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The unified method for abundant soliton solutions of local time fractional nonlinear evolution equations

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ABSTRACT

This work studies two important temporal fractional nonlinear evolution equations, namely the (2+1)-dimensional Chaffee–Infante equation and (1+1)-dimensional Zakharov equation by way of the unified method along with properties of local M-derivative. The typical structures of fractional optical soliton wave solutions are obtained in polynomial and rational forms. Further, to grant the validity of non-singular solutions are given with limitation conditions and graphically depicted in 3D. Also, to expose the effect of a local fractional parameter on expected non-singular solutions are depicted through 2D graphs. The predicted solutions are revealing that the proposed approach is straightforward and valuable to find the solitary wave solutions of other nonlinear evolution equations.

1. Introduction

Nonlinearity is a captivating factor of nature, and many researchers think about nonlinear science to be the primary domain to recognize the nature of physical phenomena. The mathematical analysis of complicated phenomena that modify with time depends on the escalation within the research of a numerous perspective of nonlinear ordinary and partial differential equations. These fashions are designed in numerous disciplines including population ecology to economics, neural networks, fluid mechanics, solid-state physics, plasma physics, fiber optics and plenty of other scientific and engineering fields. In this way, in the last few years, the discovery of soliton solutions of prior expressed marvels has been fascinating and inconceivable the subject of the study and the related issue is the development of soliton wave solutions for a wide array of nonlinear evolution equations (NLEEs). One of the interesting subjects for investigating the propagation of solitons through nonlinear fiber optics is the theory of optical solitons. The propagation of ultrashort pulses of electromagnetic radiation is a multidimensional phenomenon in a nonlinear medium. The interaction between various physical entities such as dispersion, diffraction and nonlinear response influences the pulse dynamics. Promptly, many mathematicians and physical researchers have made remarkable efforts to obtain soliton wave solutions from such NLEEs as well as a variety of capable and efficient strategies, including the first integral technique [1], the Exp-function technique [2], the homogeneous balance technique [3], the modified simple equation technique [4], the unified technique [5], the Bäcklund transformation technique [6], the exp(-P(κ))-expansion method [7], the Adomian decomposition technique [8], the bilinear transformation technique [9], the Hirota-bilinear technique [10,11], the extended trial function technique [12], extended sinh–Gordon equation expansion method [13], the Extended Fan sub-equation method [14] and many more, have been designed.

Among these strategies, the unified method is one of the foremost easy, direct and viable algebraic strategy to discover the exact solutions of NLEEs. This approach permits a researcher to discover the traveling wave solutions in two forms, the first one is polynomial, and the other is a rational form of functional solutions. This method has been frequently utilized by many researchers to retrieve the traveling wave solutions of NLEEs. In 2018, M.S. Osman uses the unified method to the conformable time fractional nonlinear Schrödinger equation with perturbation terms [15]. As a result, he obtained the valuable optical soliton solutions in solitary, periodic, elliptic and soliton wave forms in polynomial and rational forms. These days, to think about the fractional NLEEs gets to be a hot subject within the investigate zone [16–23].

In 1695, L'Hospital inquired Leibniz in a letter that whether or not the sense of expanding the order of derivative from integer to

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fraction is viable? This concept has begun with the improvement of an advanced calculus that changed into named the arbitrary order calculus and is presently extensively referred to as the fractional calculus. Fractional-order systems can describe physical phenomena in a more elaborate way of dynamical systems arising in mathematical physics, electrochemistry, biology and chaotic systems. Up to now, different types of fractional derivatives have been proposed, including from Riemann–Liouville, Caputo, Hadamard, Caputo–Hadamard, Conformable and Grunwald–Letnikov [24–27].

In 2014, Katugampola proposed a novel fractional derivative that generalizes the so-called alternative fractional derivative [28]. This new form of derivative is recognizable as a local M-derivative and guarantees some features of an integer order calculus like linear property, product rule, quotient rule, chain rule, and composition of functions. In similar to the ordinary derivative, the local M-derivative of constant function leads to zero. An interesting property of this novel fractional derivative is that when fractional parameter approaches unity, it carries on like a standard derivative.

The present paper is proposed at investigating the local temporal fractional (2+1)-dimensional Chaffee–Infante equation (CIE) and (1+1)-dimensional Zakharov equation (ZE) by using the unified method [29,30]. The CIE is a notable reaction duffing equation emerging in mathematical physics while ZE describes the interaction between low-frequency ion-acoustic waves and high-frequency Langmuir waves. As a result, the fractional soliton wave solutions are obtained. In addition to the reported solutions are plotted in 2D and 3D figures and the behavior of solitons for the local fractional parameter is demonstrated.

The remaining sections of the article are arranged as follows: Section 2 discusses the preliminary definition of the utilized fractional derivative and its properties. Section 3 includes the analysis of the suggested approach. In Section 4, we present the interpretation of the governing equations, the extraction of fractional solitons from the governing equations and the graphical portrayal of predicted non-singular solitons. Section 5 presents the physical explanation of the retrieved non-singular soliton solutions. In Section 6, we present the conclusion of this paper.

2. Description of the local M-derivative

This section contains the basic idea as well as the characteristics of local M-derivative, that removes all deficiencies of existing derivatives [28].

Definition. Let $h : [0, \infty) \to \mathbb{R}$ also $t > 0$, we define the fractional order of local M-derivative $\gamma \in (0, 1)$ of function h as

$$D_M^{\gamma;\mu}\{h(t)\} = \lim_{\epsilon \to 0} \frac{h\big(tE_\mu(\epsilon t^{-\gamma})\big) - h(t)}{\epsilon}, \quad \forall\ t > 0, \tag{1}$$

where M shows the derived function includes a function called Mittag-Leffler $(E_\mu(\cdot), \forall \mu > 0)$ alongside one parameter, that is entire function. Besides, if $h(t)$ is l- times differentiable within a specific extent of $(0, l)$, $0 > l$ and $\lim_{t\to 0^+} D_M^{\gamma;\mu}\{h(t)\}$ exists, then it leads to the following relation

$$D_M^{\gamma;\mu}\{h(0)\} = \lim_{t\to 0^+} D_M^{\gamma;\mu}\{h(t)\}, \tag{2}$$

with

$$D_M^{\gamma;\mu}\{h(t)\} = \frac{t^{1-\gamma}}{\Gamma(\mu+1)} \frac{d}{dt}\{h(t)\}, \tag{3}$$

and therefore results the relation (3),

$$D_M^{\gamma;\mu}\left(\frac{\Gamma(\mu+1)t^\gamma}{\gamma}\right) = 1. \tag{4}$$

Such a derivative of fractional order also satisfies the property mentioned below:

$$D_M^{\gamma;\mu}(g \cdot h)(l) = g'(h(l)) D_M^{\gamma;\mu} h(l), \tag{5}$$

therefore, from Eqs. (4) and (5), the corresponding relationship can be developed:

$$D_M^{\gamma;\mu} F\left[\frac{\Gamma(\mu+1)t^\gamma}{\gamma}\right] = F'\left[\frac{\Gamma(\mu+1)t^\gamma}{\gamma}\right] D_M^{\gamma;\mu}\left[\frac{\Gamma(\mu+1)t^\gamma}{\gamma}\right] = F'\left[\frac{\Gamma(\mu+1)t^\gamma}{\gamma}\right], \tag{6}$$

with

$$\eta = \frac{b}{\gamma}\Gamma\big(\mu+1\big)t^\gamma, \tag{7}$$

where b is a constant and eventually we get the relation given by

$$D_M^{\gamma;\mu}\{F(\eta)\} = bF'(\eta). \tag{8}$$

3. Analysis of the suggested approach

Take into account the overall structure of temporal fractional evolution equation in a manner given by:

$$M\left(x, y, t, \frac{\partial^\gamma p}{\partial t^\gamma}, \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial^{2\gamma} p}{\partial t^{2\gamma}}, \frac{\partial^2 p}{\partial x^2}, \dots\right) = 0, \quad 0 \le t,\ \gamma \in (0, 1]. \tag{9}$$

where M is a function in $p(x, y, t)$ with x, y, t as independent variables. While applying the fractional traveling wave variable of form $r = x + y - v\frac{t^\gamma}{\gamma}$, Eq. (9) can be yielded into the following relation

$$H\left(p, \frac{dp}{dr}, \frac{d^2p}{dr^2}, \dots\right) = 0, \tag{10}$$

where H is a function of $p(r)$ together with corresponding derivatives and v is velocity of soliton. To uncover the solution for Eq. (10) throughout the UM method, that permits to search those solutions in two ways including: rational as well as polynomial function solutions [31].

3.1. Polynomial function solution

Consider Eq. (10) has a polynomial solution as

$$p(\tau) = \sum_{i=0}^{n} \alpha_i \phi^i(\tau), \tag{11}$$

with satisfying

$$(\phi'(\tau))^\sigma = \sum_{i=0}^{\sigma k} \beta_i \phi^i(\tau), \quad \tau = x + y - v\frac{t^\gamma}{\gamma}, \quad \sigma = 1, 2. \tag{12}$$

Here, in Eqs. (11) and (12), α_i and β_i are the unspecified constants to be determined, such as solution provided by Eq. (11) satisfies the Eq. (10).

It is essential to point out that the numeric values of n and k are to be taken by applying the balance principle between the highest order of linear and nonlinear terms involved in Eq. (10) [32]. Similarly, the unknown coefficients in Eq. (11) are to be determined precisely by applying the condition that is usually referred to as the condition of consistency. Now in order to solve the Eq. (11), the UM method solves Eq. (11) for elementary or elliptic solutions when $\sigma = 1$ or $\sigma = 2$, respectively.

3.2. Rational function solution

The fundamental precept of this approach is to conclude that Eq. (10) has a rational solution as

$$p(\tau) = \frac{\sum_{i=0}^{n} P_i \phi^i(\tau)}{\sum_{i=0}^{r} Q_i \phi^i(\tau)}, \quad n \ge r, \tag{13}$$

with satisfying

$$(\phi'(\tau))^\sigma = \sum_{i=0}^{\sigma k} \beta_i \phi^i(\tau), \quad \tau = x + y - v\frac{t^\gamma}{\gamma}, \quad \sigma = 1, 2. \tag{14}$$

Here, in Eqs. (13) and (14), P_i, Q_i and β_i are the unspecified constants to be determined, such as solution provided by Eq. (13) satisfies the Eq. (10).

It is essential to point out that the numeric values of n and k are to be taken by applying the balance principle between the highest order of linear and nonlinear terms involved in Eq. (10) [32]. Similarly, the unknown coefficients in Eq. (13) are to be determined precisely by applying the condition that is usually referred to as the condition of consistency. Now in order to solve the Eq. (13), the UM method solves Eq. (13) for elementary or elliptic solutions when $\sigma = 1$ or $\sigma = 2$, respectively.

To obtain the solutions for Eq. (10) throughout the UM in two ways like polynomial function or rational function solutions, the succeeding points to be followed:

i. Solve the system of algebraic equation using any symbolic computer software.

ii. Solve the auxiliary equation.

iii. Eventually, acquire the exact solution provided by Eq. (11) (or Eq. (13)).

4. Governing equations

In this section, we discuss the two temporal fractional governing equations namely the (2+1)-dimensional CIE [29] and ZE in (1+1)-dimension [30] and obtained the fairly interesting useful solutions by using the unified method along with local M-derivative.

4.1. The temporal fractional form of (2+1)-dimensional CIE

We first consider the temporal fractional form of (2+1)-dimensional CIE [29] in the following way:

$$\left(\frac{\partial^\gamma q}{\partial t^\gamma}\right)_x + \left(-\frac{\partial^2 q}{\partial x^2} + \alpha q^3 - \alpha q\right)_x + \delta \frac{\partial^2 q}{\partial y^2} = 0, \tag{15}$$

where $q(x,y,t)$ is the wave profile and $\frac{\partial^\gamma q}{\partial t^\gamma}$ represents the temporal fraction of $q(x,y,t)$ with fraction parameter γ. Here, the diffusion term is identified by the coefficient of α, whereas δ indicates the degradation coefficient. In physical terms, the diffusion of a gas in a homogeneous medium is a significant phenomenon and the CIE supply a convenient way to recognize such phenomena. The (2+1)-dimensional CIE is a noteworthy response of Duffing equations in the physical sciences [29, 33].

Now apply the definition of novel fractional local M-derivative to the above governed equation (15) and yields the following equation

$$\left(D^{\gamma;\mu}_{M,t} q\right)_x + \left(-\frac{\partial^2 q}{\partial x^2} + \alpha q^3 - \alpha q\right)_x + \delta \frac{\partial^2 q}{\partial y^2} = 0, \tag{16}$$

where $D^{\gamma;\mu}_{M}$ specifies the fractional local M-derivative with fraction order γ. It is interesting to remember that with the aid of thinking about $\gamma = \mu = 1$, Eq. (16) can be transformed into original CIE.

In order to examine Eq. (16) via the unified approach, we apply the fractional transformation of the following form:

$$q(x,y,t) = X(\tau), \quad \tau = x + y - \frac{\upsilon}{\gamma}\Gamma(\mu+1)t^\gamma, \tag{17}$$

where τ is the fractional traveling wave variable and υ indicates the velocity of the wave profile.

Substituting Eq. (17) into Eq. (16) and performing few steps of algebra, yields

$$X''' + (\upsilon - \delta)X'' - 3\alpha X^2 X' + \alpha X' = 0,$$

and by taking into account the integration of the above ordinary differential equation and setting the integrable constant to be zero, we have

$$X'' + (\upsilon - \delta)X' + \alpha X(1 - X^2) = 0, \tag{18}$$

where $'$ shows the differentiation with respect to τ.

Now, to uncover the fairly interesting and useful solutions of Eq. (18) via the UM that permits to search those solutions in two ways including as polynomial or rational function solutions [34].

4.1.1. Polynomial function solution

For polynomial function solution, we assume the initial solution in a following way

$$X(\tau) = \sum_{i=0}^{n} \alpha_i \phi^i(\tau), \tag{19}$$

$$(\phi'(\tau))^\sigma = \sum_{i=0}^{\sigma k} \beta_i \phi^i(\tau), \quad \tau = x + y - \frac{\upsilon}{\gamma}\Gamma(\mu+1)t^\gamma, \quad \sigma = 1, 2,$$

where α_i and β_i are the unknown constants to be assessed. By employing the homogeneous balance principle [35–37] to Eq. (18) results a relation given by $n = k - 1$, for all $k \geq 2$.

Here, we have discussed two different cases $k = 2, \sigma = 1$ and $k = 2, \sigma = 2$. Hence, the Eq. (19) can be converted into

$$X(\tau) = \alpha_0 + \alpha_1 \phi(\tau), \tag{20}$$

$$(\phi'(\tau))^\sigma = \sum_{i=0}^{2\sigma} \beta_i \phi^i(\tau), \quad \sigma = 1, 2.$$

Case 1: In this case we put $\sigma = 1$ in Eq. (20) and results into the following equation

$$X(\tau) = \alpha_0 + \alpha_1 \phi(\tau), \tag{21}$$

$$\phi'(\tau) = \sum_{i=0}^{2} \beta_i \phi^i(\tau).$$

Using Eq. (21) into Eq. (18) yields a system of algebraic equations in ϕ. By pursuing some symbolic computing software to solve this system for $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \alpha, \delta, \upsilon$ and as a result, the following relation is obtained:

$$\alpha = \frac{2\beta_2^2}{\alpha_1^2}, \delta = \frac{\alpha_1 \upsilon - 3\beta_2}{\alpha_1}, \alpha_0 = \frac{\alpha_1 \beta_1 + \beta_2}{2\beta_2}, \beta_0 = \frac{\beta_1^2 \alpha_1^2 - \beta_2^2}{4\beta_2 \alpha_1^2}, \tag{22}$$

and to solve the auxiliary equation given in Eq.(21) together these values into initial polynomial solution specified by Eq. (21), we obtain the following solution of Eq. (15) for $\mu = 1$

$$q(x,y,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{1}{2}\frac{\tau \beta_2}{\alpha_1}\right)\right), \tag{23}$$

where $\tau = x + y - \upsilon\frac{t^\gamma}{\gamma}$ provided that $\upsilon > \delta$ and $\alpha_1, \beta_2 \neq 0$.

Fig. 1 shows the graphical representation of fractional optical wave solution given by Eq. (23) in 3D and 2D at suitable parameters $\upsilon = 0.06, \beta_2 = 0.58, \alpha_1 = 0.25$.

Case 2: In this case we put $\sigma = 2$ in Eq. (20) and results into the following equation

$$X(\tau) = \alpha_0 + \alpha_1 \phi(\tau), \tag{24}$$

$$\phi'(\tau) = \phi(\tau)\sqrt{\beta_0 + \beta_1 \phi(\tau) + \beta_2 \phi^2(\tau)}.$$

Using Eq. (23) into Eq. (18) in a same as we did in the last case gives:

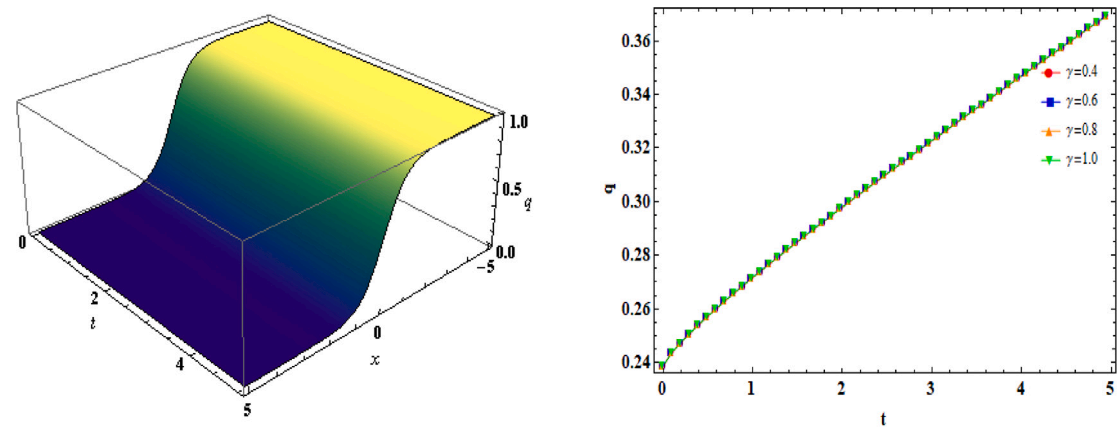
$$\alpha = \frac{2}{9}(\upsilon - \delta)^2, \alpha_0 = 1, \alpha_1 = \frac{9\beta_1}{2(\upsilon - \delta)^2}, \beta_0 = \frac{1}{9}(\upsilon - \delta)^2, \beta_2 = \frac{9\beta_1^2}{4(\upsilon - \delta)^2}, \tag{25}$$

and by solving the auxiliary equation given in Eq. (23) together these values into Eq. (23), we obtain the following solution of Eq. (15) for $\mu = 1$

$$q(x,y,t) = \frac{1}{1 - 2\beta_1 e^{\frac{1}{3}\tau(\upsilon - \delta)}}, \tag{26}$$

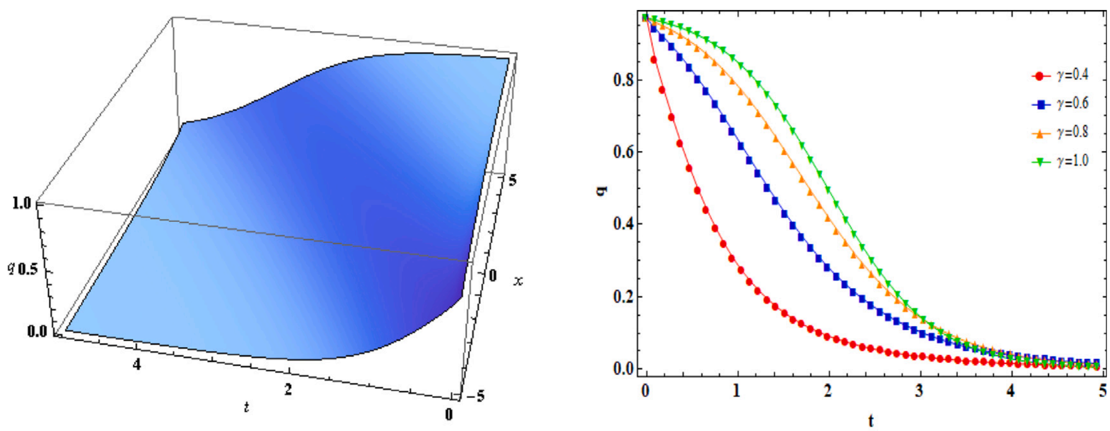
where $\tau = x + y - \upsilon\frac{t^\gamma}{\gamma}$ provided that $\upsilon > \delta$ and $\beta_1 < 0$.

Fig. 2 shows the graphical representation of fractional optical wave solution given by Eq. (26) in 3D and 2D at suitable parameters $\upsilon = 3.5, \beta_1 = -0.02, \delta = 5$.



(a) y=0.5, $\gamma = 0.8$ (b) x=0, y=0.5

Fig. 1. Graphical visualization of fractional optical wave solution given by Eq. (23) in 3D and 2D at $v = 0.06$; $\beta_2 = 0.58$; $\alpha_1 = 0.25$.



(c) y=0.5, $\gamma = 0.8$ (d) x=0, y=0.5

Fig. 2. Graphical visualization of fractional optical wave solution given by Eq. (26) in 3D and 2D at $v = 3.5$; $\beta_1 = -0.02$; $\delta = 5.0$.

4.1.2. Rational function solution

For rational function solution, we assume the initial solution in a following way

$$X(\tau) = \frac{\sum_{i=0}^{n} P_i \phi^i(\tau)}{\sum_{i=0}^{r} Q_i \phi^i(\tau)}, \quad n \geq r, \tag{27}$$

$$(\phi'(\tau))^{\sigma} = \sum_{i=0}^{\sigma k} \beta_i \phi^i(\tau), \quad \tau = x + y - \frac{\upsilon}{\gamma}\Gamma(\mu+1)t^{\gamma}, \quad \sigma = 2,$$

where P_i, Q_i and β_i are unknown constants to be assessed. Now, by utilizing the balance condition given by Lemma 2.3 [35–37], provides $k = 1$ and relation $n = r$, which permits to choose n freely.

Here, we have discussed the case when $k = 1, \sigma = 2$. Hence, the Eq. (25) can be converted into

$$X(\tau) = \frac{P_0 + P_1\phi(\tau)}{Q_0 + Q_1\phi(\tau)},$$

$$\phi'(\tau) = \sqrt{\beta_0 + \beta_1\phi(\tau) + \beta_2\phi^2(\tau)}. \tag{28}$$

Using Eq. (26) into Eq. (18) provides a system of algebraic equations in ϕ. By pursuing some symbolic computing software to solve this system for $P_0, P_1, Q_0, Q_1, \beta_0, \beta_1, \beta_2, \alpha, \sigma, \upsilon$ and we obtained the following results:

$$\alpha = \frac{2(\upsilon-\delta)^2}{9}, P_1 = \frac{2(\upsilon-\delta)^2 P_0}{\beta_1}, Q_1 = -\frac{2(\upsilon-\delta)^2 P_0}{\beta_1},$$
$$\beta_0 = \frac{9\beta_1^2}{4(\upsilon-\delta)^2}, \beta_2 = \frac{(\upsilon-\delta)^2}{9}, \tag{29}$$

and by way of solving the auxiliary equation given as Eq.(26_2) together these values into Eq. (26), we obtain the following solution of Eq. (15) for $\mu = 1$

$$q(x, y, t) = \frac{(\upsilon-\delta)P_0}{3e^{\frac{1}{3}\tau(\upsilon-\delta)}\beta_1(P_0+Q_0) - P_0(\upsilon-\delta)}, \tag{30}$$

where $\tau = x + y - \upsilon\frac{t^{\gamma}}{\gamma}$ provided that $\upsilon > \delta$ and $P_0, \beta_1 \neq 0$.

4.2. The temporal fractional form of (1+1)-dimensional ZE

To understand the ability of interaction among high-frequency Langmuir waves and low-frequency ion-acoustic waves, Zakharov [30] first developed the (1+1)-dimensional Zakharov equation (ZE). The (1+1)-dimensional form of ZE might be consider [30,38,39] in the following way

$$\imath p_t + p_{xx} + bH(|p|^2)p = pq, \tag{31}$$

$$q_{tt} - q_{xx} = (|p|^{2d}), \tag{32}$$

where $p(x, t)$ is the complex potential which indicates the envelope of high-frequency electric field [30], $q(x, t)$ is the potential function which represents the plasma density determined from its equilibrium state, whereas b, d are real parameters and H is real-valued algebraic nonlinear function.

Here, we considered the temporal fractional form of ZE in the following way

$$\imath\frac{\partial^\gamma p}{\partial t^\gamma}+\frac{\partial^2 p}{\partial x^2}+b(|p|^2)p=pq,\qquad \frac{\partial^{2\gamma} q}{\partial t^{2\gamma}}-\frac{\partial^2 q}{\partial x^2}=(|p|^2), \tag{33}$$

by taking $H(|p|^2)=|p|^2$ and $d=1$.

Now apply the definition of local M-derivative to the above governed equation (33) and yields the following equation

$$\imath D^{\gamma;\mu}_{M,t}p+\frac{\partial^2 p}{\partial x^2}+b(|p|^2)p=pq,\qquad D^{2\gamma;\mu}_{M,2t}q-\frac{\partial^2 q}{\partial x^2}=(|p|^2), \tag{34}$$

where $D^{2\gamma;\mu}_{M,2t}q$ represents the second local M-derivative of $q(x,t)$ with respect to t. It is worthwhile to keep in mind that by considering $\gamma=\mu=1$, Eq. (34) can be transformed into original ZE.

In order to examine Eq. (34) via the unified approach, we apply the complex fractional transformation of the following form:

$$p(x,t)=X(\tau)e^{\imath\phi(x,t)},\ \phi(x,t)=-ax+\frac{\omega}{\gamma}\Gamma(\mu+1)t^\gamma+\xi_0, \tag{35}$$

and

$$q(x,t)=Y(\tau),\ \tau=x-\frac{\upsilon}{\gamma}\Gamma(\mu+1)t^\gamma, \tag{36}$$

where $X(\tau)$ shows the shape of the pulse and υ is velocity of soliton, a is the wave number, while ω is the frequency and ξ_0 is phase constant.

Substituting Eqs. (35) and (36) into Eq. (34) and performing few steps of algebra, we reach into the following ordinary differential equation

$$\left(-\imath\upsilon X'-\omega X+X''-2iaX'-a^2X+bX^3-XY\right)e^{\imath\phi(x,t)}=0, \tag{37}$$

$$\upsilon^2Y''-Y''-(X^2)''=0. \tag{38}$$

After simplification, Eqs. (37) and (38) can be turned into the following two equations

$$-\imath(2a+\upsilon)X'+X''-(\omega+a^2)X+bX^3-XY=0, \tag{39}$$

$$(\upsilon^2-1)Y''-(X^2)''=0. \tag{40}$$

To set $2a+\upsilon=0$ in Eq. (39) and taking the integration of Eq. (40) twice with respect to τ, assuming constant of integration to be zero, gives

$$X''-(\omega+a^2)X+bX^3-XY=0, \tag{41}$$

$$(\upsilon^2-1)Y''-(X^2)''=0. \tag{42}$$

From Eq. (42), we obtain a relation given by

$$Y=\frac{X^2}{\upsilon^2-1}. \tag{43}$$

Now, using this relation in Eq. (41) yields the ordinary differential equation of the following form

$$X''-(\omega+a^2)X+\left(b-\frac{1}{\upsilon^2-1}\right)X^3=0, \tag{44}$$

where $'$ shows the differentiation with respect to τ.

Now, to uncover the fairly interesting and useful solutions of Eq. (44) via the UM that permits to search those solutions in two ways including as polynomial or rational function solutions [34].

4.2.1. Polynomial function solution

For polynomial function solution, we assume the initial solution in the following way

$$X(\tau)=\sum_{i=0}^{n}\alpha_i\phi^i(\tau),\qquad (\phi'(\tau))^\sigma=\sum_{i=0}^{\sigma k}\beta_i\phi^i(\tau),\quad \tau=x-\frac{\upsilon}{\gamma}\Gamma(\mu+1)t^\gamma,\quad \sigma=1,2. \tag{45}$$

where α_i and β_i are unspecified constants to be assessed. By employing the balance principle [35–37] to Eq. (44) results a relation $n=k-1$, for all $k\geq 2$.

Here, we have discussed two different cases $k=2,\sigma=1$ and $k=2,\sigma=2$. Hence, the Eq. (45) can be converted into

$$X(\tau)=\alpha_0+\alpha_1\phi(\tau),\qquad (\phi'(\tau))^\sigma=\sum_{i=0}^{2\sigma}\beta_i\phi^i(\tau),\quad \sigma=1,2. \tag{46}$$

Case 1: In this case we substitute $\sigma=1$ in Eq. (46) and results into the following equation

$$X(\tau)=\alpha_0+\alpha_1\phi(\tau),\qquad \phi'(\tau)=\sum_{i=0}^{2}\beta_i\phi^i(\tau). \tag{47}$$

Using Eq. (47) into Eq. (44) provides a system of algebraic equations in ϕ. By pursuing some symbolic computing software to solve this system for $\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2,a,b,\omega$ and as a result, the following relation is obtained:

$$b=\frac{\alpha_1^2-2\beta_2^2(\upsilon^2-1)}{\alpha_1^2(\upsilon^2-1)},\ \beta_0=\frac{2\alpha_0^2\beta_2^2+\alpha_1^2(a^2+\omega)}{\beta_2\alpha_1^2},\ \beta_1=\frac{2\alpha_0\beta_2}{\alpha_1}, \tag{48}$$

and by solving the auxiliary equation given in Eq. (47) together these values into Eq. (47), yields

$$X(\tau)=-\frac{1}{2}\frac{\alpha_1\tanh\left(\frac{1}{2}\tau\sqrt{-2(a^2+\omega)}\right)\sqrt{-2(a^2+\omega)}}{\beta_2}, \tag{49}$$

which results into the following two equations for $\mu=1$

$$p(x,t)=-\frac{1}{2}\frac{\alpha_1\tanh\left(\frac{1}{2}\tau\sqrt{-2(a^2+\omega)}\right)\sqrt{-2(a^2+\omega)}}{\beta_2}e^{\imath\phi(x,t)}, \tag{50}$$

and

$$q(x,t)=-\frac{1}{2}\frac{\alpha_1^2\tanh\left(\frac{1}{2}\tau\sqrt{-2(a^2+\omega)}\right)^2(a^2+\omega)}{\beta_2^2(\upsilon^2-1)}, \tag{51}$$

where $\tau=x-\upsilon\frac{t^\gamma}{\gamma}$ and $\phi(x,t)=-ax+\omega\frac{t^\gamma}{\gamma}+\xi_0$.

Case 2: ($\sigma=2$) In this case we substitute $\sigma=2$ in Eq. (46) and results into the following equation

$$X(\tau)=\alpha_0+\alpha_1\phi(\tau),\qquad \phi'(\tau)=\phi(\tau)\sqrt{\beta_0+\beta_1\phi(\tau)+\beta_2\phi^2(\tau)}. \tag{52}$$

Using Eq. (48) into Eq. (44) in a way as we did in the previous case gives:

$$b=\frac{\alpha_0^2+(\upsilon^2-1)(a^2+\omega)}{\alpha_0^2(\upsilon^2-1)},\ \beta_0=-2(a^2+\omega),\qquad \beta_1=-\frac{2\alpha_1(a^2+\omega)}{\alpha_0},\ \beta_2=-\frac{1}{2}\frac{\alpha_1^2(a^2+\omega)}{\alpha_0^2}, \tag{53}$$

and by solving the auxiliary equation given in Eq. (47) together these values into Eq. (47), yields

$$X(\tau)=-\frac{\alpha_0\left(-\alpha_0+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}\omega+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}a^2\right)}{\alpha_0+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}\omega+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}a^2}. \tag{54}$$

which results into the following two equations for $\mu=1$

$$p(x,t)=-\frac{\alpha_0\left(-\alpha_0+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}\omega+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}a^2\right)}{\alpha_0+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}\omega+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}a^2}e^{\imath\phi(x,t)}, \tag{55}$$

and

$$q(x,t)=\frac{\alpha_0^2\left(-\alpha_0+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}\omega+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}a^2\right)^2}{\left(\alpha_0+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}\omega+4\alpha_1e^{\tau\sqrt{-2(a^2+\omega)}}a^2\right)^2(\upsilon^2-1)}, \tag{56}$$

where $\tau = x - \upsilon \frac{t^\gamma}{\gamma}$ and $\phi(x,t) = -ax + \omega \frac{t^\gamma}{\gamma} + \xi_0$.

4.2.2. Rational function solution

For rational function solution, we assume the initial solution in a following way

$$X(\tau) = \frac{\sum_{i=0}^{n} P_i \phi^i(\tau)}{\sum_{i=0}^{r} Q_i \phi^i(\tau)}, \quad n \geq r, \tag{57}$$
$$(\phi'(\tau))^\sigma = \sum_{i=0}^{\sigma k} \beta_i \phi^i(\tau), \quad \tau = x - \frac{\upsilon}{\gamma}\Gamma(\mu+1)t^\gamma, \quad \sigma = 2.$$

where P_i, Q_i and β_i are unspecified constants to be assessed. Now, by utilizing the balance condition given by Lemma 2.3 [35–37], provides $k = 1$ and relation $n = r$, which permits to choose n freely.

Here, we have discussed a case when $k = 1, \sigma = 2$. Hence, the Eq. (25) can be converted into

$$X(\tau) = \frac{P_0 + P_1\phi(\tau)}{Q_0 + Q_1\phi(\tau)},$$
$$\phi'(\tau) = \sqrt{\beta_0 + \beta_1\phi(\tau) + \beta_2\phi^2(\tau)}. \tag{58}$$

Using Eq. (26) into Eq. (18) provides a system of algebraic equations in ϕ. By pursuing some symbolic computing software to solve this system for $P_0, P_1, Q_0, Q_1, \beta_0, \beta_1, \beta_2, a, b, \omega$, results into the following relation:

$$b = \frac{Q_1^2(a^2+\omega)}{P_1^2}, \quad \beta_0 = -\frac{1}{2}\frac{(P_0Q_1 + Q_0P_1)^2(a^2+\omega)}{P_1^2Q_1^2}, \tag{59}$$
$$\beta_1 = -\frac{2(P_0Q_1 + Q_0P_1)(a^2+\omega)}{P_1Q_1}, \quad \beta_2 = -2(a^2+\omega),$$

and by solving the auxiliary equation given in Eq. (58) together these values into Eq. (58), yields

$$X(\tau) = -\frac{\left(\sqrt{-2(a^2+\omega)}e^{\tau\sqrt{-2(a^2+\omega)}}(P_0Q_1 - Q_0P_1) + P_1Q_1\right)P_1}{\left(\sqrt{-2(a^2+\omega)}e^{\tau\sqrt{-2(a^2+\omega)}}(P_0Q_1 - Q_0P_1) - P_1Q_1\right)Q_1}, \tag{60}$$

which results into the following two equations for $\mu = 1$

$$p(x,t) = -\frac{\left(\sqrt{-2(a^2+\omega)}e^{\tau\sqrt{-2(a^2+\omega)}}(P_0Q_1 - Q_0P_1) + P_1Q_1\right)P_1}{\left(\sqrt{-2(a^2+\omega)}e^{\tau\sqrt{-2(a^2+\omega)}}(P_0Q_1 - Q_0P_1) - P_1Q_1\right)Q_1}e^{\iota\phi(x,t)}, \tag{61}$$

and

$$q(x,t) = \frac{\left(\sqrt{-2(a^2+\omega)}e^{\tau\sqrt{-2(a^2+\omega)}}(P_0Q_1 - Q_0P_1) + P_1Q_1\right)^2 P_1^2}{\left(\sqrt{-2(a^2+\omega)}e^{\tau\sqrt{-2(a^2+\omega)}}(P_0Q_1 - Q_0P_1) - P_1Q_1\right)^2 Q_1^2(\upsilon^2-1)}, \tag{62}$$

where $\tau = x - \upsilon \frac{t^\gamma}{\gamma}$ and $\phi(x,t) = -ax + \omega \frac{t^\gamma}{\gamma} + \xi_0$.

5. Graphical representation

This section contributes to the physical meaning of the fractional soliton solutions and the dynamic characteristics of these solutions. The Figs. 1–2 shows the pictorial representation of the obtained solutions of (2+1)-dimensional CIE with the effect of fractional parameters in 2D and 3D figures at some suitable parameters. The 3D Figs. 1(a) and 2(a) are the graphical visualization of the fractional soliton solutions for Eqs. (23) and (26) respectively, with $\gamma = 0.8$ and $y = 0.5$. While, the 2D figures 1(b) and 2(b) are the graphical representations of the fractional soliton solutions for Eqs. (23) and (26) with $\gamma = 0.4, 0.6, 0.8, 1$ and $y = 0.5, x = 0$. These graphs are plotted at the different values of fractional parameter γ to show the impact on obtained solutions.

6. Conclusion

In this paper, we have effectively established the fractional the soliton wave solutions in polynomial and rational forms. The unified method alongside local M-derivative is implemented to a couple of temporal fractional NLEE, (2+1)-dimensional CIE and (1+1)-dimensional ZE. The pictorial representation of the obtained solutions with the effect of fractional parameters is also shown in Figs. 1–2. The proposed approach enables us to discover these solutions in polynomial and rational forms. Further, to grant the validity of non-singular solutions are given with limitation conditions and graphically depicted in 3D. Similarly, to show the effect of fractional parameter on predicted non-singular solutions is portrayed through 2D graphs. The obtained solutions are revealing that the suggested methodology is straightforward and valuable to recover the solitary wave solutions of other nonlinear evolution equations and helps to give a valuable addition in solitary wave theory.

CRediT authorship contribution statement

Nauman Raza: Conceptualization, Methodology, Writing - original draft, Software. **Muhammad Hamza Rafiq:** Methodology, Writing - original draft, Visualization, Investigation. **Melike Kaplan:** Methodology, Writing - original draft. **Sunil Kumar:** Supervision, Validation, Project administration, Formal analysis. **Yu-Ming Chu:** Project administration, Writing - review & editing, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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