



Auxiliary equation method for time-fractional differential equations with conformable derivative

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ARTICLE INFO

Article history:

Received 21 April 2017

Received in revised form 17 July 2017

Accepted 15 October 2017

Available online 7 November 2017

Keywords:

Conformable fractional derivative

Auxiliary equation method

Time-fractional differential equations

ABSTRACT

In this paper, the auxiliary equation method is applied to obtain analytical solutions of $(2 + 1)$ -dimensional time-fractional Zoomeron equation and the time-fractional third order modified KdV equation in the sense of the conformable fractional derivative. Given equations are converted to the nonlinear ordinary differential equations of integer order; and then, the resulting equations are solved using a novel analytical method called the auxiliary equation method. As a result, some exact solutions for them are successfully established. The exact solutions obtained by the proposed method indicate that the approach is easy to implement and effective.

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1. Introduction

The fractional calculus, which was introduced by Leibnitz in 1695, has attracted much attention. The impact of this fractional calculus in both pure and applied science and engineering disciplines started to increase substantially during the past twenty years seemingly [1,2].

Noticing remarkable various applications of fractional derivative, it is then apparent to search proper definition and method of solutions to the resulted fractional differential equations. Accordingly, many proposed definitions from different point of view of applications have been introduced in the literature such as Hilfer, Riemann–Liouville, Caputo and so on.

The exact solutions of nonlinear fractional differential equations play a crucial role in the mathematical physics. It is used to describe the observed various qualitative and quantitative features of nonlinear phenomena in many fields of mathematical physics and nonlinear sciences. The availability of symbolic computation packages can facilitate many direct and powerful approaches to establish exact solutions to fractional differential equations. Researchers have presented various methods to search exact solutions such as modified simple equation method [3,4], trial equation method [5–7], first integral method [8–12], (G'/G) -expansion method [13,14], sub equation method [15], exp-function method [16], homotopy perturbation method [17], homotopy analysis method [18], generalized Kudrayshov method [19–21], functional variable method [22] and so on [23–25].

In [26], a new approach has been proposed based on the extended tanh-function method which was called a new auxiliary equation method. By using this method, Sirendaoreji obtained some new exact travelling wave solutions with the constraint $c = 0$ of some nonlinear partial differential equations [26–29]. This approach has been applied to nonlinear fractional differential equations with Riemann–Liouville derivative [30]. In this paper, our goal is to apply the algorithm to nonlinear fractional differential equations with conformable fractional derivative.

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2. Brief of conformable fractional derivative

Recently, the authors Khalil et al. introduced a new simple and intriguing definition of the fractional derivative called conformable fractional derivative [31]. This derivative is well-behaved and obeys the Leibniz rule and chain rule. Here we give the definition of the conformable fractional derivative and some useful properties of this new derivative [32].

Definition 1. Suppose $f : [0, \infty) \rightarrow R$ be a function. Then, the conformable fractional derivative of f of order $\alpha, 0 < \alpha \leq 1$, is defined as

$$(T_\alpha f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$. Some useful properties can be listed as follows:

1. Linearity: $T_\alpha(af + bg) = a(T_\alpha f) + b(T_\alpha g)$, for all $a, b \in R$
2. Leibniz rule: $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$
3. Chain rule: Let $f : (0, \infty) \rightarrow R$ be a differentiable and α -differentiable function, g be a differentiable function defined in the range of f .

$$T_\alpha(f \circ g)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

Moreover, the following rules hold.

$$T_\alpha(t^p) = pt^{p-\alpha}, \text{ for all } p \in R$$

$$T_\alpha(\lambda) = 0, \text{ for all constant functions } f(t) = \lambda$$

$$T_\alpha(f/g) = \frac{g(T_\alpha f) - f(T_\alpha g)}{g^2}.$$

Additively, if f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

3. Auxiliary equation method

Suppose that we have a nonlinear fractional differential equation with the conformable time-fractional derivative, say in the independent variables x and t and dependent variable u , given by

$$F(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \dots) = 0, \tag{1}$$

where F is a polynomial of $u(x, t)$ and its various partial conformable derivatives including the highest order derivatives and nonlinear terms.

First, by using the new definition for travelling wave variable

$$u(x, t) = u(\xi), \quad \xi = kx - l \frac{t^\alpha}{\alpha}, \tag{2}$$

where k and l are nonzero arbitrary constants to be determined later, Eq. (1) can be rewritten as a nonlinear ordinary differential equation (ODE) as follows:

$$Q(u, u', u'', \dots) = 0. \tag{3}$$

According to the algorithm, we can seek for the solutions of Eq. (3) in the form:

$$u(\xi) = \sum_{i=0}^{2N} a_i F^i(\xi), \tag{4}$$

where $a_i (i = 0, \dots, 2N)$ are constants to be determined later and the positive integer N can be determined by using homogeneous balance method between the highest order derivatives and the nonlinear terms appearing in Eq. (3). Here $F(\xi)$ satisfies the following variable separated ordinary differential equation:

$$(F')^2(\xi) = aF^2(\xi) + bF^4(\xi) + cF^6(\xi), \tag{5}$$

where a, b, c are parameters to be determined.

The positive integer N can be determined by considering the homogeneous balance between the highest order derivative term with the highest order nonlinear term appearing in Eq. (3). Then by substituting Eq. (4) along with Eq. (5) into Eq. (3) and equating the coefficients of all powers of $F(\xi)$ to zero yields a set of algebraic equations for unknowns $a, b, c; a_i (i = 0, \dots, 2N), k$ and l . We find the solutions of the set of algebraic equations by Maple and substitute the obtained solutions in this step back into Eq. (3) so as to obtain the exact solutions of Eq. (1) [33–35].

4. Applications

In this section, we have dealt with two fractional differential equations as an application of the auxiliary equation method.

4.1. (2 + 1)-dimensional conformable time-fractional Zoomeron equation

First, we consider the (2 + 1)-dimensional conformable time-fractional Zoomeron equation [36]:

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} \left[\frac{u_{xy}}{u} \right] - \frac{\partial^2 u}{\partial x^2} \left[\frac{u_{xy}}{u} \right] + 2 \frac{\partial^\alpha u}{\partial t^\alpha} [u^2]_x = 0, \quad 0 < \alpha \leq 1. \tag{6}$$

We use the following transformation for converting Eq. (6) to an ODE

$$u(x, y, t) = u(\xi), \quad \xi = kx + my - l \frac{t^{\alpha_1}}{\alpha_1}, \tag{7}$$

where k, m and l are nonzero constants.

$$km l^2 \left(\frac{u''}{u} \right)'' - k^3 m \left(\frac{u''}{u} \right)'' - 2kl(u^2)'' = 0. \tag{8}$$

Then, we integrate Eq. (8) twice with respect to ξ to find

$$km (l^2 - k^2) u'' - 2klu^3 - Au = 0, \tag{9}$$

where A is a nonzero constant of integration, while the second constant of integration is vanishing. Here, it is obvious from the homogeneous balance principle that the balancing number is 1. From there, we deduced that the exact solution of Eq. (9) is

$$u(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi), \tag{10}$$

where a_0, a_1 and a_2 are constants to be determined later. From the substitution of this solution we can verify that

$$u''(\xi) = \frac{1}{2} (2a_1 F + 4bF^3 + 6cF^5) (a_1 + 2a_2 F) + 2a_2 (aF^2 + bF^4 + cF^6), \tag{11}$$

and

$$u^3(\xi) = (a_0 + a_1 F + a_2 F^2)^3. \tag{12}$$

Substituting Eqs. (10)–(12) with the condition Eq. (5) into Eq. (9) and collecting all terms with the same power of $F^i(\xi)$, ($i = 0, 1, \dots, 6$), we obtain the following system:

$$\begin{aligned} F^6(\xi) &: -8k^3 m a_2 c - 2kla_2^3 + 8kml^2 a_2 c = 0, \\ F^5(\xi) &: -6kla_1 a_2^2 - 3k^3 m a_1 c + 3kml^2 a_1 c = 0, \\ F^4(\xi) &: -6kla_1^2 a_2 + 6kml^2 a_2 b - 6k^3 m a_2 b - 6kla_0 a_2^2 = 0, \\ F^3(\xi) &: -2k^3 m a_1 b - 2kla_1^3 + 2kml^2 a_1 b - 12kla_0 a_1 a_2 = 0, \\ F^2(\xi) &: 4kml^2 a_2 a - 4k^3 m a_2 a - Aa_2 - 6kla_0^2 a_2 - 6kla_0 a_1^2 = 0, \\ F^1(\xi) &: -Aa_1 + kml^2 a_1 a - 6kla_0^2 a_1 - k^3 m a_1 a = 0, \\ F^0(\xi) &: -Aa_0 - 2kla_0^3 = 0. \end{aligned}$$

From the solutions of the system above, we get

$$\begin{aligned} a &= \frac{A}{2km(k^2 - l^2)}, \quad b = b, \quad c = \frac{(k^2 - l^2) mb^2 k}{2A}, \\ a_0 &= \mp \frac{1}{\sqrt{-\frac{2kl}{Al}}}, \quad a_1 = 0, \quad a_2 = \pm \sqrt{-\frac{2k}{Al}} (k^2 - l^2) mb, \end{aligned}$$

Therefore, we find two different cases for $F(\xi)$.

Case 1:

$$F_1(\xi) = \exp \left(-\frac{\sqrt{2A}(-\xi + C_1)}{2\sqrt{kmA}(k^2 - l^2)} \right) \sqrt{\frac{Ae^\psi}{bkme^\psi (k^2 - l^2) - e^\psi}},$$

where

$$\Phi = \frac{\sqrt{2}\xi A}{\sqrt{mAk(k^2 - l^2)}}, \quad \Psi = \frac{\sqrt{2}C_1 A}{\sqrt{mAk(k^2 - l^2)}}.$$

Then, the exact solutions of (2 + 1)-dimensional conformable time-fractional Zoomeron equation can be written as

$$u_{1,2}(x, y, t) = \pm \frac{1}{\sqrt{2}} \frac{bkme^\Phi(k^2 - l^2) + e^\Psi}{\sqrt{-\frac{k}{Al}l(bkme^\Phi(k^2 - l^2) - e^\Psi)}}.$$

where $\xi = kx + my - l\frac{t^{\alpha_1}}{\alpha_1}$.

Case 2:

$$F_2(\xi) = \exp\left(\frac{\sqrt{2}A(-\xi + C_1)}{2\sqrt{kmA(k^2 - l^2)}}\right) \sqrt{\frac{Ae^\Phi}{bkme^\Psi(k^2 - l^2) - e^\Phi}},$$

and therefore the exact solutions of (2 + 1)-dimensional conformable time-fractional Zoomeron equation can be written as

$$u_{3,4}(x, y, t) = \pm \frac{1}{\sqrt{2}} \frac{bkme^\Psi(k^2 - l^2) + e^\Phi}{\sqrt{-\frac{k}{Al}l(bkme^\Psi(k^2 - l^2) - e^\Phi)}},$$

where $\xi = kx + my - l\frac{t^{\alpha_1}}{\alpha_1}$.

The obtained solutions can be comparable with the results in the literature. We can note that comparing our solutions with [36], it can be seen that by choosing suitable values for the parameters, similar solutions can be verified. Also in [37] Alquran and Al-Khaled have investigated analytical solutions of (2 + 1)-dimensional conformable time-fractional Zoomeron equation by using extended tanh method, exponential function method and $\text{sech}^p - \tanh^p$ method. Note that our solutions are different.

4.2. Conformable time-fractional third order modified KdV equation

The new exact solutions of the conformable time-fractional third order modified KdV equation [38] are desired to obtain

$$D_t^\alpha u + pu^2u_x + qu_{xxx} = 0, \quad 0 < \alpha \leq 1, \tag{13}$$

where p, q are arbitrary constants. Similarly, we use the transformation

$$u(x, t) = u(\xi), \quad \xi = kx - l\frac{t^{\alpha_1}}{\alpha_1}, \tag{14}$$

where k and l are nonzero arbitrary constants.

$$-lu' + pku^2u' + qk^3u''' = 0. \tag{15}$$

By integrating once with respect to ξ

$$-lu + \frac{1}{3}pku^3 + qk^3u'' = 0. \tag{16}$$

It is easy to see that the balancing number is 1. So the solution takes the same form of Eq. (9). Substituting Eq. (9) into Eq. (16) and setting the coefficients of $F^i(\xi)$, ($i = 0, 1, \dots, 6$) to be zero, we find

$$\begin{aligned} F^6(\xi) &: \frac{pka_2^3}{3} + 8qk^3a_2c = 0, \\ F^5(\xi) &: pka_1a_2^2 + 3qk^3a_1c = 0, \\ F^4(\xi) &: pka_0a_2^2 + pka_1^2a_2 + 6qk^3a_2b = 0, \\ F^3(\xi) &: 2pka_0a_1a_2 + 2qk^3a_1b + \frac{pka_1^3}{3} = 0, \\ F^2(\xi) &: -la_2 + pka_0a_1^2 + 4qk^3a_2a + pka_0^2a_2 = 0, \\ F^1(\xi) &: -la_1 + pka_0^2a_1 + qk^3a_1a = 0, \\ F^0(\xi) &: -la_0 + \frac{pka_0^3}{3} = 0. \end{aligned}$$

From the solutions of the nonlinear algebraic system above we obtain different cases as follows:

$$\begin{aligned}
 \text{(I)} \quad & a = \frac{l}{qk^3}, \quad b = b, \quad c = 0, \quad a_0 = 0, \\
 & a_1 = \mp \sqrt{-\frac{6qb}{p}}, \quad a_2 = 0, \\
 \text{(II)} \quad & a = -\frac{l}{2qk^3}, \quad b = b, \quad c = -\frac{k^3qb^2}{2l}, \quad a_0 = \mp \frac{\sqrt{3}}{p\sqrt{\frac{k}{lp}}}, \\
 & a_1 = 0, \quad a_2 = \pm 2\sqrt{3}k^2bq\sqrt{\frac{k}{lp}}.
 \end{aligned} \tag{17}$$

The corresponding functions of the values above can be written as follows, respectively:

(I) We obtain two different values for $F(\xi)$:

Case 1.1:

$$F_1(\xi) = -\frac{4qkl \exp\left(\frac{\sqrt{qkl}(\xi+C_1)}{qk^2}\right)}{4q^3k^5bl \exp\left(\frac{2\sqrt{qkl}\xi}{qk^2}\right) - \exp\left(\frac{2\sqrt{qkl}C_1}{qk^2}\right)}. \tag{18}$$

Therefore, the new exact solutions of the conformable time-fractional third order modified KdV equation are obtained as follows:

$$u_{1,2}(\xi) = \pm \frac{4\sqrt{6}\sqrt{-\frac{qb}{p}}k^2ql \exp\left(\frac{\sqrt{qkl}(\xi+C_1)}{qk^2}\right)}{4q^3k^5bl \exp\left(\frac{2\sqrt{qkl}\xi}{qk^2}\right) - \exp\left(\frac{2\sqrt{qkl}C_1}{qk^2}\right)},$$

where $\xi = kx - l\frac{t^{\alpha_1}}{\alpha_1}$.

Case 1.2:

$$F_2(\xi) = -\frac{4qkl \exp\left(\frac{\sqrt{qkl}(\xi+C_1)}{qk^2}\right)}{4q^3k^5bl \exp\left(\frac{2\sqrt{qkl}C_1}{qk^2}\right) - \exp\left(\frac{2\sqrt{qkl}\xi}{qk^2}\right)}.$$

Therefore, the new exact solutions of the conformable time-fractional third order modified KdV equation are obtained as follows:

$$u_{3,4} = \pm \frac{4\sqrt{6}\sqrt{-\frac{qb}{p}}k^2ql \exp\left(\frac{\sqrt{qkl}(\xi+C_1)}{qk^2}\right)}{4q^3k^5bl \exp\left(\frac{2\sqrt{qkl}C_1}{qk^2}\right) - \exp\left(\frac{2\sqrt{qkl}\xi}{qk^2}\right)},$$

where $\xi = kx - l\frac{t^{\alpha_1}}{\alpha_1}$.

(II) Similarly, we obtain two different values for $F(\xi)$:

Case 2.1:

$$F_3(\xi) = \sqrt{\frac{l \exp\left(\frac{\sqrt{-2qkl}\xi}{qk^2}\right)}{\exp\left(\frac{\sqrt{-2qkl}C_1}{qk^2}\right) - k^3qb \exp\left(\frac{\sqrt{-2qkl}\xi}{qk^2}\right)}}.$$

Therefore, the new exact solutions of the conformable time-fractional third order modified KdV equation are obtained as follows:

$$u_{5,6}(x, t) = \mp \frac{\sqrt{3} \left(\exp\left(\frac{\sqrt{-2qkl}C_1}{qk^2}\right) + k^3qb \exp\left(\frac{\sqrt{-2qkl}\xi}{qk^2}\right) \right)}{p\sqrt{\frac{k}{lp}} \left(\exp\left(\frac{\sqrt{-2qkl}C_1}{qk^2}\right) - k^3qb \exp\left(\frac{\sqrt{-2qkl}\xi}{qk^2}\right) \right)},$$

where $\xi = kx - l\frac{t^{\alpha_1}}{\alpha_1}$.

Case 2.2:

$$F_4(\xi) = \exp\left(\frac{\sqrt{-qkl}(C_1 - \xi)}{\sqrt{2}qk^2}\right) \sqrt{\frac{l \exp\left(\frac{\sqrt{-2qkl}\xi}{qk^2}\right)}{k^3qb \exp\left(\frac{\sqrt{-2qkl}C_1}{qk^2}\right) - \exp\left(\frac{\sqrt{-2qkl}\xi}{qk^2}\right)}}.$$

So that, the new exact solutions of the conformable time-fractional third order modified KdV equation are obtained as follows:

$$u_{7,8}(x, t) = \pm \frac{\sqrt{3} \left(\exp \left(\frac{\sqrt{-2q} k \xi}{qk^2} \right) + k^3 q b \exp \left(\frac{\sqrt{-2q} k C_1}{qk^2} \right) \right)}{p \sqrt{\frac{k}{ip} \left(\exp \left(\frac{\sqrt{-2q} k \xi}{qk^2} \right) + k^3 q b \exp \left(\frac{\sqrt{-2q} k C_1}{qk^2} \right) \right)}},$$

where $\xi = kx - t^{\frac{\alpha_1}{\alpha_1}}$.

Finally, we can note that our solutions can be comparable with the results in the literature. In [38], the exact solutions of time fractional modified KdV equation by using (G'/G) -expansion method and improved (G'/G) -expansion method. It can be seen that by choosing suitable values for the parameters, some similar solutions can be verified whereas some solutions are different.

5. Conclusions

To summarize, based on the auxiliary equation method, we have obtained many new exact solutions of the two nonlinear fractional equation with conformable derivative. We have discussed the new definition of wave transformation and new conformable fractional derivative for converting the nonlinear fractional differential equations into the ODEs. We have constructed a variety of new solutions of the mentioned equations. Since the technique is direct and powerful it can be used to handle a variety of equations which appears in applications in several branches of the nonlinear sciences. Also the correctness of the solutions has been checked by putting them back into the original equation with the aid of Maple. Therefore we have provided an extra measure of confidence in the results.

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