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Split Pell and Pell–Lucas Quaternions

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Abstract. The aim of this work is to introduce split Pell and split Pell–Lucas quaternions. We give generating functions and Binet formulas for these numbers. Also, we obtain many identities for split Pell and split Pell–Lucas quaternions including Catalan's identity, Cassini's identity and d'Ocagne's identity.

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1. Introduction

Sir W.R. Hamilton discovered the quaternions in 1843. Six years later, J. Cockle introduced split-quaternions which were called coquaternions by him.

Let F be a arbitrary field (its characteristic is not two) and F' be multiplicative group of F. The quaternion algebra $(\frac{a,b}{F})$ is defined to be the F algebra on generators i and j with the relations

$$i^2 = a, j^2 = b, ij = -ji.$$

If we take $k := ij$, then we have $k^2 = -ab \in F'$ and $ik = -ki = aj$, $kj = -jk = bi$.

In the case $a = b = -1$ and $F = \mathbb{R}$, $(\frac{-1,-1}{\mathbb{R}})$ is the ring of quaternions over the reals (for details, see [15]). Similarly the algebra $(\frac{-1,1}{\mathbb{R}})$ is ring of split quaternions over reals. Split quaternions are also called coquaternions, para-quaternions, anti-quaternions, pseudo-quaternions or hyperbolic quaternions. We show this ring as follows

$$P_{\mathbb{R}} = \{\alpha = a + bi + cj + dk \,|a, b, c, d \in \mathbb{R}, i^2 = -1, j^2 = k^2 = 1, ij = k = -ji\}.$$

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The conjugate of split quaternion $\alpha = a + bi + cj + dk$ is defined by

$$\overline{\alpha} = a + bi - cj - dk$$

and the norm of α is given by

$$\begin{aligned} N(\alpha) &= \alpha\overline{\alpha} \\ &= a^2 + b^2 - c^2 - d^2. \end{aligned}$$

The famous integer sequence, Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$, is defined with numbers which satisfy the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers have delighted the mathematicians for centuries with their properties and applications in many interesting areas (see [13] for details). Another well-known sequence is the Lucas sequence $\{L_n\}_{n=0}^{\infty}$. Lucas numbers satisfy the same recurrence relation with the Fibonacci numbers except initial conditions. Namely, $L_n = L_{n-1} + L_{n-2}$ and the initial conditions are $L_0 = 2$ and $L_1 = 1$. Sometimes, Lucas numbers are defined with the well-known properties $L_n = F_{n-1} + F_{n+1}$ between Fibonacci and Lucas numbers.

The Pell sequence $\{P_n\}_{n=0}^{\infty}$ is another famous sequence among integer sequences which satisfies the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \tag{1.1}$$

with initial conditions $P_0 = 0$ and $P_1 = 1$. Pell–Lucas sequence $\{PL_n\}_{n=0}^{\infty}$ satisfies the same recurrence relation with Pell sequence except initial conditions $PL_0 = 1$ and $PL_1 = 1$. Some authors define the Pell–Lucas sequence by taking the initial conditions $PL_0 = 2$ and $PL_1 = 2$. In such a case, the Pell–Lucas sequence is called modified Pell sequence.

The Binet formula for the Pell and Pell–Lucas numbers are, respectively

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

and

$$PL_n = \frac{\gamma^n + \delta^n}{2}$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are the solutions of the characteristic equation $x^2 - 2x - 1 = 0$ (see [14] for details). The positive root γ is known as silver ratio and it plays a similar role with the golden ratio for Fibonacci and Lucas numbers.

There are many relations between Pell and Pell–Lucas numbers such as

$$PL_{n+1} = P_{n+1} + P_n, \tag{1.2}$$

$$PL_n = P_{n+1} - P_n, \tag{1.3}$$

$$2PL_n = P_{n-1} + P_{n+1}, \tag{1.4}$$

$$2P_n = PL_{n+1} - PL_n. \tag{1.5}$$

Pell and Pell–Lucas numbers appear in many subjects of mathematics. For example they appear as solutions of the Pell equation $x^2 - 2y^2 = (-1)^n$. The solutions of this equation are (PL_n, P_n).

Horadam [10] defined the Fibonacci and Lucas quaternions with the following relations

$$Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

and

$$K_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k$$

respectively, where F_n and L_n are nth Fibonacci and Lucas numbers. He also defined a generalization of Fibonacci quaternions by

$$P_n = H_n + H_{n+1}i + H_{n+2}j + H_{n+3}k$$

where H_n is the nth generalized Fibonacci number defined by $H_1 = p$, $H_2 = p + q$, $H_n = H_{n-1} + H_{n-2}$. Harman [9] defined complex Fibonacci numbers by the following two-dimensional recurrence relation:

$$\begin{aligned} G(n+2,m) &= G(n+1,m) + G(n,m),\\ G(n,m+2) &= G(n,m+1) + G(n,m) \end{aligned}$$

where $G(0,0) = 0$, $G(1,0) = 1$, $G(0,1) = i$, and $G(1,1) = 1 + i$. Extending this idea, Horadam [11] defined the recurrence relations

$$\begin{aligned} G(h+2,l,m,n) &= G(h+1,l,m,n) + G(h,l,m,n),\\ G(h,l+2,m,n) &= G(h,l+1,m,n) + G(h,l,m,n),\\ G(h,l,m+2,n) &= G(h,l,m+1,n) + G(h,l,m,n),\\ G(h,l,m,n+2) &= G(h,l,m,n+1) + G(h,l,m,n) \end{aligned}$$

with initial conditions

$$\begin{aligned} &G(0,0,0,0) = 0, G(1,0,0,0) = 1, G(0,1,0,0) = i, G(0,0,1,0) = j,\\ &\qquad\qquad G(0,0,0,1) = k,\\ &G(1,1,0,0) = 1 + i, \ldots, G(0,0,1,1) = j + k,\\ &G(1,1,1,0) = 1 + i + j, \ldots, G(0,1,1,1) = i + j + k,\\ &G(1,1,1,1) = 1 + i + j + k \end{aligned}$$

and gave many properties between Fibanacci numbers and $G(a,b,c,d)$.

Swamy [19] obtained some properties of the Fibonacci quaternions Q_n and generalized Fibonacci quaternions P_n which were defined by Horadam. For example, he gave

$$P_{2n+1} = F_{n+1}P_{n+1} + F_nP_n$$

and

$$P_{2n} = F_{n+1}P_n + F_nP_{n-1}.$$

Iyer [12] gave the connections between the Fibonacci and Lucas quaternions. Halici [8] gave some properties of the Fibonacci and Lucas quaternions including Binet formulas.

Akyigit et al. [1], Flaut and Shpakivskyi [7] gave some properties of Fibonacci quaternions Q_n and generalized Fibonacci quaternions P_n which were defined by Horadam over the generalized real quaternion algebra. The basis elements of this quaternion algebra satisfy the following multiplication table:

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	$-\beta_1$	k	$-\beta_1 j$
j	j	$-k$	$-\beta_2$	$\beta_2 i$
k	k	$\beta_1 j$	$-\beta_2 i$	$-\beta_1\beta_2$

Stakhov and Rozin [18] defined Fibonacci p-numbers. For any integer $p \geq 0, n \in \mathbb{Z}$ and $n > p$, the nth Fibonacci p-number is given by the recurrence relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with initial conditions $F_p(0) = 0,\ F_p(1) = F_p(2) = \cdots = F_p(p) = 1$. Tasci and Yalcin [21] investigated properties of Fibonacci p-quaternions which are defined with the help of G-notations of Harman and Horadam. Fibonacci p quaternion $G_p(h,l,m,n)$ satisfy the recurrence relations:

$$\begin{aligned} G_p(h+1,l,m,n) &= G_p(h,l,m,n) + G_p(h-p,l,m,n),\ h > p\\ G_p(h,l+1,m,n) &= G_p(h,l,m,n) + G_p(h,l-p,m,n),\ l > p\\ G_p(h,l,m+1,n) &= G_p(h,l,m,n) + G_p(h,l,m-p,n),\ m > p\\ G_p(h,l,m,n+1) &= G_p(h,l,m,n) + G_p(h,l,m,n-1),\ h > p \end{aligned}$$

with initial conditions $G_p(r,s,t,u) = F_{p,r} + iF_{p,s} + jF_{p,t} + kF_{p,u}$ where $r,s,t,u \in \{0,1,\ldots,p\}$.

For any real number k, Falcon and Plaza [5] defined k-Fibonacci numbers and Falcon [6] defined k-Lucas numbers as follows:

$$F_{k,0} = 0, F_{k,1} = 1 \text{ and } F_{k,n+1} = kF_{k,n} + F_{k,n-1}$$

and

$$L_{k,0} = 2, L_{k,1} = k \text{ and } L_{k,n+1} = kL_{k,n} + L_{k,n-1}.$$

Ramirez [17] defined k-Fibonacci and k-Lucas quaternions by

$$D_{k,n} = F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + \kappa F_{k,n+3}$$

and

$$P_{k,n} = L_{k,n} + LF_{k,n+1} + jL_{k,n+2} + \kappa L_{k,n+3}$$

where $F_{k,n}$ and $L_{k,n}$ are the nth k-Fibonacci and k-Lucas numbers, respectively.

Polatli et al. [16] and studied split k-Fibonacci and k-Lucas Quaternions. Akyigit et al. [2] worked on split Fibonacci and split Lucas quaternions. They obtained some identities of these numbers.

Cimen and Ipek [4] defined Pell and Pell–Lucas quaternions by

$$QP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$$

and

$$QPL_n = PL_n + iPL_{n+1} + jPL_{n+2} + kPL_{n+3}$$

where P_n and PL_n are nth Pell and Pell–Lucas numbers. They obtained many properties of these quaternions including Binet formulas and Cassini's identity. Szynal-Liana and Wloch [20] gave some identities for the Pell and Pell–Lucas quaternions such as norm and some matrix properties.

Modified k-Pell sequence is defined by

$$q_{k,0} = q_{k,1} = 1 \text{ and } q_{k,n+1} = 2q_{k,n} + q_{n-1},\ n \geq 1.$$

In a recent work, Cataniro [3] worked on modified Pell and modified k-Pell quaternions which are defined by

$$MP_n = q_n + iq_{n+1} + jq_{n+2} + kq_{n+3}$$

and

$$MP_{k,n} = q_{k,n} + iq_{k,n+1} + jq_{k,n+2} + kq_{k,n+3}$$

where q_n and $q_{k,n}$ are the nth Pell–Lucas and modified k-Pell numbers, respectively.

2. Split Pell and Split Pell–Lucas Quaternions

Now we can focus on split Pell and split Pell–Lucas quaternions. For $n \geq 0$, any nth Split Pell and Split Pell–Lucas quaternions defined by

$$SP_n = P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k$$

and

$$SPL_n = PL_n + PL_{n+1}i + PL_{n+2}j + PL_{n+3}k,$$

respectively, where P_n and PL_n are nth Pell and Pell–Lucas numbers, and $\{1, i, j, k\}$ is the standard basis of split quaternions.

For any positive integer n, we obtain norm of SP_n as follows:

$$\begin{aligned}
N(SP_n) &= (P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k)(P_n + P_{n+1}i - P_{n+2}j - P_{n+3}k) \\
&= P_n^2 + P_{n+1}^2 - P_{n+2}^2 - P_{n+3}^2 \\
&= (P_n^2 - P_{n+2}^2) + (P_{n+1}^2 - P_{n+3}^2) \\
&= (P_n - P_{n+2})(P_n + P_{n+2}) + (P_{n+1} - P_{n+3})(P_{n+1} + P_{n+3}).
\end{aligned}$$

Using the recurrence relation (1.1), we obtain $-2P_{n+1} = P_n - P_{n+2}$. With the help of this equation and Eq. (1.4), we find

$$N(SP_n) = -4(P_{n+1}PL_{n+1} + P_{n+2}PL_{n+2}).$$

Similarly, we obtain norm of split Pell–Lucas quaternions as follows

$$N(SPL_n) = -8(P_{n+1}PL_{n+1} + P_{n+2}PL_{n+2}).$$

From definitions, we can easily see that

$$SP_n := 2SP_{n-1} + SP_{n-2}$$

and

$$SPL_n := 2SPL_{n-1} + SPL_{n-2}.$$

For split Pell and split Pell–Lucas quaternions with negative indexes, we need the following lemma.

Lemma 2.1. *For any integer n, we have*

$$SP_{m+n} = P_{n+1}SP_m + P_nSP_{m-1}. \tag{2.1}$$

Proof.

$$\begin{aligned} SP_{m+n} &= P_{m+n} + P_{m+n+1}i + P_{m+n+2}j + P_{m+n+3}k \\ &= P_mP_{n+1} + P_{m-1}P_n + (P_{m+1}P_{n+1} + P_mP_n)i \\ &\quad +(P_{m+2}P_{n+1} + P_{m+1}P_n)j + (P_{m+3}P_{n+1} + P_{m+2}P_n)k \\ &= P_{n+1}(P_m + P_{m+1}i + P_{m+2}j + P_{m+3}k) \\ &\quad +P_n(P_{m-1} + P_mi + P_{m+1}j + P_{m+2}k) \\ &= P_{n+1}SP_m + P_nSP_{m-1.} \end{aligned}$$

□

If we take $m \to 1$ and $n \to n-1$ and use the identity $P_{-n} = (-1)^{n+1}P_n$, we have

$$SP_{-n} = (-1)^n(P_{n+1}SP_0 - P_nSP_1).$$

Similarly, one can obtain

$$SPL_{-n} = (-1)^n(P_{n+1}SPL_0 - P_nSPL_1).$$

Having Binet-like formula for an integer sequence is important to investigate its properties by a direct computational access. The next theorem gives Binet formulas for the split Pell and split Pell–Lucas quaternions.

Theorem 2.2. *For any integer n, nth split Pell and split Pell–Lucas quaternions are*

$$SP_n = \frac{\gamma^n\gamma^* - \delta^n\delta^*}{\gamma - \delta} \tag{2.2}$$

and

$$SPL_n = \frac{\gamma^n\gamma^* + \delta^n\delta^*}{2} \tag{2.3}$$

respectively, where $\gamma = 1 + \sqrt{2}$, $\delta = 1 - \sqrt{2}$, $\gamma^* = 1 + \gamma i + \gamma^2 j + \gamma^3 k$ and $\delta^* = 1 + \delta i + \delta^2 j + \delta^3 k$.

Proof. Let us consider the followings for Eq. (2.2):

$$\begin{aligned} \gamma SP_n + SP_{n-1} &= \gamma P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k + P_{n-1} + P_ni + P_{n+1}j \\ &\quad +P_{n+2}k \\ &= \gamma P_n + P_{n-1} + (\gamma P_{n+1} + P_n)i + (\gamma P_{n+2} + P_{n+1})j \\ &\quad +(\gamma P_{n+3} + P_{n+2})k \end{aligned}$$

By the help of the identity $\gamma P_n + P_{n-1}\gamma^n$, we get

$$\gamma SP_n + SP_{n-1} = \gamma^n\gamma^*. \tag{2.4}$$

Similarly, using the identity $\delta^n = \delta P_n + P_{n-1}$, we have

$$\delta SP_n + SP_{n-1} = \delta^n \delta^*. \tag{2.5}$$

From the Eqs. (2.4) and (2.5), we obtain

$$SP_n = \frac{\gamma^n \gamma^* - \delta^n \delta^*}{\gamma - \delta}.$$

Equation (2.3) can be obtained similarly. □

Ordinary generating functions for split Pell and split Pell–Lucas quaternions are given in the following theorem.

Theorem 2.3. *The generating functions of split Pell and split Pell–Lucas quaternions are*

$$P(x) = \frac{SP_0 + x(SP_1 - 2SP_0)}{1 - 2x - x^2}$$

and

$$PL(x) = \frac{SPL_0 + x(SPL_1 - 2SPL_0)}{1 - 2x - x^2},$$

respectively.

Now, we give some useful identities without proof because they are very simple.

Lemma 2.4. *We have the followings*

$$\gamma^{*^2} = 112 + 80\sqrt{2} + 2SPL_0 + 2\sqrt{2}SP_0, \tag{2.6}$$

$$\delta^{*^2} = 112 - 80\sqrt{2} + 2SPL_0 - 2\sqrt{2}SP_0, \tag{2.7}$$

$$\gamma^*\delta^* = 2SPL_0 + 2\sqrt{2}\lambda, \tag{2.8}$$

$$\delta^*\gamma^* = 2SPL_0 - 2\sqrt{2}\lambda \tag{2.9}$$

where $\lambda = i - 2j + k$.

These identities play very important roles throughout the paper for all calculations. Accordingly, we need the two following identities:

$$\gamma^*\delta^* + \delta^*\gamma^* = 4SPL_0$$

and

$$\gamma^{*^2} + \delta^{*^2} = 224 + 4SPL_0.$$

The following theorem gives Catalan's identities for split Pell and split Pell–Lucas quaternions.

Theorem 2.5. *For every integers* n *and* r*, we have*

$$SP_{n+r}SP_{n-r} - SP_n^2 = (-1)^{n-r+1}[2P_r^2 SPL_0 + \lambda P_{2r}]$$

and

$$\begin{aligned} SPL_{n+r}SPL_{n-r} - SPL_n^2 &= -2[SP_{n+r}SP_{n-r} - SP_n^2] \\ &= -2(-1)^{n-r}[2P_r^2 SPL_0 + \lambda P_{2r}]. \end{aligned}$$

Proof. By using the Binet formula of split Pell quaternions, we have

$$\begin{aligned} &SP_{n+r}SP_{n-r} - SP_n^2 \\ &= \left(\frac{\gamma^{n+r}\gamma^* - \delta^{n+r}\delta^*}{\gamma-\delta}\right)\left(\frac{\gamma^{n-r}\gamma^* - \delta^{n-r}\delta^*}{\gamma-\delta}\right) - \left(\frac{\gamma^{n}\gamma^* - \delta^{n}\delta^*}{\gamma-\delta}\right)^2 \\ &= \frac{1}{8}(-\gamma^*\delta^*\gamma^{n+r}\delta^{n-r} - \delta^*\gamma^*\gamma^{n-r}\delta^{n+r} + \gamma^*\delta^*\gamma^n\delta^n + \delta^*\gamma^*\gamma^n\delta^n) \\ &= \frac{1}{8}((\gamma\delta)^{n-r+1}(\gamma^*\delta^*\gamma^{2r} + \delta^*\gamma^*\delta^{2r}) + (-1)^n(\gamma^*\delta^* + \delta^*\gamma^*)). \end{aligned}$$

After using Eqs. (2.8), (2.9) and some elementary operations, we have

$$SP_{n+r}SP_{n-r} - SP_n^2 = (-1)^{n-r+1}\left(\frac{1}{2}SPL_0PL_{2r} + \lambda P_{2r}\right) + \frac{1}{2}(-1)^n SPL_0$$

and using the identity $4P_r^2 = PL_{2r} - (-1)^r$, we obtain

$$SP_{n+r}SP_{n-r} - SP_n^2 = (-1)^{n-r+1}[2P_r^2 SPL_0 + \lambda P_{2r}].$$

The second part of the theorem can be obtained easily. □

Let $\alpha = a + ib + jc + dk \in \mathbb{P}_{\mathbb{Z}}$. If $\gcd(a, b, c, d) = 1$, then α is called primitive. By using Cassini's identities for Pell and Pell–Lucas numbers, it is seen that two consecutive Pell and Pell–Lucas numbers are relatively prime. Hence, every split Pell and split Pell–Lucas quaternion is primitive.

For $r = 1$, Theorem 2.5 gives Cassini's identities for split Pell and split Pell–Lucas quaternions.

Corollary 2.6. *For any integer n, we get*

$$SP_{n+1}SP_{n-1} - SP_n^2 = (-1)^n[2SPL_0 + 2\lambda]$$

and

$$SPL_{n+1}SPL_{n-1} - SPL_n^2 = (-1)^{n-1}[4SPL_0 + 4\lambda].$$

D'Ocagne's identities for split Pell and split Pell–Lucas quaternions are given in the following theorem:

Theorem 2.7. *For any integers n and m we have*

$$SP_{m+1}SP_n - SP_mSP_{n+1} = 2(-1)^m(P_{n-m}SPL_0 - \lambda PL_{n-m})$$

and

$$SPL_{m+1}SPL_n - SPL_mSPL_{n+1} = 4(-1)^{m+1}(P_{n-m}SPL_0 - \lambda PL_{n-m}),$$

Proof. By using the Binet formula in Theorem 2.2, we obtain

$$\begin{aligned} &SPL_{m+1}SPL_n - SPL_mSPL_{n+1} \\ &= \left(\frac{\gamma^{m+1}\gamma^* + \delta^{m+1}\delta^*}{2}\right)\left(\frac{\gamma^{n}\gamma^* + \delta^{n}\delta^*}{2}\right) \\ &\quad - \left(\frac{\gamma^{m}\gamma^* + \delta^{m}\delta^*}{2}\right)\left(\frac{\gamma^{n+1}\gamma^* + \delta^{n+1}\delta^*}{2}\right) \\ &= \frac{1}{4}\left[\gamma^*\delta^*\gamma^{m+1}\delta^n + \delta^*\gamma^*\gamma^n\delta^{m+1} - \gamma^*\delta^*\gamma^m\delta^{n+1} - \delta^*\gamma^*\gamma^{n+1}\delta^m\right] \end{aligned}$$

$$= \frac{2\sqrt{2}}{4}(-1)^{m+1}\left(\delta^*\gamma^*\gamma^{n-m} - \gamma^*\delta^*\delta^{n-m}\right).$$

If we substitute Eqs. (2.8) and (2.9) into the last equation and after doing some elementary operations, we have

$$SPL_{m+1}SPL_n - SPL_mSPL_{n+1} = 4(-1)^{m+1}(P_{n-m}SPL_0 - \lambda PL_{n-m}).$$

The first identity in theorem can be proven similarly. □

3. Some Identities on Split Pell and Split Pell–Lucas Quaternions

We give the four identities, which are the relations between the split Pell and split Pell–Lucas quaternions as can be seen from definitions directly

$$SPL_{n+1} = SP_{n+1} + SP_n, \tag{3.1}$$
$$SPL_n = SP_{n+1} - SP_n, \tag{3.2}$$
$$2SPL_n = SP_{n-1} + SP_{n+1}, \tag{3.3}$$
$$2SP_n = SPL_{n+1} - SPL_n. \tag{3.4}$$

Sometimes, Lucas numbers are defined by the identity $L_n = F_{n-1} + F_{n+1}$. This identity holds for Pell and Pell–Lucas numbers, namely $PL_n = P_{n-1} + P_{n+1}$. As it can be seen in Eq. (3.3), similar relation between split Pell and split Pell–Lucas numbers holds.

Finally, we give the following identities between split Pell and split Pell–Lucas quaternions:

$$SPL_n^2 - 2SP_n^2 = 2(-1)^n SPL_0, \tag{3.5}$$
$$SP_n^2 + SPL_n^2 = 3\left(28PL_{2n} + 40P_{2n} + \frac{1}{2}SPL_0PL_{2n} + SP_0P_{2n}\right) + \frac{(-1)^n}{2}SPL_0, \tag{3.6}$$
$$SP_n^2 - SPL_n^2 = -\frac{1}{2}\left[56PL_{2n} + 80P_{2n} + SPL_0PL_{2n} + 2SP_0P_{2n} + 3(-1)^n SPL_0\right], \tag{3.7}$$
$$SP_nSPL_n = 56P_{2n} + 40PL_{2n} + P_{2n}SPL_0 + PL_{2n}SP_0 + (-1)^n\lambda, \tag{3.8}$$
$$SP_{n+r}SPL_{n+s} - SP_{n+s}SPL_{n+r} = 2(-1)^{n+r+1}P_{s-r}SPL_0, \tag{3.9}$$
$$SPL_{m+n} + (-1)^n SPL_{m-n} = 2PL_nSPL_m, \tag{3.10}$$
$$SP_{m+n} + (-1)^n SP_{m-n} = 2PL_nSP_m, \tag{3.11}$$
$$SP_{m-n} = (-1)^n\left[P_{n-1}SP_m - P_nSP_{m-1}\right], \tag{3.12}$$
$$SP_{2n} = P_{n+1}SP_n + P_nSP_{n-1}, \tag{3.13}$$
$$SP_{2n+1} = P_{n+1}SP_{n+1} + P_nSP_n. \tag{3.14}$$

We prove some of these identities in the next section.

4. Proofs of Identities

Proof of identity (3.5)

We need the Binet formulas given in Theorem 2.2:

$$\begin{aligned} SPL_n^2 - 2SP_n^2 &= \frac{1}{4}(\gamma^*\gamma^n + \delta^*\delta^n)^2 - 2\frac{1}{8}(\gamma^*\gamma^n - \delta^*\delta^n)^2 \\ &= \frac{1}{4}({\gamma^*}^2\gamma^{2n} + \gamma^n\delta^n\gamma^*\delta^* + \gamma^n\delta^n\delta^*\gamma^* + {\delta^*}^2\delta^{2n}) \\ &\quad -\frac{1}{4}({\gamma^*}^2\gamma^{2n} - \gamma^n\delta^n\gamma^*\delta^* - \gamma^n\delta^n\delta^*\gamma^* + {\delta^*}^2\delta^{2n}) \\ &= \frac{1}{2}(\gamma^*\delta^*\gamma^n\delta^n + \delta^*\gamma^*\gamma^n\delta^n). \end{aligned}$$

Now we need Eqs. (2.6)–(2.9). Using these in the last equation and doing some elementary operation, we obtain

$$\begin{aligned} SPL_n^2 - 2SP_n^2 &= \frac{1}{2}(-1)^n 4SPL_0 \\ &= 2(-1)^n SPL_0. \end{aligned}$$

Proof of identity (3.6)

$$\begin{aligned} SP_n^2 + SPL_n^2 &= \frac{1}{8}(\gamma^*\gamma^n - \delta^*\delta^n)^2 + \frac{1}{4}(\gamma^*\gamma^n + \delta^*\delta^n)^2 \\ &= \frac{1}{8}[{\gamma^*}^2\gamma^{2n} + {\delta^*}^2\delta^{2n} - (-1)^n 4SPL_0] \\ &\quad +\frac{1}{4}[{\gamma^*}^2\gamma^{2n} + {\delta^*}^2\delta^{2n} + (-1)^n 4SPL_0] \\ &= \frac{1}{8}[3{\gamma^*}^2\gamma^{2n} + 3{\delta^*}^2\delta^{2n} + (-1)^n 4SPL_0] \\ &= \frac{3}{8}({\gamma^*}^2\gamma^{2n} + {\delta^*}^2\delta^{2n}) + \frac{(-1)^n}{2}SPL_0 \end{aligned}$$

If we substitute ${\gamma^*}^2$ and ${\delta^*}^2$, namely Eqs. (2.6) and (2.7) in Lemma 2.4, into the last equation, we obtain

$$\begin{aligned} SP_n^2 + SPL_n^2 = 3\left(28PL_{2n} + 40P_{2n} + \frac{1}{2}SPL_0PL_{2n} + SP_0P_{2n}\right) \\ + \frac{(-1)^n}{2}SPL_0. \end{aligned}$$

Proof of identity (3.7)

$$\begin{aligned} SP_n^2 - SPL_n^2 &= \frac{1}{8}(\gamma^*\gamma^n - \delta^*\delta^n)^2 - \frac{1}{4}(\gamma^*\gamma^n + \delta^*\delta^n)^2 \\ &= -\frac{1}{8}[{\gamma^*}^2\gamma^{2n} + {\delta^*}^2\delta^{2n} + 12(-1)^n SPL_0] \end{aligned}$$

Again Eqs. (2.6) and (2.7) in Lemma 2.4, we have

$$\begin{aligned} SP_n^2 - SPL_n^2 = -\frac{1}{2}[56PL_{2n} + 80P_{2n} + SPL_0PL_{2n} + 2SP_0P_{2n} \\ + 3(-1)^n SPL_0]. \end{aligned}$$

Proof of identity (3.8)

$$\begin{aligned} SP_nSPL_n &= \left(\frac{\gamma^*\gamma^n - \delta^*\delta^n}{2\sqrt{2}}\right)\left(\frac{\gamma^*\gamma^n + \delta^*\delta^n}{2}\right) \\ &= \frac{1}{4\sqrt{2}}({\gamma^*}^2\gamma^{2n} + \gamma^*\delta^*\gamma^n\delta^n - \delta^*\gamma^*\gamma^n\delta^n - {\delta^*}^2\delta^{2n}) \\ &= \frac{1}{4\sqrt{2}}[{\gamma^*}^2\gamma^{2n} + (-1)^n(\gamma^*\delta^* - \delta^*\gamma^*) - {\delta^*}^2\delta^{2n}] \end{aligned}$$

If we use Eqs. (2.6)–(2.9) in Lemma 2.4, we get

$$SP_nSPL_n = 56P_{2n} + 40PL_{2n} + P_{2n}SPL_0 + PL_{2n}SP_0 + (-1)^n\lambda.$$

Proof of identity (3.9)

$$\begin{aligned} &SP_{n+r}SPL_{n+s} - SP_{n+s}SPL_{n+r} \\ &= \left(\frac{\gamma^*\gamma^{n+r} - \delta^*\delta^{n+r}}{2\sqrt{2}}\right)\left(\frac{\gamma^*\gamma^{n+s} + \delta^*\delta^{n+s}}{2}\right) \\ &\quad - \frac{(\gamma^*\gamma^{n+s} - \delta^*\delta^{n+s})}{2\sqrt{2}}\frac{(\gamma^*\gamma^{n+r} + \delta^*\delta^{n+r})}{2} \\ &= \frac{1}{4\sqrt{2}}(\gamma^*\delta^*\gamma^{n+r}\delta^{n+s} - \delta^*\gamma^*\gamma^{n+s}\delta^{n+r} \\ &\quad -\gamma^*\delta^*\gamma^{n+s}\delta^{n+r} + \delta^*\gamma^*\gamma^{n+r}\delta^{n+s}) \\ &= \frac{1}{4\sqrt{2}}(\gamma^{n+r}\delta^{n+s}4SPL_0 - \gamma^{n+s}\delta^{n+r}4SPL_0) \\ &= \frac{1}{\sqrt{2}}SPL_0\gamma^{n+r}\delta^{n+r}(\delta^{s-r} - \gamma^{s-r}) \\ &= 2(-1)^{n+r+1}P_{s-r}SPL_0. \end{aligned}$$

Proof of identity (3.10)

$$\begin{aligned} &SPL_{m+n} + (-1)^nSPL_{m-n} \\ &= \frac{1}{2}\left[\gamma^*\gamma^{m+n} + \delta^*\delta^{m+n} + (-1)^n\gamma^*\gamma^{m-n} + (-1)^n\delta^*\delta^{m-n}\right] \\ &= \frac{1}{2}\left\{\gamma^*\gamma^{m-n}[\gamma^{2n} + (-1)^n] + \delta^*\delta^{m-n}[\delta^{2n} + (-1)^n]\right\} \\ &= \frac{1}{2}\left[\gamma^*\gamma^m(\gamma^n + \delta^n) + \delta^*\delta^m(\delta^n + \gamma^n)\right] \\ &= 2PL_nSPL_m. \end{aligned}$$

Identities (3.12)–(3.14) can be proven by substituting $n \to -n$, $m \to n$ and $m \to n+1$ into Eq. (2.1).

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