = 1$, $a = 2$, $b = 2$, $c = 0$, $p = 2$, $y = 0$, $y = 0$

$$w_2(x,y,t) = -b\cosh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) - b\sinh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) + p. \tag{27}$$

By substituting Eq. (27) into Eq. (18), we find solitary wave solution of nonlinear evolution Eq. (1) as follows

$$u_2(x,y,t) = 2bk\frac{\sinh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) + \cosh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right)}{-b\left(\cosh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right) + \sinh\left(-kx - ly + \frac{k^3}{2}t - \xi_0\right)\right) + p} + c. \tag{28}$$

2. Let us consider $w(x,y,t)$ of the form

$$w(x,y,t) = a\cos(kx + ly + rt + \xi_0) + b\sin(kx + ly + rt + \xi_0) + p, \tag{29}$$

where a, b, k, l, r, ξ_0 and p are constants be determined later and ξ_0 is an arbitrary constant. Substituting (29) into (19) and (20), we get a system of algebraic equations. Solving the system of equations, we find:

$$a = bi, \quad b = b, \quad k = k, \quad l = l, \quad p = p, \quad r = \frac{k^3}{2}, \tag{30}$$

where b, k, l and p are arbitrary non-zero parameters. Then, we obtain

$$w_3(x,y,t) = bi\cos\left(kx + ly + \frac{k^3t}{2} + \xi_0\right) + b\sin\left(kx + ly + \frac{k^3t}{2} + \xi_0\right) + p \tag{31}$$

By substituting Eq. (31) into Eq. (18), we find periodic wave solution of nonlinear evolution Eq. (1) as follows (Fig. 3)

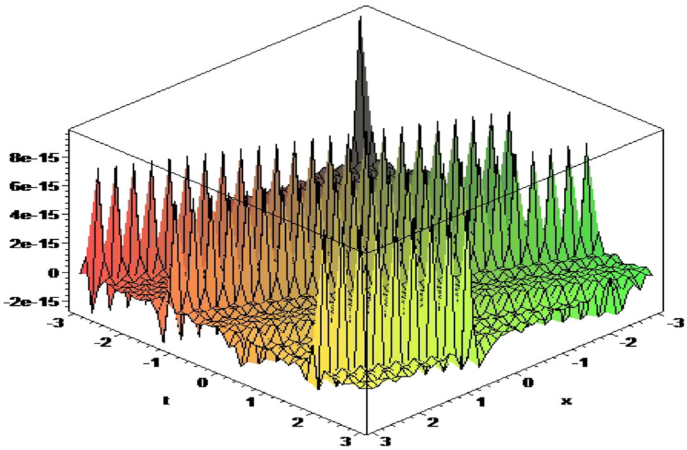

Fig. 3 The traveling solution for $u(x, y, t)$ obtained in (2), when $k = 1$, $l = 3$, $\xi_0 = 1$, $a = 2$, $b = 2$, $c = 0$, $p = 2$, $y = 0$, $y = 0$

$$u_3(x,y,t) = \frac{2bk\left(-\sin\left(kx+ly+\frac{k^3t}{2}+\xi_0\right)i+\cos\left(kx+ly+\frac{k^3t}{2}+\xi_0\right)\right)}{bi\cos\left(kx+ly+\frac{k^3t}{2}+\xi_0\right)+b\sin\left(kx+ly+\frac{k^3t}{2}+\xi_0\right)+p}+c \quad (32)$$

3. Now we are seeking the solution of Eq. (1) as in the following form

$$w(x,y,t) = A\exp(k_1x+l_1y+r_1t+\xi_{10})+B\exp(k_2x+l_2y+r_2t+\xi_{20})+C, \quad (33)$$

where A, B, C, k_i, l_i, r_i $(i = 1, 2)$ are constants to be determined later and ξ_{i0} $(i = 1, 2)$ are arbitrary constants. By substituting Eq. (33) into Eqs. (19) and (20), we get

$$k_1 = k_2, \quad l_1 = l_1, \quad l_2 = l_2, \quad r_1 = -\frac{k_2^3}{2}, \quad r_2 = -\frac{k_2^3}{2}, \quad (34)$$

where k_2, l_1, l_2 and k_2 are arbitrary non-zero parameters. Therefore, we find

$$w_4(x,y,t) = A\exp\left(k_2x+l_2y-\frac{k_2^3}{2}t+\xi_{10}\right)+B\exp\left(k_2x+l_2y-\frac{k_2^3}{2}t+\xi_{20}\right)+C \quad (35)$$

Then by substituting Eq. (35) into Eq. (18), we obtain exponential solution for non-linear evolution Eq. (1) as follows (Fig. 4)

$$u_4(x,y,t) = 2k_2\frac{A\exp\left(k_2x+l_2y-\frac{k_2^3}{2}t+\xi_{10}\right)+B\exp\left(k_2x+l_2y-\frac{k_2^3}{2}t+\xi_{20}\right)}{A\exp\left(k_2x+l_2y-\frac{k_2^3}{2}t+\xi_{10}\right)+B\exp\left(k_2x+l_2y-\frac{k_2^3}{2}t+\xi_{20}\right)+C}+c. \quad (36)$$

Note that, if we substitute zero instead of arbitrary constant c in the obtained solutions, we can easily verify the exact explicit solutions of (2 + 1)-dimensional nonlinear evolution equation for Cole–Hopf transformation Eq. (21).

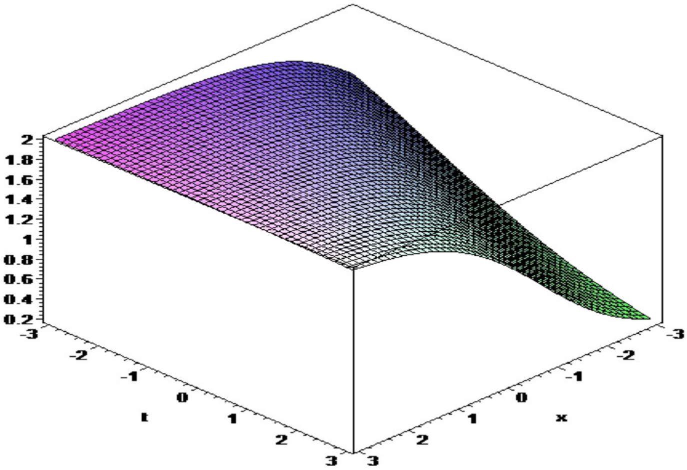

Fig. 4 The traveling solution for $u(x, y, t)$ obtained in (3),when $\xi_{10} = 1$, $\xi_{20} = 2$, $a = 2$, $b = 2$, $c = 0$, $p = 2$, $k_2 = 1$, $, l_2 = 3$, $y = 0$

4 Conclusion

In this work, we derive Auto-Bäcklund transformations of a nonlinear evolution equation by using Maple and concept of the extended homogeneous balance method. Also, the exact explicit solutions of the mentinoned equation are verified by using these transformations. These explicit solutions include exponential function, rational function, and periodic wave solutions. Also we can note that the obtained solutions are different form the ones appear in literature (Kaplan and Ozer 2018). This algorithm is useful and reliable. The obtained solutions are likely to be important for the explanation of some practical physical problems. All solutions have been checked by symbolic computation programming.

5 Physical explanations of obtained solutions

In the current work, we have obtained three types of analytical solutions, namely trigonometric, hyperbolic and exponential function solutions. We present some graphs of these types of solutions by setting special values as parameters to visualize the underlying mechanism of the original equations. The figures give a good impression about the change of amplitude and width of the wave. Finally, we note that the obtained solutions are periodic and kink type solutions.

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