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3-Parameter Generalized Quaternions

Tuncay Deniz Şentürk[1] · **Zafer Ünal**[2]

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Abstract

In this article, we give a general form of the quaternions algebra depending on 3-parameters. We define 3-parameter generalized quaternions (3PGQs) and study various properties and applications. Firstly we present the definiton, the multiplication table and algebraic properties of 3PGQs. We give matrix representation and Hamilton operators for 3PGQs. We derive the polar represenation, De Moivre's and Euler's formulas with the matrix representations for 3PGQs. Additionally, we derive relations between the powers of the matrices associated with 3PGQs. Finally, Lie groups and Lie algebras are studied and their matrix representations are given. Also the Lie multiplication and the Killing bilinear form are given.

Keywords 3-Parameter generalized quaternion · Lie algebra · De Moivre's formula · Matrix representation of quaternions · Euler formula

Mathematics Subject Classification 14A20 · 14A22 · 15A66 · 70G55 · 70G65

1 Introduction

Irish mathematician Sir William Rowan Hamilton started working on the complex numbers in 1830. Hamilton wanted to generalize these numbers. Firstly, he wanted to express these numbers as compositions of two imaginary numbers and one real number. So in the beginning he hoped to expand the complex numbers into three-

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✉ Tuncay Deniz Şentürk
tuncaydenizsenturk@gmail.com

Zafer Ünal
zunal@kastamonu.edu.tr

[1] Göl Anatolian Highschool, 37150 Kastamonu, Turkey

[2] Department of Mathematics, Faculty of Arts and Science, Kastamonu University, 37150 Kastamonu, Turkey

dimensional space. Although he could do addition and subtraction with these triples, he could not define a norm on these triples. For years he thought about this issue and made various investigations. Finally, on 16 October 1843, he defined real quaternions as:

$$\begin{aligned}\mathbb{H} = \big\{a + be_1 + ce_2 + de_3 \mid\ & a, b, c, d \in \mathbb{R},\\ & e_1^2 = e_2^2 = e_3^2 = -1, e_1e_2e_3 = -1\big\}\end{aligned}$$

[8–14]. Also in [32], all the properties of quaternions, quaternion algebra and applications are explained by Ward. Following the identification of the real quaternions, in 1849, split-quaternions, also known as para-quaternions, co-quaternions, pseudo-quaternions in the literature, were defined by Sir James Cockle in [4]:

$$\begin{aligned}\mathbb{P} = \big\{a + be_1 + ce_2 + de_3 \mid\ & a, b, c, d \in \mathbb{R},\\ & -e_1^2 = e_2^2 = e_3^2 = 1, e_1e_2e_3 = 1\big\}.\end{aligned}$$

Cockle brought a new perspective to the quaternions. Hamilton quaternions with complex coefficients are called biquaternions. The biquaternions were described by Sir William Clifford in 1871 [3]. In 1924 and 1928, Leonard Eugene Dickson and Lois Wilfred Griffiths wrote two articles on generalized quaternions [6, 7]. The set of generalized quaternions with two parameters is given by

$$\begin{aligned}\mathbb{H}_{\lambda,\mu} = \big\{a + be_1 + ce_2 + de_3 \mid\ & a, b, c, d, \lambda, \mu \in \mathbb{R},\\ & e_1^2 = -\lambda, e_2^2 = -\mu, e_3^2 = -\lambda\mu, e_1e_2e_3 = -\lambda\mu\big\}.\end{aligned}$$

These quaternions are known as generalized quaternions in the literature. Throughout the article, we will refer to these as 2-parameters generalized quaternions (2PGQs) for shortness of notation. In the set of the 2PGQs, if $\lambda = \mu = 1$ is taken, then we obtain the Hamilton quaternions. If $\lambda = -\mu = 1$ is taken, then we achieve the set of split-quaternions.

It is possible to see the effects of Hamilton's discovery, which is about two centuries old, in many areas from physics to computer graphics. In the current literature, quaternions are also associated with number sequences. These studies can be found in [1, 5, 15, 27–31].

In this article, we will go far beyond the generalization mentioned above and we will give the most general form of the quaternions algebra depending on 3-parameters.

2 3-Parameter Generalized Quaternions

In this section, we define the 3-parameter generalized quaternions and form the algebra, inspired by the work of Hamilton, Cockle, Dickson and Griffiths.

Definition 2.1 The following set is called the set of 3-parameter generalized quaternions (3PGQs):

$$\mathbb{K} = \big\{a_0 + a_1e_1 + a_2e_2 + a_3e_3 \mid a_0, a_1, a_2, a_3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$
$$e_1^2 = -\lambda_1\lambda_2, e_2^2 = -\lambda_1\lambda_3, e_3^2 = -\lambda_2\lambda_3, e_1e_2e_3 = -\lambda_1\lambda_2\lambda_3\big\}.$$

Each element $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ of the set $\mathbb{K}$ is called a 3-parameter generalized quaternion (3PGQ). Here the real numbers a_0, a_1, a_2, a_3 are called components of p. The base vectors $1, e_1, e_2, e_3$ of the 3PGQs comply with the following multiplication table:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\lambda_1\lambda_2$	λ_1e_3	$-\lambda_2e_2$
e_2	e_2	$-\lambda_1e_3$	$-\lambda_1\lambda_3$	λ_3e_1
e_3	e_3	λ_2e_2	$-\lambda_3e_1$	$-\lambda_2\lambda_3$

According to this multiplication table, $\mathbb{K} = Sp\{1, e_1, e_2, e_3\}$.

Special cases:

i. If $\lambda_1 = 1, \lambda_2 = \lambda, \lambda_3 = \mu$, then the algebra of 2PGQs is obtained.
ii. $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$, gives us the algebra of split quaternions.
iii. If $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$, then the algebra of Hamilton quaternions is achieved.
iv. If $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0$, then the algebra of semi-quaternions is attained.
v. $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$, then we get the algebra of split semi-quaternions.
vi. If $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$, then the algebra of $1/4$-quaternions is achieved.

Of course, it is possible to work in more specific quaternion algebras according to $\lambda_{i\in\{1,2,3\}}$.

Throughout the article, we will consider special cases for the $\lambda_{i\in\{1,2,3\}}$ values given above.

Any 3PGQ $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ consists of two parts, the scalar and the vector part:

$$p = S_p + V_p$$

where

$$S_p = a_0, \;\; V_p = a_1e_1 + a_2e_2 + a_3e_3.$$

Definition 2.2 Let p be a 3PGQ. If $S_p = 0$, then p is called a 3-parameter generalized pure-quaternion (3PGPQ) or 3-parameter generalized vector (3PGV). Let us show that the set of 3-parameter generalized vectors is as follows:

$$\mathrm{Im}\,(\mathbb{K}) = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1, a_2, a_3 \in \mathbb{R}\}\,.$$

Equality, addition, multiplication by scalar and multiplication operations are defined on $\mathbb{K}$ as following:

Let $p = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ and $q = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3$ be 3PGQs and α be a real number.

- Equality: $p = q \Leftrightarrow a_0 = b_0, a_1 = b_1, a_2 = b_2, a_3 = b_3$.
- Addition: $p + q = \left(S_p + S_q\right) + \left(V_p + V_q\right) = (a_0 + b_0) + (a_1 + b_1) e_1 + (a_2 + b_2) e_2 + (a_3 + b_3) e_3$.
- Multiplication by scalar: The following operation is called multiplication by scalar or external operation:

$$\begin{aligned} \odot : \mathbb{R} \times \mathbb{K} &\to \mathbb{K} \\ (c, p) &\to c \odot p =: cp = ca_0 + ca_1 e_1 + ca_2 e_2 + ca_3 e_3 \end{aligned}$$

- Multiplication:

$$\begin{aligned} \times : \mathbb{K} \times \mathbb{K} &\to \mathbb{K} \\ (p, q) &\to p \times q = pq \end{aligned}$$

if p and q are multiplied according to the multiplication table, then we have:

$$\begin{aligned} pq = {} & (a_0 b_0 - \lambda_1 \lambda_2 a_1 b_1 - \lambda_1 \lambda_3 a_2 b_2 - \lambda_2 \lambda_3 a_3 b_3) \\ & + e_1 \left(a_0 b_1 + b_0 a_1 + \lambda_3 \left(a_2 b_3 - a_3 b_2\right)\right) \\ & + e_2 \left(a_0 b_2 + b_0 a_2 + \lambda_2 \left(a_3 b_1 - a_1 b_3\right)\right) \\ & + e_3 \left(a_0 b_3 + a_3 b_0 + \lambda_1 \left(a_1 b_2 - a_2 b_1\right)\right). \end{aligned}$$

We can formulate this result as follows:

$$\begin{aligned} pq &= \left(S_p + V_p\right)\left(S_q + V_q\right) \\ &= S_p S_q + S_p V_q + S_q V_p + V_p V_q \\ &= S_p S_q - f\left(V_p, V_q\right) + S_p V_p + S_q V_q + V_p \bar{\wedge} V_q, \end{aligned}$$

where

$$\begin{aligned} & f : \operatorname{Im}(\mathbb{K}) \times \operatorname{Im}(\mathbb{K}) \to \mathbb{R} \\ & \left(V_p, V_q\right) \to f\left(V_p, V_q\right) = \lambda_1 \lambda_2 a_1 b_1 + \lambda_1 \lambda_3 a_2 b_2 + \lambda_2 \lambda_3 a_3 b_3 \end{aligned}$$

and

$$\begin{aligned} \bar{\wedge} : \operatorname{Im}(\mathbb{K}) \times \operatorname{Im}(\mathbb{K}) &\to \operatorname{Im}(\mathbb{K}) \\ \left(V_p, V_q\right) \to V_p \bar{\wedge} V_q &= \begin{vmatrix} \lambda_3 e_1 & \lambda_2 e_2 & \lambda_1 e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ = \lambda_3 \left(a_2 b_3 - a_3 b_2\right) e_1 & \\ + \lambda_2 \left(a_3 b_1 - a_1 b_3\right) e_2 + \lambda_1 \left(a_1 b_2 - a_2 b_1\right) e_3. & \end{aligned}$$

If $p = V_p = a_1e_1 + a_2e_2 + a_3e_3$ and $q = V_q = b_1e_1 + b_2e_2 + b_3e_3$ then the multiplication of p and q is:

$$\begin{aligned} &\times : \mathbb{K} \times \mathbb{K} \to \mathbb{K} \\ &\quad (V_p, V_q) \to V_p \times V_q =: V_pV_q = -f\left(V_p, V_q\right) + V_p \bar{\wedge} V_q \end{aligned}$$

There are two special cases:

i. If $V_p \perp V_q$, then $V_pV_q = V_p\bar{\wedge}V_q$,
ii. If $V_p \parallel V_q$, then $V_pV_q = -f\left(V_p, V_q\right)$.

Theorem 2.3 *Let p, q and r be 3PGVs. The following equations are satisfied:*

i. $p\bar{\wedge}\left(q\bar{\wedge}r\right) = f(p, r)\, q - f(p, q)\, r,$
ii. $\left(p\bar{\wedge}q\right)\bar{\wedge}r = f(p, r)\, q - f(q, r)\, p.$

Proof i. If $p = a_1e_1 + a_2e_2 + a_3e_3$, $q = b_1e_1 + b_2e_2 + b_3e_3$, $r = c_1e_1 + c_2e_2 + c_3e_3$ then we get

$$\begin{aligned} p\bar{\wedge}\left(q\bar{\wedge}r\right) = &\, e_1\left(a_2\left(b_1c_2 - b_2c_1\right)\lambda_1\lambda_3 + a_3\left(b_1c_3 - b_3c_1\right)\lambda_2\lambda_3\right) \\ &+ e_2\left(a_1\left(b_2c_1 - b_1c_2\right)\lambda_1\lambda_2 + a_3\left(b_2c_3 - b_3c_2\right)\lambda_2\lambda_3\right) \\ &+ e_3\left(a_1\left(b_3c_1 - b_1c_3\right)\lambda_1\lambda_2 + a_2\left(b_3c_2 - b_2c_3\right)\lambda_1\lambda_3\right) \end{aligned} \tag{2.1}$$

on the other hand we have

$$\begin{aligned} f(p, r)\, q - f(p, q)\, r = &\, e_1\left(x_2\left(y_1z_2 - y_2z_1\right)\lambda_1\lambda_3 + x_3\left(y_1z_3 - y_3z_1\right)\lambda_2\lambda_3\right) \\ &+ e_2\left(x_1\left(y_2z_1 - y_1z_2\right)\lambda_1\lambda_2 + x_3\left(y_2z_3 - y_3z_2\right)\lambda_2\lambda_3\right) \\ &+ e_3\left(x_1\left(y_3z_1 - y_1z_3\right)\lambda_1\lambda_2 + x_2\left(y_3z_2 - y_2z_3\right)\lambda_1\lambda_3\right) \end{aligned} \tag{2.2}$$

according to Eqs. (2.1) and (2.2), the result is obtained.

ii. Similar to i, the existence of a proof is readily seen.

Corollary 2.4 *Let p and q be two 3PGVs. Then*

$$S(pq) = -f(p, q).$$

Proof If $p = a_1e_1 + a_2e_2 + a_3e_3$ and $q = b_1e_1 + b_2e_2 + b_3e_3$, then

$$S(pq) = -\lambda_1\lambda_2a_1b_1 - \lambda_1\lambda_3a_2b_2 - \lambda_2\lambda_3a_3b_3 = -f(p, q).$$

Corollary 2.5 i. $(\mathbb{K}, +)$ *is an Abelian group.*
ii. *The abelian group $(\mathbb{K}, +)$ is a vector space on the field $\mathbb{R}$ with the external operation $\odot$.*
iii. $\{\mathbb{K}, +, \times\}$ *is a ring with unity.*
iv. $\{\mathbb{K}, +, \times\}$ *is not a commutative ring.*
v. $\{\mathbb{K}, +, \times\}$ *is not an integral domain.*

vi. $\{\mathbb{K}, +, \times\}$ *is not a field.*
vii. $\{\mathbb{K}, +, \mathbb{R}, +, \cdot, \odot\}$ *is a vector space.*
viii. $\{\mathbb{K}, +, \mathbb{R}, +, \cdot, \odot, \times\}$ *is a non-commutative algebra. This algebra is called* 3-parameter generalized quaternion algebra.

Proof The reader can easily verify all of the items above.

Definition 2.6 The conjugate of a 3PGQ p is defined as follows

$$\begin{aligned} C : \mathbb{K} &\to \mathbb{K} \\ p &\to C(p) =: \bar{p} = S_p - V_p \end{aligned}$$

If $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, then $\bar{p} = a_0 - a_1e_1 - a_2e_2 - a_3e_3$.

Theorem 2.7 i. *For all p, q in $\mathbb{K}$ and all c_1, c_2 in $\mathbb{R}$, $\overline{c_1p + c_2q} = \overline{c_1p} + \overline{c_2q}$,*
ii. *For all p, q in $\mathbb{K}$, $\overline{pq} = \bar{q}\bar{p}$,*
iii. *For all p in $\mathbb{K}$, $\overline{\overline{p}} = p$.*

Proof i. For $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $q = b_0 + b_1e_1 + b_2e_2 + b_3e_3$,

$$\begin{aligned} \overline{c_1p + c_2q} &= (c_1a_0 + c_2b_0) - (c_1a_1 + c_2b_1)\, e_1 \\ &\quad - (c_1a_2 + c_2b_2)\, e_2 - (c_1a_3 + c_2b_3)\, e_3 \\ &= c_1\,(a_0 - a_1e_1 - a_2e_2 - a_3e_3) + c_2\,(b_0 - b_1e_1 - b_2e_2 - b_3e_3) \\ &= \overline{c_1p} + \overline{c_2q} \end{aligned}$$

ii. and iii. can be proven in a similar way.

Theorem 2.8 *For any two 3PGVs p, q, we have*

$$p\bar{\wedge}q = \frac{q\bar{p} - p\bar{q}}{2}.$$

Proof If $p = a_1e_1 + a_2e_2 + a_3e_3$ and $q = b_1e_1 + b_2e_2 + b_3e_3$, then

$$\begin{aligned} p\bar{\wedge}q &= \begin{vmatrix} \lambda_3e_1 & \lambda_2e_2 & \lambda_1e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \lambda_3\,(a_2b_3 - a_3b_2)\, e_1 + \lambda_2\,(a_3b_1 - a_1b_3)\, e_2 + \lambda_1\,(a_1b_2 - a_2b_1)\, e_3 \\ &= \frac{1}{2}\,(\bar{q}p - p\bar{q})\,. \end{aligned}$$

Definition 2.9

$$\begin{aligned} N : \mathbb{K} &\to \mathbb{R} \\ p &\to N_p = p\overline{p} = \overline{p}p \end{aligned}$$

The function N is called the norm operation on $\mathbb{K}$. The norm of any 3PGQ p is calculated as follows:

$$N_p = p\overline{p} = a_0^2 + \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2 = S_p S_p + f(V_p, V_p). \quad (2.3)$$

Let p be 3PGQ. If $N_p = 1$, then p is called a 3-parameter generalized unit quaternion (3PGUQ).

Theorem 2.10 *For all p, q in $\mathbb{K}$ and all c in $\mathbb{R}$,*

i. $N_p N_q = N_{pq}$,
ii. $N_{cp} = c^2 N_p$.

Proof The reader can easily prove i and ii using Eq. (2.3)

Definition 2.11 The following function is called an inverse operation on $\mathbb{K}$:

$$\begin{aligned} I &: \mathbb{K} \to \mathbb{R} \\ &p \to I(p) =: p^{-1} = \frac{\overline{p}}{N_p},\ N_p \neq 0. \end{aligned}$$

Let p be a nonzero 3PGQ. If $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, then the inverse of p is as follows:

$$p^{-1} = \frac{\overline{p}}{N_p} = \frac{a_0 - a_1e_1 - a_2e_2 - a_3e_3}{a_0^2 + \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}. \quad (2.4)$$

Theorem 2.12 *For any two nonzero 3PGQs p and q, and any nonzero real number c, we have the following:*

i. $(pq)^{-1} = q^{-1}p^{-1}$,
ii. $(cp)^{-1} = \frac{1}{c}p^{-1}$.

Proof The proof can easily be proved by using Eq. (2.4)

Definition 2.13 Let $p = S_p + V_p$ and $q = S_q + V_q$ be any two 3PGQs. The multiplication defined as follows is called the scalar multiplication of two 3PGQs:

$$\begin{aligned} \langle , \rangle &: \mathbb{K} \times \mathbb{K} \to \mathbb{R} \\ &(p, q) \to \langle p, q \rangle = S_p S_q + f\left(V_p, V_q\right) \end{aligned} \quad (2.5)$$

Also if $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $q = b_0 + b_1e_1 + b_2e_2 + b_3e_3$ then

$$\langle p, q \rangle = a_0b_0 + \lambda_1\lambda_2 a_1b_1 + \lambda_1\lambda_3 a_2b_2 + \lambda_2\lambda_3 a_3b_3 = S(p\bar{q}).$$

Lemma 2.14 *For the metric on $\mathbb{K}$ we have $S(p\bar{q}) = S(\bar{q}p)$ for all p, q in $\mathbb{K}$.*

Proof For $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $q = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{K}$, we obtain

$$\begin{aligned} p\bar{q} = &(a_0b_0 + \lambda_1\lambda_2a_1b_1 + \lambda_1\lambda_3a_2b_2 + \lambda_2\lambda_3a_3b_3) \\ &+e_1\left(-a_0b_1 + b_0a_1 + \lambda_3\left(-a_2b_3 + a_3b_2\right)\right) \\ &+e_2\left(-a_0b_2 + b_0a_2 + \lambda_2\left(-a_3b_1 + a_1b_3\right)\right) \\ &+e_3\left(-a_0b_3 + a_3b_0 + \lambda_1\left(-a_1b_2 + a_2b_1\right)\right) \end{aligned}$$

and thus $S(p\bar{q}) = \langle p, q\rangle$ is seen. Also we find that

$$\langle q, p\rangle = b_0a_0 + \lambda_1\lambda_2b_1a_1 + \lambda_1\lambda_3b_2a_2 + \lambda_2\lambda_3b_3a_3 = S\left(\bar{q}p\right). \tag{2.6}$$

The existence of a proof is apparent from Eq. (2.5) and Eq. (2.6)

Theorem 2.15 *For the metric on* $\mathbb{K}$ *the equalities below hold for all* p, q, r *in* $\mathbb{K}$.

i. $\langle rp, rq\rangle = N_r\langle p, q\rangle$,
ii. $\langle pr, qr\rangle = N_r\langle p, q\rangle$,
iii. $\langle pq, r\rangle = N_r\langle q, \bar{p}r\rangle$,
iv. $\langle pq, r\rangle = N_r\langle p, r\bar{q}\rangle$.

Proof We prove the theorem by using Lemma 2.13 and Eq. (2.5). We will establish the first equation. The proof of the other item has been left to the reader.

$$\langle rp, rq\rangle = S(rp\overline{rq}) = S(rp\bar{q}\bar{r}) = S(\overline{q}\bar{r}rp) = N_rS(\overline{q}p) = N_rS(p\overline{q}) = N_r\left\langle p, q\right\rangle.$$

3 Hamilton Operators and Matrices Associated with 3PGQs

We can not think of quaternions independently of matrices. Real, split and 2PGQs have been also expressed by matrices and various applications have been made with them. Some algebraic properties of Hamilton operators for both 2PGQs and dual quaternions are found in [16, 19, 22, 25]. In this section we associate 3PGQs with matrices.

3.1 Obtaining the Fundamental Matrices

In order to obtain the matrix $\mathcal{M}$, a 3PGQ $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ is multiplied from the left side by $1, e_1, e_2, e_3$,

$$\begin{aligned} (a_0 + a_1e_1 + a_2e_2 + a_3e_3)\,1 &= a_0 + a_1e_1 + a_2e_2 + a_3e_3 \\ (a_0 + a_1e_1 + a_2e_2 + a_3e_3)\,e_1 &= a_0e_1 + a_1e_1^2 + a_2e_2e_1 + a_3e_3e_1 \\ &= -\lambda_1\lambda_2a_1 + a_0e_1 + \lambda_2a_3e_2 - \lambda_1a_2e_3 \\ (a_0 + a_1e_1 + a_2e_2 + a_3e_3)\,e_2 &= a_0e_2 + a_1e_1e_2 + a_2e_2^2 + a_3e_3e_2 \\ &= -\lambda_1\lambda_3a_2 - \lambda_3a_3e_1 + a_0e_2 + \lambda_1a_1e_3 \\ (a_0 + a_1e_1 + a_2e_2 + a_3e_3)\,e_3 &= a_0e_3 + a_1e_1e_3 + a_2e_2e_3 + a_3e_3^2 \\ &= -\lambda_2\lambda_3a_3 + \lambda_3a_2e_1 - \lambda_2a_1e_2 + a_0e_3. \end{aligned}$$

The coefficients of the equations in the above rows are the column elements of the matrix $\mathcal{M}$:

$$\mathcal{M} = \begin{bmatrix} a_0 & -\lambda_1\lambda_2 a_1 & -\lambda_1\lambda_3 a_2 & -\lambda_2\lambda_3 a_3 \\ a_1 & a_0 & -\lambda_3 a_3 & \lambda_3 a_2 \\ a_2 & \lambda_2 a_3 & a_0 & -\lambda_2 a_1 \\ a_3 & -\lambda_1 a_2 & \lambda_1 a_1 & a_0 \end{bmatrix}.$$

If we set $\lambda_1 = 1, \lambda_2 = \lambda, \lambda_3 = \mu$, then we find the fundamental matrix for 2PGQs.

Taking $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$ in $\mathcal{M}$, we have the fundamental matrix for split quaternions.

Similarly by setting $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$, a Hamilton matrix is attained.

Accordingly, multiplication of two 3PGQs as $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $q = b_0 + b_1e_1 + b_2e_2 + b_3e_3$ can be obtained as follows:

$$\begin{aligned} pq &= \begin{bmatrix} a_0 & -\lambda_1\lambda_2 a_1 & -\lambda_1\lambda_3 a_2 & -\lambda_2\lambda_3 a_3 \\ a_1 & a_0 & -\lambda_3 a_3 & \lambda_3 a_2 \\ a_2 & \lambda_2 a_3 & a_0 & -\lambda_2 a_1 \\ a_3 & -\lambda_1 a_2 & \lambda_1 a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= \begin{bmatrix} a_0b_0 - \lambda_1\lambda_2 a_1b_1 - \lambda_1\lambda_3 a_2b_2 - \lambda_2\lambda_3 a_3b_3 \\ a_0b_1 + a_1b_0 + \lambda_3 a_2b_3 - \lambda_3 a_3b_2 \\ a_0b_2 + a_2b_0 - \lambda_2 a_1b_3 + \lambda_2 a_3b_1 \\ a_0b_3 + b_0a_3 + \lambda_1 a_1b_2 - \lambda_1 a_2b_1 \end{bmatrix}. \end{aligned}$$

Theorem 3.1 *The 3PGQ ring* $\mathbb{K}$ *is isomorphic to a subring of the ring* $\mathbb{M}_4(\mathbb{R})$.

Proof Let us define the mapping $\phi : (\mathbb{K}, +, \times) \to (\mathbb{M}_4(\mathbb{R}), \oplus, \otimes)$, where

$$\phi(a_0 + a_1e_1 + a_2e_2 + a_3e_3) \to \begin{bmatrix} a_0 & -\lambda_1\lambda_2 a_1 & -\lambda_1\lambda_3 a_2 & -\lambda_2\lambda_3 a_3 \\ a_1 & a_0 & -\lambda_3 a_3 & \lambda_3 a_2 \\ a_2 & \lambda_2 a_3 & a_0 & -\lambda_2 a_1 \\ a_3 & -\lambda_1 a_2 & \lambda_1 a_1 & a_0 \end{bmatrix}.$$

Let us prove that the mapping ϕ is a ring isomorphism. Taking into account the addition and multiplication operations achieved for the 3PGQ, the following equalities can easily be shown:

$$\begin{aligned} \phi(p+q) &= \phi(p) \oplus \phi(q), \\ \phi(pq) &= \phi(p) \otimes \phi(q). \end{aligned}$$

Now let us show that ϕ is bijective. Since

$$\operatorname{Ker}\phi = \{p : \phi(p) = 0\} = \left\{ p : 0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} = \{0\},$$

ϕ is one-to-one.

$$\phi(\mathbb{K}) = \{\phi(p) : p \in \mathbb{K}\} = \left\{ \begin{bmatrix} a_0 & -\lambda_1\lambda_2 a_1 & -\lambda_1\lambda_3 a_2 & -\lambda_2\lambda_3 a_3 \\ a_1 & a_0 & -\lambda_3 a_3 & \lambda_3 a_2 \\ a_2 & \lambda_2 a_3 & a_0 & -\lambda_2 a_1 \\ a_3 & -\lambda_1 a_2 & \lambda_1 a_1 & a_0 \end{bmatrix} : a_i \in \mathbb{R} \right\}.$$

If we take the restriction

$$\phi : \mathbb{K} \to \phi(\mathbb{K}) \subset \mathbb{M}_4(\mathbb{R})$$

because of our choice of the value set, the mapping ϕ is bijective.

In order to obtain the matrix $\mathcal{N}$, any 3PGQ $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ is multiplied from right side by $1, e_1, e_2, e_3$. Similarly to the creation of $\mathcal{M}$, the $\mathcal{N}$ matrix is produced:

$$\mathcal{N} = \begin{bmatrix} a_0 & -\lambda_1\lambda_2 a_1 & -\lambda_1\lambda_3 a_2 & -\lambda_2\lambda_3 a_3 \\ a_1 & a_0 & \lambda_3 a_3 & -\lambda_3 a_2 \\ a_2 & -\lambda_2 a_3 & a_0 & \lambda_2 a_1 \\ a_3 & \lambda_1 a_2 & -\lambda_1 a_1 & a_0 \end{bmatrix}.$$

As a result, there are two fundamental matrices that give the algebra of 3PGQs: $\mathcal{M}$ and $\mathcal{N}$. Throughout the article, since all the operations with $\mathcal{M}$ and $\mathcal{N}$ matrices will proceed in a similar way, we will give definitions, theorems and explanations for only the matrix $\mathcal{M}$ and note that the same can be done for the matrix $\mathcal{N}$ in a similar way.

3.2 Obtaining the Multiplication Table with the Help of Fundamental Matrices

From the matrix $\mathcal{M}$ that we have obtained in the previous section, we achieve the base elements e_0, e_1, e_2, e_3 as follows

$$e_0 = 1 \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_0 = I_4,\ e_1 \leftrightarrow \begin{bmatrix} 0 & -\lambda_1\lambda_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 \\ 0 & 0 & \lambda_1 & 0 \end{bmatrix} = E_1,$$

$$e_2 \leftrightarrow \begin{bmatrix} 0 & 0 & -\lambda_1\lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \\ 1 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 \end{bmatrix} = E_2,\ e_3 \leftrightarrow \begin{bmatrix} 0 & 0 & 0 & -\lambda_2\lambda_3 \\ 0 & 0 & -\lambda_3 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = E_3.$$

where $\{E_0, E_1, E_2, E_3\}$ is the set of base matrices which correspond to the base elements $1, e_1, e_2, e_3$. Accordingly, multiplying these matrices by each other yields the following:

$$\begin{array}{lll}
e_1^2 \leftrightarrow -\lambda_1\lambda_2 I_4, & e_2^2 \leftrightarrow -\lambda_1\lambda_3 I_4, & e_3^2 \leftrightarrow -\lambda_2\lambda_3 I_4, \\
e_1e_2 \leftrightarrow \lambda_1 E_3, & e_2e_1 \leftrightarrow -\lambda_1 E_3, & e_2e_3 \leftrightarrow \lambda_3 E_1, \\
e_3e_2 \leftrightarrow -\lambda_3 E_1, & e_1e_3 \leftrightarrow -\lambda_2 E_2, & e_3e_1 \leftrightarrow \lambda_2 E_2, \\
e_1e_2e_3 \leftrightarrow -\lambda_1\lambda_2\lambda_3 I_4, & e_2e_3e_1 \leftrightarrow -\lambda_1\lambda_2\lambda_3 I_4, & \\
e_3e_1e_2 \leftrightarrow -\lambda_1\lambda_2\lambda_3 I_4, & e_1e_3e_2 \leftrightarrow \lambda_1\lambda_2\lambda_3 I_4, & \\
e_2e_1e_3 \leftrightarrow \lambda_1\lambda_2\lambda_3 I_4, & e_3e_2e_1 \leftrightarrow \lambda_1\lambda_2\lambda_3 I_4 &
\end{array}$$

which gives us the multiplication table in Definition 2.1.

3.3 Determinant, Characteristic Polynomial, Characteristic Equation, Eigenvalues and Eigenvectors of the Matrix $\mathcal{M}$

The determinant of the matrix $\mathcal{M}$ is calculated as follows:

$$|\mathcal{M}| = \begin{vmatrix} a_0 & -\lambda_1\lambda_2 a_1 & -\lambda_1\lambda_3 a_2 & -\lambda_2\lambda_3 a_3 \\ a_1 & a_0 & -\lambda_3 a_3 & \lambda_3 a_2 \\ a_2 & \lambda_2 a_3 & a_0 & -\lambda_2 a_1 \\ a_3 & -\lambda_1 a_2 & \lambda_1 a_1 & a_0 \end{vmatrix} = \left(N_p\right)^2,$$

where $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$.

The characteristic polynomial of the matrix $\mathcal{M}$ is

$$P_{\mathcal{M}}(t) = \left(t^2 - 2ta_0 + a_0^2 + \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2\right)^2.$$

The characteristic equation of the matrix $\mathcal{M}$ is

$$\begin{aligned}
\det\left(\mathcal{M} - tI_4\right) &= 0 \\
0 &= \begin{vmatrix} a_0 - t & -\lambda_1\lambda_2 a_1 & -\lambda_1\lambda_3 a_2 & -\lambda_2\lambda_3 a_3 \\ a_1 & a_0 - t & -\lambda_3 a_3 & \lambda_3 a_2 \\ a_2 & \lambda_2 a_3 & a_0 - t & -\lambda_2 a_1 \\ a_3 & -\lambda_1 a_2 & \lambda_1 a_1 & a_0 - t \end{vmatrix} \\
0 &= \left(t^2 - 2ta_0 + a_0^2 + \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2\right)^2.
\end{aligned}$$

The four eigenvalues are both coincident in pairs and each other's conjugate:

$$\begin{aligned}
t_{1,2} &= a_0 + \sqrt{-\lambda_1\lambda_2 a_1^2 - \lambda_1\lambda_3 a_2^2 - \lambda_2\lambda_3 a_3^2} \\
t_{3,4} &= a_0 - \sqrt{-\lambda_1\lambda_2 a_1^2 - \lambda_1\lambda_3 a_2^2 - \lambda_2\lambda_3 a_3^2}.
\end{aligned}$$

Multiplication of the eigenvalues is achieved as

$$a_0^2 + \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2 = N(q).$$

Also there are two eigenvectors corresponding to the eigenvalue

$$a_0 + \sqrt{-\lambda_1\lambda_2 a_1^2 - \lambda_1\lambda_3 a_2^2 - \lambda_2\lambda_3 a_3^2}$$

and these are

$$\begin{bmatrix} \frac{\lambda_1 a_2\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}-\lambda_1\lambda_2 a_1 a_3}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ \frac{a_3\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}+\lambda_1 a_1 a_2}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \frac{\lambda_2 a_3\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}+\lambda_1\lambda_2 a_1 a_2}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ -\frac{a_2\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}-\lambda_2 a_1 a_3}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue

$$a_0 - \sqrt{-\lambda_1\lambda_2 a_1^2 - \lambda_1\lambda_3 a_2^2 - \lambda_2\lambda_3 a_3^2}$$

are

$$\begin{bmatrix} -\frac{\lambda_1 a_2\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}+\lambda_1\lambda_2 a_1 a_3}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ -\frac{a_3\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}-\lambda_1 a_1 a_2}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{\lambda_2 a_3\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}-\lambda_1\lambda_2 a_1 a_2}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ \frac{a_2\sqrt{-\lambda_1\lambda_2 a_1^2-\lambda_1\lambda_3 a_2^2-\lambda_2\lambda_3 a_3^2}+\lambda_2 a_1 a_3}{\lambda_1 a_2^2+\lambda_2 a_3^2} \\ 0 \\ 1 \end{bmatrix}.$$

4 Polar Representation, De Moivre's and Euler's Formulas for 3PGQs

Euler's and De Moivre's formulas in complex number are generalized for Hamilton quaternions in [2]. This has also been studied for split and dual quaternions in [21, 26]. Recently, De Moivre's and Euler's formulas have been derived for matrices associated with real, dual quaternions [17, 18]. In the generalized quaternion algebra, De Moivre's and Euler's formulas are studied in [23]. In this section, the polar representation of 3PGQs is studied. And the polar matrix representation of the fundamental matrix $\mathcal{M}$ is created and De Moivre's and Euler's formulas are composed for 3PGQs and the matrix $\mathcal{M}$.

4.1 Polar Representation of 3PGQs and the Matrix $\mathcal{M}$

We can associate an angle θ with a 3PGQ $p = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ as

$$\cos\theta = \frac{a_0}{\sqrt{N(p)}} \quad \text{and} \quad \sin\theta = \frac{\sqrt{\lambda_1\lambda_2a_1^2 + \lambda_1\lambda_3a_2^2 + \lambda_2\lambda_3a_3^2}}{\sqrt{N(p)}}.$$

Definition 4.1 Any 3PGQ p can be written in polar form as follows:

$$p = \sqrt{N(p)}\left(\cos\theta + \hat{p}\sin\theta\right) \tag{4.1}$$

where

$$\hat{p} = \frac{(a_1, a_2, a_3)}{\sqrt{\lambda_1\lambda_2a_1^2 + \lambda_1\lambda_3a_2^2 + \lambda_2\lambda_3a_3^2}}$$

is a 3-parameter generalized unit vector (3PGUV). We will use $\hat{p} = (p_1, p_2, p_3)$ in order to be simpler and more concise, where

$$p_1 = \frac{a_1}{\sqrt{\lambda_1\lambda_2a_1^2 + \lambda_1\lambda_3a_2^2 + \lambda_2\lambda_3a_3^2}},$$
$$p_2 = \frac{a_2}{\sqrt{\lambda_1\lambda_2a_1^2 + \lambda_1\lambda_3a_2^2 + \lambda_2\lambda_3a_3^2}},$$
$$p_3 = \frac{a_3}{\sqrt{\lambda_1\lambda_2a_1^2 + \lambda_1\lambda_3a_2^2 + \lambda_2\lambda_3a_3^2}}.$$

Indeed, the form of $\hat{p}$ Eq. (4.1) is given as follows:

$$\begin{aligned}
p &= a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \\
&= \sqrt{N(p)}\left(\frac{a_0}{\sqrt{N(p)}} + \frac{1}{\sqrt{N(p)}}\left(a_1 e_1 + a_2 e_2 + a_3 e_3\right)\right) \\
&= \sqrt{N(p)}\Bigg(\frac{a_0}{\sqrt{N(p)}} + \frac{\sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}}{\sqrt{N(p)}}. \\
&\quad \times\Bigg(\frac{a_1}{\sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}} e_1 + \frac{a_2}{\sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}} e_2 \\
&\quad + \frac{a_3}{\sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}} e_3\Bigg)\Bigg) \\
&= \sqrt{N(p)}\,(\cos\theta + (p_1, p_2, p_3)\sin\theta.) \\
&= \sqrt{N(p)}\left(\cos\theta + \hat{p}\sin\theta\right).
\end{aligned}$$

4.2 Polar Representation of the Matrix $\mathcal{M}$

Let p be a 3PGUQ. We can write

$$\begin{aligned}
p &= a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \\
&= \cos\theta + \hat{p}\sin\theta \\
&= \cos\theta + (p_1, p_2, p_3)\sin\theta \\
&= \cos\theta + p_1\sin\theta + p_2\sin\theta + p_3\sin\theta \\
&= (\cos\theta, p_1\sin\theta, p_2\sin\theta, p_3\sin\theta)
\end{aligned}$$

and the polar form of the matrix $\mathcal{M}$ is obtained as follows:

$$\mathcal{M} = \begin{bmatrix} \cos\theta & -\lambda_1\lambda_2 p_1\sin\theta & -\lambda_1\lambda_3 p_2\sin\theta & -\lambda_2\lambda_3 p_3\sin\theta \\ p_1\sin\theta & \cos\theta & -\lambda_3 p_3\sin\theta & \lambda_3 p_2\sin\theta \\ p_2\sin\theta & \lambda_2 p_3\sin\theta & \cos\theta & -\lambda_2 p_1\sin\theta \\ p_3\sin\theta & -\lambda_1 p_2\sin\theta & \lambda_1 p_1\sin\theta & \cos\theta \end{bmatrix}.$$

4.3 De Moivre's Formula for 3PGQs

Let us represent the set of 3PGUQs as $S_{\mathbb{K}}$ and the set of 3-parameter generalized unit vectors (3PGUVs) as $S_{\mathbb{K}}^2$. Namely

$$S_{\mathbb{K}} = \left\{p \in \mathbb{K} : N_p = 1\right\}, \quad S_{\mathbb{K}}^2 = \{h \in \mathrm{Im}(\mathbb{K}) : N_h = 1\}.$$

Lemma 4.2 *If* $v \in S^2_{\mathbb{K}}$*, then*

$$(\cos\alpha + v\sin\alpha)(\cos\beta + v\sin\beta) = \cos(\alpha+\beta) + v\sin(\alpha+\beta).$$

Proof The proof is similar to the proof in [24].

Theorem 4.3 *For* $p \in S_{\mathbb{K}}$*, if* $p = \cos\theta + \hat{p}\sin\theta$*, then*

$$p^n = \left(\cos\theta + \hat{p}\sin\theta\right)^n = \cos(n\theta) + \hat{p}\sin(n\theta).$$

Proof The theorem is easily proved by using Lemma 4.2 and the induction method, similar to the proof in [24].

4.4 De Moivre's Formula for Matrices Associated with 3PGQs

We will obtain De Moivre's formula for the matrices corresponding to the 3PGQ p. Let $p = \cos\alpha + \hat{p}\sin\alpha$ be the polar representation of a 3PGQ, where $\hat{p}$ is a 3PGUQ.

Lemma 4.4

$$P = \begin{bmatrix} \cos\alpha & -\lambda_1\lambda_2 p_1\sin\alpha & -\lambda_1\lambda_3 p_2\sin\alpha & -\lambda_2\lambda_3 p_3\sin\alpha \\ p_1\sin\alpha & \cos\alpha & -\lambda_3 p_3\sin\alpha & \lambda_3 p_2\sin\alpha \\ p_2\sin\alpha & \lambda_2 p_3\sin\alpha & \cos\alpha & -\lambda_2 p_1\sin\alpha \\ p_3\sin\alpha & -\lambda_1 p_2\sin\alpha & \lambda_1 p_1\sin\alpha & \cos\alpha \end{bmatrix}$$

$$Q = \begin{bmatrix} \cos\beta & -\lambda_1\lambda_2 p_1\sin\beta & -\lambda_1\lambda_3 p_2\sin\beta & -\lambda_2\lambda_3 p_3\sin\beta \\ p_1\sin\beta & \cos\beta & -\lambda_3 p_3\sin\beta & \lambda_3 p_2\sin\beta \\ p_2\sin\beta & \lambda_2 p_3\sin\beta & \cos\beta & -\lambda_2 p_1\sin\beta \\ p_3\sin\beta & -\lambda_1 p_2\sin\beta & \lambda_1 p_1\sin\alpha & \cos\beta \end{bmatrix}$$

the matrix PQ *is achieved as*

$$\begin{bmatrix} \cos(\alpha+\beta) & -\lambda_1\lambda_2 p_1\sin(\alpha+\beta) & -\lambda_1\lambda_3 p_2\sin(\alpha+\beta) & -\lambda_2\lambda_3 p_3\sin(\alpha+\beta) \\ p_1\sin(\alpha+\beta) & \cos(\alpha+\beta) & -\lambda_3 p_3\sin(\alpha+\beta) & \lambda_3 p_2\sin(\alpha+\beta) \\ p_2\sin(\alpha+\beta) & \lambda_2 p_3\sin(\alpha+\beta) & \cos(\alpha+\beta) & -\lambda_2 p_1\sin(\alpha+\beta) \\ p_3\sin(\alpha+\beta) & -\lambda_1 p_2\sin(\alpha+\beta) & \lambda_1 p_1\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}.$$

Proof Let $PQ = \left[a_{ij}\right]_{4\times4}$.

$$\begin{aligned} a_{11} &= a_{22} = a_{33} = a_{44} \\ &= \cos\alpha\cos\beta - \lambda_1\lambda_2 p_1^2\sin\alpha\sin\beta - \lambda_1\lambda_3 p_2^2\sin\alpha\sin\beta \\ &\quad -\lambda_2\lambda_3 p_3^2\sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta - \left(\lambda_1\lambda_2 p_1^2 + \lambda_1\lambda_3 p_2^2 + \lambda_2\lambda_3 p_3^2\right)\sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta) \end{aligned}$$

and

$$\begin{aligned} a_{12} &= -\lambda_1\lambda_2 p_1 \cos\alpha \sin\beta - \lambda_1\lambda_2 p_1 \sin\alpha \cos\beta \\ &\quad -\lambda_1\lambda_2\lambda_3 p_2 p_3 \sin\alpha \sin\beta + \lambda_1\lambda_2\lambda_3 p_2 p_3 \sin\alpha \sin\beta \\ &= -\lambda_1\lambda_2 p_1 \left(\cos\alpha \sin\beta + \sin\alpha \cos\beta\right) \\ &\quad -\lambda_1\lambda_2\lambda_3 p_2 p_3 \left(\sin\alpha \sin\beta - \sin\alpha \sin\beta\right) \\ &= -\lambda_1\lambda_2 p_1 \sin\left(\alpha+\beta\right). \end{aligned}$$

Similar calculations are used to derive the other entries of the matrix.

Theorem 4.5 *For any integer n, if*

$$P = \begin{bmatrix} \cos\alpha & -\lambda_1\lambda_2 p_1 \sin\alpha & -\lambda_1\lambda_3 p_2 \sin\alpha & -\lambda_2\lambda_3 p_3 \sin\alpha \\ p_1 \sin\alpha & \cos\alpha & -\lambda_3 p_3 \sin\alpha & \lambda_3 p_2 \sin\alpha \\ p_2 \sin\alpha & \lambda_2 p_3 \sin\alpha & \cos\alpha & -\lambda_2 p_1 \sin\alpha \\ p_3 \sin\alpha & -\lambda_1 p_2 \sin\alpha & \lambda_1 p_1 \sin\alpha & \cos\alpha \end{bmatrix},$$

then the nth power of the matrix P is obtained as

$$\begin{bmatrix} \cos(n\alpha) & -\lambda_1\lambda_2 p_1 \sin(n\alpha) & -\lambda_1\lambda_3 p_2 \sin(n\alpha) & -\lambda_2\lambda_3 p_3 \sin(n\alpha) \\ p_1 \sin(n\alpha) & \cos(n\alpha) & -\lambda_3 p_3 \sin(n\alpha) & \lambda_3 p_2 \sin(n\alpha) \\ p_2 \sin(n\alpha) & \lambda_2 p_3 \sin(n\alpha) & \cos(n\alpha) & -\lambda_2 p_1 \sin(n\alpha) \\ p_3 \sin(n\alpha) & -\lambda_1 p_2 \sin(n\alpha) & \lambda_1 p_1 \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}.$$

Proof We can prove this by use of induction. First, let us show the correctness of the theorem for $n \geq 2$. For $n = 2$, by using Lemma 5.4 and replacing the Q matrix with the P matrix, we obtain

$$P^2 = \begin{bmatrix} \cos(2\alpha) & -\lambda_1\lambda_2 p_1 \sin(2\alpha) & -\lambda_1\lambda_3 p_2 \sin(2\alpha) & -\lambda_2\lambda_3 p_3 \sin(2\alpha) \\ p_1 \sin(2\alpha) & \cos(2\alpha) & -\lambda_3 p_3 \sin(2\alpha) & \lambda_3 p_2 \sin(2\alpha) \\ p_2 \sin(2\alpha) & \lambda_2 p_3 \sin(2\alpha) & \cos(2\alpha) & -\lambda_2 p_1 \sin(2\alpha) \\ p_3 \sin(2\alpha) & -\lambda_1 p_2 \sin(2\alpha) & \lambda_1 p_1 \sin(2\alpha) & \cos(2\alpha) \end{bmatrix}.$$

For $n = k$, assume the formula is correct. For $n = k + 1$, by using $P^{k+1} = P^k P$ and Lemma 5.4, P^{k+1} is obtained as

$$\begin{bmatrix} \cos((k+1)\alpha) & -\lambda_1\lambda_2 p_1 \sin((k+1)\alpha) & -\lambda_1\lambda_3 p_2 \sin((k+1)\alpha) & -\lambda_2\lambda_3 p_3 \sin((k+1)\alpha) \\ p_1 \sin((k+1)\alpha) & \cos((k+1)\alpha) & -\lambda_3 p_3 \sin((k+1)\alpha) & \lambda_3 p_2 \sin((k+1)\alpha) \\ p_2 \sin((k+1)\alpha) & \lambda_2 p_3 \sin((k+1)\alpha) & \cos((k+1)\alpha) & -\lambda_2 p_1 \sin((k+1)\alpha) \\ p_3 \sin((k+1)\alpha) & -\lambda_1 p_2 \sin((k+1)\alpha) & \lambda_1 p_1 \sin((k+1)\alpha) & \cos((k+1)\alpha) \end{bmatrix}.$$

We find the matrix P^{-1} by calculation of the inverse matrix as

$$P^{-1} = \begin{bmatrix} \cos\alpha & \lambda_1\lambda_2 p_1 \sin\alpha & \lambda_1\lambda_3 p_2 \sin\alpha & \lambda_2\lambda_3 p_3 \sin\alpha \\ -p_1 \sin\alpha & \cos\alpha & \lambda_3 p_3 \sin\alpha & -\lambda_3 p_2 \sin\alpha \\ -p_2 \sin\alpha & -\lambda_2 p_3 \sin\alpha & \cos\alpha & \lambda_2 p_1 \sin\alpha \\ -p_3 \sin\alpha & \lambda_1 p_2 \sin\alpha & -\lambda_1 p_1 \sin\alpha & \cos\alpha \end{bmatrix}.$$

Since cosine is an even function and sine is an odd function, the matrix P^{-1} is written as

$$P^{-1} = \begin{bmatrix} \cos(-\alpha) & -\lambda_1\lambda_2 p_1 \sin(-\alpha) & -\lambda_1\lambda_3 p_2 \sin(-\alpha) & -\lambda_2\lambda_3 p_3 \sin(-\alpha) \\ p_1 \sin(-\alpha) & \cos(-\alpha) & -\lambda_3 p_3 \sin(-\alpha) & \lambda_3 p_2 \sin(-\alpha) \\ p_2 \sin(-\alpha) & \lambda_2 p_3 \sin(-\alpha) & \cos(-\alpha) & -\lambda_2 p_1 \sin(-\alpha) \\ p_3 \sin(-\alpha) & -\lambda_1 p_2 \sin(-\alpha) & \lambda_1 p_1 \sin(-\alpha) & \cos(-\alpha) \end{bmatrix}.$$

According to this, P^{-n} is achieved as

$$\begin{bmatrix} \cos(-n\alpha) & -\lambda_1\lambda_2 p_1 \sin(-n\alpha) & -\lambda_1\lambda_3 p_2 \sin(-n\alpha) & -\lambda_2\lambda_3 p_3 \sin(-n\alpha) \\ p_1 \sin(-n\alpha) & \cos(-n\alpha) & -\lambda_3 p_3 \sin(-n\alpha) & \lambda_3 p_2 \sin(-n\alpha) \\ p_2 \sin(-n\alpha) & \lambda_2 p_3 \sin(-n\alpha) & \cos(-n\alpha) & -\lambda_2 p_1 \sin(-n\alpha) \\ p_3 \sin(-n\alpha) & -\lambda_1 p_2 \sin(-n\alpha) & \lambda_1 p_1 \sin(-n\alpha) & \cos(-n\alpha) \end{bmatrix}.$$

Example 1 Let a 3PGUQ be

$$p = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}}, \frac{1}{\sqrt{\lambda_1\lambda_3}}, \frac{1}{\sqrt{\lambda_2\lambda_3}}\right).$$

Polar representation of p can be expressed as

$$p = \cos\frac{2\pi}{3} + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}}, \frac{1}{\sqrt{\lambda_1\lambda_3}}, \frac{1}{\sqrt{\lambda_2\lambda_3}}\right)\sin\frac{2\pi}{3}.$$

If

$$\hat{p} = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}}, \frac{1}{\sqrt{\lambda_1\lambda_3}}, \frac{1}{\sqrt{\lambda_2\lambda_3}}\right)$$

then

$$p_1 = \frac{1}{\sqrt{3\lambda_1\lambda_2}}, \quad p_2 = \frac{1}{\sqrt{3\lambda_1\lambda_3}}, \quad p_3 = \frac{1}{\sqrt{3\lambda_2\lambda_3}}.$$

The matrix representation of p is

$$A=\begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{\lambda_1\lambda_2}}{2} & \frac{-\sqrt{\lambda_1\lambda_3}}{2} & \frac{-\sqrt{\lambda_2\lambda_3}}{2} \\ \frac{1}{2\sqrt{\lambda_1\lambda_2}} & -\frac{1}{2} & \frac{-\sqrt{\lambda_3}}{2\sqrt{\lambda_2}} & \frac{\sqrt{\lambda_3}}{2\sqrt{\lambda_1}} \\ \frac{1}{2\sqrt{\lambda_1\lambda_3}} & \frac{\sqrt{\lambda_2}}{2\sqrt{\lambda_3}} & -\frac{1}{2} & \frac{-\sqrt{\lambda_2}}{2\sqrt{\lambda_3}} \\ \frac{1}{2\sqrt{\lambda_2\lambda_3}} & \frac{-\sqrt{\lambda_1}}{2\sqrt{\lambda_3}} & \frac{\sqrt{\lambda_1}}{2\sqrt{\lambda_2}} & -\frac{1}{2} \end{bmatrix}.$$

and the matrix polar representation of p is

$$A=\begin{bmatrix} \cos\frac{2\pi}{3} & -\lambda_1\lambda_2 p_1\sin\frac{2\pi}{3} & -\lambda_1\lambda_3 p_2\sin\frac{2\pi}{3} & -\lambda_2\lambda_3 p_3\sin\frac{2\pi}{3} \\ p_1\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & -\lambda_3 p_3\sin\frac{2\pi}{3} & \lambda_3 p_2\sin\frac{2\pi}{3} \\ p_2\sin\frac{2\pi}{3} & \lambda_2 p_3\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & -\lambda_2 p_1\sin\frac{2\pi}{3} \\ p_3\sin\frac{2\pi}{3} & -\lambda_1 p_2\sin\frac{2\pi}{3} & \lambda_1 p_1\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}.$$

5th and 21st powers of p are

$$\begin{aligned} p^5 &= \cos\left(5\cdot\frac{2\pi}{3}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}},\frac{1}{\sqrt{\lambda_1\lambda_3}},\frac{1}{\sqrt{\lambda_2\lambda_3}}\right)\sin\left(5\cdot\frac{2\pi}{3}\right) \\ &= \cos\left(\frac{\pi}{3}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}},\frac{1}{\sqrt{\lambda_1\lambda_3}},\frac{1}{\sqrt{\lambda_2\lambda_3}}\right)\sin\left(\frac{\pi}{3}\right) \\ &= \frac{1}{2}+\frac{1}{2}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}},\frac{1}{\sqrt{\lambda_1\lambda_3}},\frac{1}{\sqrt{\lambda_2\lambda_3}}\right) \end{aligned}$$

and

$$\begin{aligned} p^{21} &= \cos\left(21\cdot\frac{2\pi}{3}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}},\frac{1}{\sqrt{\lambda_1\lambda_3}},\frac{1}{\sqrt{\lambda_2\lambda_3}}\right)\sin\left(21\cdot\frac{2\pi}{3}\right) \\ &= \cos 0+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}},\frac{1}{\sqrt{\lambda_1\lambda_3}},\frac{1}{\sqrt{\lambda_2\lambda_3}}\right)\sin 0 \\ &= 1. \end{aligned}$$

The matrix form is easily calculated as

$$A^5=\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{\lambda_1\lambda_2}}{2} & \frac{-\sqrt{\lambda_1\lambda_3}}{2} & \frac{-\sqrt{\lambda_2\lambda_3}}{2} \\ \frac{1}{2\sqrt{\lambda_1\lambda_2}} & \frac{1}{2} & \frac{-\sqrt{\lambda_3}}{2\sqrt{\lambda_2}} & \frac{\sqrt{\lambda_3}}{2\sqrt{\lambda_1}} \\ \frac{1}{2\sqrt{\lambda_1\lambda_3}} & \frac{\sqrt{\lambda_2}}{2\sqrt{\lambda_3}} & \frac{1}{2} & \frac{-\sqrt{\lambda_2}}{2\sqrt{\lambda_3}} \\ \frac{1}{2\sqrt{\lambda_2\lambda_3}} & \frac{-\sqrt{\lambda_1}}{2\sqrt{\lambda_3}} & \frac{\sqrt{\lambda_1}}{2\sqrt{\lambda_2}} & \frac{1}{2} \end{bmatrix}$$

and

$$A^{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.5 Euler's Formula for 3PGQs

For any $v \in S^2_{\mathbb{K}}$, we know $v^2 = -1$. Then

$$v^3 = -v, \quad v^4 = 1, \quad v^5 = v, \quad v^6 = -1, \ldots$$

Euler's formula for 3PGQs with any angle θ is obtained as

$$\begin{aligned} e^{v\theta} &= 1 + v\theta + v^2\frac{\theta^2}{2} + v^3\frac{\theta^3}{3!} + v^4\frac{\theta^4}{4!} + \cdots \\ & 1 + v\theta - \frac{\theta^2}{2} - v\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + v\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + v\sin\theta. \end{aligned}$$

4.6 Euler's Formula for Associated Matrices with 3PGQs

Let us choose a matrix $\mathcal{P}$ as follows:

$$\mathcal{P} = \begin{bmatrix} 0 & -\lambda_1\lambda_2 p_1 & -\lambda_1\lambda_3 p_2 & -\lambda_2\lambda_3 p_3 \\ p_1 & 0 & -\lambda_3 p_3 & \lambda_3 p_2 \\ p_2 & \lambda_2 p_3 & 0 & -\lambda_2 p_1 \\ p_3 & -\lambda_1 p_2 & \lambda_1 p_1 & 0 \end{bmatrix}$$

Since $\lambda_1\lambda_2 p_1^2 + \lambda_1\lambda_3 p_2^2 + \lambda_2\lambda_3 p_3^2 = 1$, $\mathcal{P}^2 = -I_4$ is easy to see. Then

$$\begin{aligned} e^{\mathcal{P}\alpha} &= I_4 + \mathcal{P}\alpha + \frac{(\mathcal{P}\alpha)^2}{2!} + \frac{(\mathcal{P}\alpha)^3}{3!} + \frac{(\mathcal{P}\alpha)^4}{4!} \\ &= I_4\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right) + \mathcal{P}\left(\alpha - \frac{\alpha^3}{2!} + \frac{\alpha^5}{3!} - \cdots\right) \\ &= \cos\alpha + \mathcal{P}\sin\alpha \\ &= \cos\alpha + \begin{bmatrix} 0 & -\lambda_1\lambda_2 p_1 & -\lambda_1\lambda_3 p_2 & -\lambda_2\lambda_3 p_3 \\ p_1 & 0 & -\lambda_3 p_3 & \lambda_3 p_2 \\ p_2 & \lambda_2 p_3 & 0 & -\lambda_2 p_1 \\ p_3 & -\lambda_1 p_2 & \lambda_1 p_1 & 0 \end{bmatrix}\sin\alpha \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} \cos\alpha & -\lambda_1\lambda_2 p_1 \sin\alpha & -\lambda_1\lambda_3 p_2 \sin\alpha & -\lambda_2\lambda_3 p_3 \sin\alpha \\ p_1 \sin\alpha & \cos\alpha & -\lambda_3 p_3 \sin\alpha & \lambda_3 p_2 \sin\alpha \\ p_2 \sin\alpha & \lambda_2 p_3 \sin\alpha & \cos\alpha & -\lambda_2 p_1 \sin\alpha \\ p_3 \sin\alpha & -\lambda_1 p_2 \sin\alpha & \lambda_1 p_1 \sin\alpha & \cos\alpha \end{bmatrix} \\
&= P.
\end{aligned}
$$

4.7 The *n*th Roots of Matrices Associated with 3PGQs

$$
A = \begin{bmatrix} \cos(\alpha+2k\pi) & -\lambda_1\lambda_2 p_1 \sin(\alpha+2k\pi) & -\lambda_1\lambda_3 p_2 \sin(\alpha+2k\pi) & -\lambda_2\lambda_3 p_3 \sin(\alpha+2k\pi) \\ p_1 \sin(\alpha+2k\pi) & \cos(\alpha+2k\pi) & -\lambda_3 p_3 \sin(\alpha+2k\pi) & \lambda_3 p_2 \sin(\alpha+2k\pi) \\ p_2 \sin(\alpha+2k\pi) & \lambda_2 p_3 \sin(\alpha+2k\pi) & \cos(\alpha+2k\pi) & -\lambda_2 p_1 \sin(\alpha+2k\pi) \\ p_3 \sin(\alpha+2k\pi) & -\lambda_1 p_2 \sin(\alpha+2k\pi) & \lambda_1 p_1 \sin(\alpha+2k\pi) & \cos(\alpha+2k\pi) \end{bmatrix}
$$

where $k \in \mathbb{Z}$. The equation $X^n = A$ has n roots. These roots are found as

$$
A_k^{1/n} = \begin{bmatrix} \cos\left(\frac{\alpha+2k\pi}{n}\right) & -\lambda_1\lambda_2 p_1 \sin\left(\frac{\alpha+2k\pi}{n}\right) & -\lambda_1\lambda_3 p_2 \sin\left(\frac{\alpha+2k\pi}{n}\right) & -\lambda_2\lambda_3 p_3 \sin\left(\frac{\alpha+2k\pi}{n}\right) \\ p_1 \sin\left(\frac{\alpha+2k\pi}{n}\right) & \cos\left(\frac{\alpha+2k\pi}{n}\right) & -\lambda_3 p_3 \sin\left(\frac{\alpha+2k\pi}{n}\right) & \lambda_3 p_2 \sin\left(\frac{\alpha+2k\pi}{n}\right) \\ p_2 \sin\left(\frac{\alpha+2k\pi}{n}\right) & \lambda_2 p_3 \sin\left(\frac{\alpha+2k\pi}{n}\right) & \cos\left(\frac{\alpha+2k\pi}{n}\right) & -\lambda_2 p_1 \sin\left(\frac{\alpha+2k\pi}{n}\right) \\ p_3 \sin\left(\frac{\alpha+2k\pi}{n}\right) & -\lambda_1 p_2 \sin\left(\frac{\alpha+2k\pi}{n}\right) & \lambda_1 p_1 \sin\left(\frac{\alpha+2k\pi}{n}\right) & \cos\left(\frac{\alpha+2k\pi}{n}\right) \end{bmatrix}.
$$

For $k = 0$, the first root is

$$
A_0^{1/n} = \begin{bmatrix} \cos\left(\frac{\alpha}{n}\right) & -\lambda_1\lambda_2 p_1 \sin\left(\frac{\alpha}{n}\right) & -\lambda_1\lambda_3 p_2 \sin\left(\frac{\alpha}{n}\right) & -\lambda_2\lambda_3 p_3 \sin\left(\frac{\alpha}{n}\right) \\ p_1 \sin\left(\frac{\alpha}{n}\right) & \cos\left(\frac{\alpha}{n}\right) & -\lambda_3 p_3 \sin\left(\frac{\alpha}{n}\right) & \lambda_3 p_2 \sin\left(\frac{\alpha}{n}\right) \\ p_2 \sin\left(\frac{\alpha}{n}\right) & \lambda_2 p_3 \sin\left(\frac{\alpha}{n}\right) & \cos\left(\frac{\alpha}{n}\right) & -\lambda_2 p_1 \sin\left(\frac{\alpha}{n}\right) \\ p_3 \sin\left(\frac{\alpha}{n}\right) & -\lambda_1 p_2 \sin\left(\frac{\alpha}{n}\right) & \lambda_1 p_1 \sin\left(\frac{\alpha}{n}\right) & \cos\left(\frac{\alpha}{n}\right) \end{bmatrix},
$$

for $k = 1$, the second root is

$$
A_1^{1/n} = \begin{bmatrix} \cos\left(\frac{\alpha+2\pi}{n}\right) & -\lambda_1\lambda_2 p_1 \sin\left(\frac{\alpha+2\pi}{n}\right) & -\lambda_1\lambda_3 p_2 \sin\left(\frac{\alpha+2\pi}{n}\right) & -\lambda_2\lambda_3 p_3 \sin\left(\frac{\alpha+2\pi}{n}\right) \\ p_1 \sin\left(\frac{\alpha+2\pi}{n}\right) & \cos\left(\frac{\alpha+2\pi}{n}\right) & -\lambda_3 p_3 \sin\left(\frac{\alpha+2\pi}{n}\right) & \lambda_3 p_2 \sin\left(\frac{\alpha+2\pi}{n}\right) \\ p_2 \sin\left(\frac{\alpha+2\pi}{n}\right) & \lambda_2 p_3 \sin\left(\frac{\alpha+2\pi}{n}\right) & \cos\left(\frac{\alpha+2\pi}{n}\right) & -\lambda_2 p_1 \sin\left(\frac{\alpha+2\pi}{n}\right) \\ p_3 \sin\left(\frac{\alpha+2\pi}{n}\right) & -\lambda_1 p_2 \sin\left(\frac{\alpha+2\pi}{n}\right) & \lambda_1 p_1 \sin\left(\frac{\alpha+2\pi}{n}\right) & \cos\left(\frac{\alpha+2\pi}{n}\right) \end{bmatrix}.
$$

Similarly, for $k = n - 1$, the nth root is obtained.

4.8 Relations among the Powers of Matrices Associated with 3PGQs

Theorem 4.6 *Let $m = 2\pi/\theta \in \mathbb{Z}^+ - \{1\}$ and the polar expression of a 3PGUQ p be $p = \cos\theta + v\sin\theta$. Then $n \equiv s \ (mod\ m)$ if and only if $p^n = p^s$.*

Proof Let $n \equiv s\ (mod\ m)$.

$$\begin{aligned}
p^n &= \left(\cos\theta + \hat{p}\sin\theta\right)^n \\
&= \cos(n\theta) + \hat{p}\sin(n\theta) \\
&= \cos((mk+s)\theta) + \hat{p}\sin((mk+s)\theta) \\
&= \cos\left(\left(\frac{2\pi}{\theta}k+s\right)\theta\right) + \hat{p}\sin\left(\left(\frac{2\pi}{\theta}k+s\right)\theta\right) \\
&= \cos(2\pi k + s\theta) + \hat{p}\sin(2\pi k + s\theta) \\
&= \cos(s\theta) + \hat{p}\sin(s\theta) \\
&= \left(\cos\theta + \hat{p}\sin\theta\right)^s \\
&= p^s.
\end{aligned}$$

On the other hand, let $p^n = \cos(n\theta) + \hat{p}\sin(n\theta)$ and $p^s = \cos(s\theta) + \hat{p}\sin(s\theta$. Since $p^n = p^s$, $\cos(n\theta) = \cos(s\theta)$ and $\sin(n\theta) = \sin(s\theta)$ are found. This also requires the equation

$$n\theta = s\theta + 2k\pi, \quad k \in \mathbb{Z}.$$

Thus

$$n = \frac{2\pi}{\theta}k + s, \quad n \equiv s\ (\mathrm{mod}\ m)$$

is attained.

Example 2 Let

$$p = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}}, \frac{1}{\sqrt{\lambda_1\lambda_3}}, \frac{1}{\sqrt{\lambda_2\lambda_3}}\right)$$

be a 3PGUQ. We have expressed the polar form of p in the previous example. Since $\varphi = 2\pi/3$ from Theorem 4.6 we find $m = 2\pi/(2\pi/3) = 3$. Then we have

$$\begin{aligned}
p &= p^4 = p^7 = \cdots \\
p^2 &= p^5 = p^8 = \cdots \\
p^3 &= p^6 = p^9 = \cdots = 1.
\end{aligned}$$

Theorem 4.7 *Let the expression of the 3PGUQ* p *in the polar form be* $p = \cos\theta + v\sin\theta$, $m = 2\pi/\theta \in \mathbb{Z}^+ - \{1\}$ *and let* A *be the matrix representation of* p. *Accordingly,* $n \equiv s\ (mod\ m)$ *if and only if* $A^n = A^s$.

Proof Similar to the proof of Theorem 4.6.

Theorem 4.8 *Let the polar representation of the 3PGUQ* p *be*

$$p = \sqrt{N_p}\left(\cos\theta + \hat{p}\sin\theta\right)$$

and $2\pi/\theta = m \in \mathbb{Z}^+ - \{1\}$. *Then* $n \equiv s \pmod{m}$ *if and only if*

$$p^n = \left(\sqrt{N_p}\right)^{n-s} p^s.$$

Proof Similar to the proof of the theorem in [24].

5 Lie Algebra and Matrix Representations of 3PGQs

In [22], Karger and Novak show that the set of all unit quaternions is a three-dimensional Lie group and also study the Lie algebra. Jafari and Yaylı conducted the same study on 2-parameter generalized unit quaternions in [20]. In this section we will show that the set of 3PGUQs is a Lie group and give the properties of the Lie algebra, adjoint mappings, Lie multpilication and Killing bi-linear form for 3PGQs.

5.1 Lie Group of 3PGQs

Theorem 5.1 $S_{\mathbb{K}} = \left\{p \in \mathbb{K} : N_p = 1\right\}$ *is a three-dimensional Lie group.*

Proof The set $S_{\mathbb{K}}$ is a unity group, together with multiplication on 3PGQ. The unit element of the group $S_{\mathbb{K}}$ is $e = 1$. Let us define the function f as

$$\begin{gathered} f : \mathbb{K} \to \mathbb{R} \\ p \to f(p) = a_0 + \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2. \end{gathered}$$

The function f is expressed as coordinate functions as the following:

$$f = x_0^2 + \lambda_1\lambda_2 x_1^2 + \lambda_1\lambda_3 x_2^2 + \lambda_2\lambda_3 x_3^2$$

The Jacobi matrix of the function f can be written as

$$J(f) = \left[\, 2x_0 \; 2\lambda_1\lambda_2 x_1 \; 2\lambda_1\lambda_3 x_2 \; 2\lambda_2\lambda_3 x_3 \,\right]$$

$Rank\, J(f) = 1$. $f^{-1}(1)$ is a submanifold of $\mathbb{K}$. It can be shown that the mappings which are defined as follows are differentiable:

$$\begin{aligned} \gamma &: \mathbb{K} \times \mathbb{K} \to \mathbb{K} \\ &(p, q) \to \gamma(p, q) = pq \end{aligned} \quad \text{and} \quad \begin{aligned} \eta &: \mathbb{K} \to \mathbb{K} \\ &p \to \eta(p) = p^{-1} = \bar{p} \end{aligned}.$$

$S_{\mathbb{K}}$ is a three-dimensional Lie group.

Theorem 5.2 $\operatorname{Im}(\mathbb{K})$ *is a Lie algebra of the Lie group* $S_{\mathbb{K}}$.

Proof Let $T_{S_{\mathbb{K}}}(e)$ be the set of velocity vectors through the point e and $v_e \in T_{S_{\mathbb{K}}}(e)$. Let us define a curve ρ on $S_{\mathbb{K}}$ as:

$$\begin{aligned}&\rho : I \subset \mathbb{R} \to S_{\mathbb{K}}\\ &\rho(s) \to \rho(s) = a_0(s) + a_1(s)e_1 + a_2(s)e_2 + a_3(s)e_3.\end{aligned}$$

Let the curve ρ accept v_e as a velocity vector:

$$\rho(0) = 1 \text{ and } \rho^{'}(0) = v_e.$$

Since $\rho(s) \in S_{\mathbb{K}}$,

$$a_0^2(s) + \lambda_1\lambda_2 a_1^2(s) + \lambda_1\lambda_3 a_2^2(s) + \lambda_2\lambda_3 a_3^2(s) = 1. \tag{5.1}$$

At the point $s = 0$, the derivative of Eq. (5.1) is

$$2a_0^{'}(s)a_0(s) + 2\lambda_1\lambda_2 a_1^{'}(s)a_1(s) + 2\lambda_1\lambda_3 a_2^{'}(s)a_2(s) + 2\lambda_2\lambda_3 a_3^{'}(s)a_3(s) = 0.$$

Then

$$a_0(0) = 1, \quad a_1(0) = 0, \quad a_2(0) = 0, \quad a_3(0) = 0, \quad a_0^{'}(0) = 0.$$

All of the vectors in $T_{S_{\mathbb{K}}}(e)$ can be written as a linear combination of the vectors in the base

$$\left.\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}\right|_{s=0}$$

of the tangent space at the point e of $\mathrm{Im}(\mathbb{K})$. Then the velocity vector $a_0^{'}(0) = 0$ is written as

$$a_0^{'} = a_0^{'}(0)\frac{\partial}{\partial x_0} + a_1^{'}(0)\frac{\partial}{\partial x_1} + a_2^{'}(0)\frac{\partial}{\partial x_2} + a_3^{'}(0)\frac{\partial}{\partial x_3}.$$

Since $a_0^{'}(0) = 0$,

$$T_{S_{\mathbb{K}}}(e) \subset Sp\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}$$

is established. Also since $\mathrm{boy} S_{\mathbb{K}} = \mathrm{boy} T_{S_{\mathbb{K}}}(e) = 3$, we obtain

$$T_{S_{\mathbb{K}}}(e) = \mathrm{Sp}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}.$$

Therefore, the Lie algebra of the Lie group $S_{\mathbb{K}}$ is $\mathrm{Im}(\mathbb{K})$.

Corollary 5.3 $T_{S_{\mathbb{K}}}(e)$ *is isomorphic to*

$$\mathrm{Im}(\mathbb{K}) = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1, a_2, a_3 \in \mathbb{R}\}.$$

5.2 Adjoint Mappings for Lie Algebra and Lie Group on 3PGQs

5.2.1 Matrix Representation for Lie Group of $S_{\mathbb{K}}$

For $S_{\mathbb{K}} = \left\{q \in \mathbb{K} : N_q = 1\right\}$ and $k \in S_{\mathbb{K}}$, let us define the function g_k that is bijective and differentiable as

$$\begin{aligned} &g_k : S_{\mathbb{K}} \to S_{\mathbb{K}} \\ &x \to g_k(x) = kxk^{-1}. \end{aligned}$$

Let us consider the derivative map of the function and its restriction about the unit $e = 1$ point of the group. In this case, for any p in $S_{\mathbb{K}}$, the following mapping is called the adjoint mapping:

$$\begin{aligned} &\mathcal{A}dp : T_{S_{\mathbb{K}}}(e) \to T_{S_{\mathbb{K}}}(e) \\ &q \to pqp^{-1}. \end{aligned}$$

Since $T_{S_{\mathbb{K}}}(e) = Sp\{e_1, e_2, e_3\}$, according to the base $\{e_1, e_2, e_3\}$ we have

$$\begin{aligned} \mathcal{A}dp(e_1) &= \left(a_0^2 + \lambda_1\lambda_2a_1^2 - \lambda_1\lambda_3a_2^2 - \lambda_2\lambda_3a_3^2\right)e_1 \\ &\quad + (2\lambda_1\lambda_2a_1a_2 + 2\lambda_2a_0a_3)e_2 + (2\lambda_1\lambda_2a_1a_2 - 2\lambda_1a_0a_2)e_3 \\ \mathcal{A}dp(e_2) &= (2\lambda_1\lambda_3a_1a_2 - 2\lambda_3a_0a_3)e_1 \\ &\quad + \left(a_0^2 - \lambda_1\lambda_2a_1^2 + \lambda_1\lambda_3a_2^2 - \lambda_2\lambda_3a_3^2\right)e_2 \\ &\quad + (2\lambda_1\lambda_3a_2a_3 - 2\lambda_1a_0a_1)e_3 \\ \mathcal{A}dp(e_3) &= (2\lambda_2\lambda_3a_1a_3 + 2\lambda_3a_0a_2)e_1 + (2\lambda_2\lambda_3a_2a_3 - 2\lambda_2a_0a_1)e_2 \\ &\quad + \left(a_0^2 - \lambda_1\lambda_2a_1^2 - \lambda_1\lambda_3a_2^2 + \lambda_2\lambda_3a_3^2\right)e_3. \end{aligned}$$

Consequently, the matrix is obtained as

$$Adp = \begin{bmatrix} a_0^2 + \lambda_1\lambda_2a_1^2 - \lambda_1\lambda_3a_2^2 - \lambda_2\lambda_3a_3^2 & 2\lambda_1\lambda_3a_1a_2 - 2\lambda_3a_0a_3 & 2\lambda_2\lambda_3a_1a_3 + 2\lambda_3a_0a_2 \\ 2\lambda_1\lambda_2a_1a_2 + 2\lambda_2a_0a_3 & a_0^2 - \lambda_1\lambda_2a_1^2 + \lambda_1\lambda_3a_2^2 - \lambda_2\lambda_3a_3^2 & 2\lambda_2\lambda_3a_2a_3 - 2\lambda_2a_0a_1 \\ 2\lambda_1\lambda_2a_1a_3 - 2\lambda_1a_0a_2 & 2\lambda_1\lambda_3a_2a_3 + 2\lambda_1a_0a_1 & a_0^2 - \lambda_1\lambda_2a_1^2 - \lambda_1\lambda_3a_2^2 + \lambda_2\lambda_3a_3^2 \end{bmatrix}.$$

Theorem 5.4 *If*

$$\varepsilon = \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{bmatrix}$$

then $\mathcal{A}dp^T \varepsilon \mathcal{A}dp = \varepsilon$.

Proof If the matrix $\mathcal{A}dp^T$ is multiplied by the matrix ε then

$$\mathcal{A}dp^T \varepsilon = \begin{bmatrix} \lambda_1\lambda_2 a_0^2 + \lambda_1^2\lambda_2^2 a_1^2 - \lambda_1^2\lambda_2\lambda_3 a_2^2 - \lambda_1\lambda_2^2\lambda_3 a_3^2 & 2\lambda_1\lambda_2^2 a_0 a_3 + 2\lambda_1^2\lambda_2^2 a_1 a_2 & 2\lambda_1^2\lambda_2^2 a_1 a_3 - 2\lambda_1^2\lambda_2 a_0 a_2 \\ 2\lambda_1^2\lambda_3^2 a_1 a_2 - 2\lambda_1\lambda_3^2 a_0 a_3 & \lambda_1\lambda_3 a_0^2 - \lambda_1^2\lambda_2\lambda_3 a_1^2 + \lambda_1^2\lambda_3^2 a_2^2 - \lambda_1\lambda_2\lambda_3^2 a_3^2 & 2\lambda_1^2\lambda_3 a_0 a_1 + 2\lambda_1^2\lambda_3^2 a_2 a_3 \\ 2\lambda_2\lambda_3^2 a_0 a_2 + 2\lambda_2^2\lambda_3^2 a_1 a_3 & 2\lambda_2^2\lambda_3^2 a_2 a_3 - 2\lambda_2^2\lambda_3 a_0 a_1 & \lambda_2\lambda_3 a_0^2 - \lambda_1\lambda_2^2\lambda_3 a_1^2 - \lambda_1\lambda_2\lambda_3^2 a_2^2 + \lambda_2^2\lambda_3^2 a_3^2 \end{bmatrix}$$

is found. From here

$$\mathcal{A}dp^T \varepsilon \mathcal{A}dp = \left(N_p\right)^2 \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{bmatrix} = \varepsilon$$

is achieved. The matrix $\mathcal{A}dp$ is orthogonal. In addition to

$$\det \mathcal{A}dp = \left(a_0^2 + \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2\right)^3 = \left(N_p\right)^3 = 1$$

is obtained. For this reason, the linear mapping $\mathcal{A}dp$ is an isometry on

$$T_G(e) = \operatorname{Im}(\mathbb{K})$$

.

Theorem 5.5 *Let p be a 3PGUQ. For* $i \in \{1, 2, 3\}$, *if* $\lambda_i > 0$ *then*

$$\mathcal{A}dp = I + \sin\theta S + (1 - \cos\theta) S^2.$$

Proof For $i \in \{1, 2, 3\}$, $\lambda_i > 0$ and any $V_p \in \operatorname{Im}(\mathbb{K})$, $V_p \neq 0$,

$$f\left(V_p, V_p\right) = \lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2 > 0$$

is found. Therefore the function f is positive definite. If

$$p = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \text{ and } N_p = 1$$

then

$$
\begin{aligned}
p &= a_0 + \sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}\,\frac{a_1e_1 + a_2e_2 + a_3e_3}{\sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}}\\
p &= \cos\frac{\theta}{2} + \hat{p}\sin\frac{\theta}{2}
\end{aligned}
$$

where

$$
\begin{aligned}
\cos\frac{\theta}{2} &= a_0, \sin\frac{\theta}{2} = \sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2},\\
\hat{p} &= \frac{a_1e_1 + a_2e_2 + a_3e_3}{\sqrt{\lambda_1\lambda_2 a_1^2 + \lambda_1\lambda_3 a_2^2 + \lambda_2\lambda_3 a_3^2}} \in S^2_{\mathbb{K}}.
\end{aligned}
$$

Firstly, we need to find the skew-symmetric matrix of the vector $\hat{p}$. If

$$
\varepsilon = \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{bmatrix}
$$

then a matrix S that satisfies the proposition $\varepsilon S = S^T\left(-\varepsilon\right)$ must exist;

$$
\begin{aligned}
\varepsilon S &= \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{bmatrix} \begin{bmatrix} 0 & -\lambda_3 s_3 & \lambda_3 s_2 \\ \lambda_2 s_3 & 0 & -\lambda_2 s_1 \\ -\lambda_1 s_2 & \lambda_1 s_1 & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & -\lambda_1\lambda_2\lambda_3 s_3 & \lambda_1\lambda_2\lambda_3 s_2 \\ \lambda_1\lambda_2\lambda_3 s_3 & 0 & -\lambda_1\lambda_2\lambda_3 s_1 \\ -\lambda_1\lambda_2\lambda_3 s_2 & \lambda_1\lambda_2\lambda_3 s_1 & 0 \end{bmatrix},\\
S^T\left(-\varepsilon\right) &= \begin{bmatrix} 0 & \lambda_2 s_3 & -\lambda_1 s_2 \\ -\lambda_3 s_3 & 0 & \lambda_1 s_1 \\ \lambda_3 s_2 & -\lambda_2 s_1 & 0 \end{bmatrix} \begin{bmatrix} -\lambda_1\lambda_2 & 0 & 0 \\ 0 & -\lambda_1\lambda_3 & 0 \\ 0 & 0 & -\lambda_2\lambda_3 \end{bmatrix}\\
&= \begin{bmatrix} 0 & -\lambda_1\lambda_2\lambda_3 s_3 & \lambda_1\lambda_2\lambda_3 s_2 \\ \lambda_1\lambda_2\lambda_3 s_3 & 0 & -\lambda_1\lambda_2\lambda_3 s_1 \\ -\lambda_1\lambda_2\lambda_3 s_2 & \lambda_1\lambda_2\lambda_3 s_1 & 0 \end{bmatrix}.
\end{aligned}
$$

The matrix S is attained as follows:

$$
S = \begin{bmatrix} 0 & -\lambda_3 s_3 & \lambda_3 s_2 \\ \lambda_2 s_3 & 0 & -\lambda_2 s_1 \\ -\lambda_1 s_2 & \lambda_1 s_1 & 0 \end{bmatrix} \leftrightarrow \hat{p} = \left(p_1, p_2, p_3\right).
$$

Let two skew-symmetric matrices be

$$S=\begin{bmatrix} 0 & -\lambda_3 s_3 & \lambda_3 s_2 \\ \lambda_2 s_3 & 0 & -\lambda_2 s_1 \\ -\lambda_1 s_2 & \lambda_1 s_1 & 0 \end{bmatrix} \quad \text{and} \quad T=\begin{bmatrix} 0 & -\lambda_3 t_3 & \lambda_3 t_2 \\ \lambda_2 t_3 & 0 & -\lambda_2 t_1 \\ -\lambda_1 t_2 & \lambda_1 t_1 & 0 \end{bmatrix}.$$

$(T \leftrightarrow \hat{q}=(q_1,q_2,q_3)$) Then

$$ST-TS=\begin{bmatrix} 0 & \lambda_1\lambda_3\left(s_2t_1-s_1t_2\right) & \lambda_2\lambda_3\left(s_3t_1-s_1t_3\right) \\ \lambda_1\lambda_2\left(s_1t_2-s_2t_1\right) & 0 & \lambda_2\lambda_3\left(s_3t_2-s_2t_3\right) \\ \lambda_1\lambda_2\left(s_1t_3-s_3t_1\right) & \lambda_1\lambda_3\left(s_2t_3-s_3t_2\right) & 0 \end{bmatrix}$$

is found. Hence

$$ST-TS \leftrightarrow \left(\lambda_3\left(s_2t_3-s_3t_2\right), \lambda_2\left(s_3t_1-s_1t_3\right), \lambda_1\left(s_1t_2-s_2t_1\right)\right)=\hat{p}\wedge\hat{q}$$

is obtained. If

$$\cos\frac{\theta}{2}=a_0, \quad s_1\sin\frac{\theta}{2}=a_1, \quad s_2\sin\frac{\theta}{2}=a_2 \quad \text{and} \quad s_3\sin\frac{\theta}{2}=a_3$$

then the matrix $\mathcal{A}dp$ is attained as follows:

$$\begin{bmatrix} \cos^2\frac{\theta}{2}+\left(\lambda_1\lambda_2s_1^2-\lambda_1\lambda_3s_2^2-\lambda_2\lambda_3s_3^2\right)\sin^2\frac{\theta}{2} & 2\lambda_1\lambda_3s_1s_2\sin^2\frac{\theta}{2}-2\lambda_3s_3\cos\frac{\theta}{2}\sin\frac{\theta}{2} & 2\lambda_2\lambda_3s_1s_3\sin^2\frac{\theta}{2}+2\lambda_3s_2\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ 2\lambda_1\lambda_2s_1s_2\sin^2\frac{\theta}{2}+2\lambda_2s_3\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \cos^2\frac{\theta}{2}+\left(-\lambda_1\lambda_2s_1^2+\lambda_1\lambda_3s_2^2-\lambda_2\lambda_3s_3^2\right)\sin^2\frac{\theta}{2} & 2\lambda_2\lambda_3s_2s_3\sin^2\frac{\theta}{2}-2\lambda_2s_1\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ 2\lambda_1\lambda_2s_1s_3\sin^2\frac{\theta}{2}-2\lambda_1s_2\cos\frac{\theta}{2}\sin\frac{\theta}{2} & 2\lambda_1\lambda_3s_2s_3\sin^2\frac{\theta}{2}+2\lambda_1s_1\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \cos^2\frac{\theta}{2}+\left(-\lambda_1\lambda_2s_1^2-\lambda_1\lambda_3s_2^2+\lambda_2\lambda_3s_3^2\right)\sin^2\frac{\theta}{2} \end{bmatrix}.$$

Let us edit the above expression, $\mathcal{A}dp$ as follows

$$I+\begin{bmatrix} \left(\lambda_1\lambda_2s_1^2-\lambda_1\lambda_3s_2^2-\lambda_2\lambda_3s_3^2-1\right)\sin^2\frac{\theta}{2} & 2\lambda_1\lambda_3s_1s_2\sin^2\frac{\theta}{2}-\lambda_3s_3\sin\theta & 2\lambda_2\lambda_3s_1s_3\sin^2\frac{\theta}{2}+\lambda_3s_2\sin\theta \\ 2\lambda_1\lambda_2s_1s_2\sin^2\frac{\theta}{2}+\lambda_2s_3\sin\theta & \left(-\lambda_1\lambda_2s_1^2+\lambda_1\lambda_3s_2^2-\lambda_2\lambda_3s_3^2-1\right)\sin^2\frac{\theta}{2} & 2\lambda_2\lambda_3s_2s_3\sin^2\frac{\theta}{2}-\lambda_2s_1\sin\theta \\ 2\lambda_1\lambda_2s_1s_3\sin^2\frac{\theta}{2}-\lambda_1s_2\sin\theta & 2\lambda_1\lambda_3s_2s_3\sin^2\frac{\theta}{2}+\lambda_1s_1\sin\theta & \left(-\lambda_1\lambda_2s_1^2-\lambda_1\lambda_3s_2^2+\lambda_2\lambda_3s_3^2-1\right)\sin^2\frac{\theta}{2} \end{bmatrix}.$$

Here if we use the equation

$$2\sin^2\frac{\theta}{2}=1-\cos\theta \text{ ve } \lambda_1\lambda_2s_1^2+\lambda_1\lambda_3s_2^2+\lambda_2\lambda_3s_3^2=1$$

then

$$\mathcal{A}dp = I + \sin\theta \begin{bmatrix} 0 & -\lambda_3 s_3 & \lambda_3 s_2 \\ \lambda_2 s_3 & 0 & -\lambda_2 s_1 \\ -\lambda_1 s_2 & \lambda_1 s_1 & 0 \end{bmatrix} + (1-\cos\theta) \\ \cdot \begin{bmatrix} -\lambda_1\lambda_3 s_2^2 - \lambda_2\lambda_3 s_3^2 & \lambda_1\lambda_3 s_1 s_2 & \lambda_2\lambda_3 s_1 s_3 \\ \lambda_1\lambda_2 s_1 s_2 & -\lambda_1\lambda_2 s_1^2 - \lambda_2\lambda_3 s_3^2 & \lambda_2\lambda_3 s_2 s_3 \\ \lambda_1\lambda_2 s_1 s_3 & \lambda_1\lambda_3 s_2 s_3 & -\lambda_1\lambda_2 s_1^2 - \lambda_1\lambda_3 s_2^2 \end{bmatrix}$$

is obtained which means that

$$\mathcal{A}dp = I + \sin\theta S + (1-\cos\theta)\, S^2.$$

For $i \in \{1, 2, 3\}$, let $\lambda_i = 1$. Then $\mathcal{A}dp$ is the matrix that rotates about an axis on $\mathbb{R}^3$ by the angle θ.

5.2.2 Lie Multiplication

For the point e of the Lie group $S_{\mathbb{K}} = \left\{p \in \mathbb{K} : N_p = 1\right\}$, let us show that the set of left invariant vector fields

$$\mathcal{X}_l\,(S_{\mathbb{K}}) = \left\{X \in \mathcal{X}\,(S_{\mathbb{K}}) \mid \left(l_p\right)_*(X) = X\right\}$$

$\mathcal{X}_l\,(S_{\mathbb{K}})$ is isomorphic to the tangent space at the point e. In that case $\mathcal{X}_l\,(S_{\mathbb{K}}) \cong T_{S_{\mathbb{K}}}(e)$. The following multiplication is a Lie multiplication:

$$\begin{aligned} [,] : T_{S_{\mathbb{K}}}(e) \times T_{S_{\mathbb{K}}}(e) &\to T_{S_{\mathbb{K}}}(e) \\ (X, Y) &\to [X, Y] = D_X Y - D_Y X \end{aligned}$$

where $D_X Y$ is the covariant derivative of Y with respect to X. $\left(T_{S_{\mathbb{K}}}(e), [,]\right)$ is the Lie algebra of the Lie group $S_{\mathbb{K}}$. Let us find the rule of the Lie multiplication:

At the point $s = 0$, let us take a curve passing through the point e that is $\gamma_1^{'}(0) = e_1$:

$$\begin{aligned} \gamma_1 &: I \to G \\ s &\to \gamma_1(s) \end{aligned}\ .$$

Let $p \in S_{\mathbb{K}}$. We have

$$\begin{aligned} \vartheta_1 &: I \to G \\ s &\to \vartheta_l(s) \end{aligned}$$

so that $\left(l_p\right)(\gamma_1(s)) = \vartheta_1(s)$ and $\left(l_p\right)_*\left(\gamma_1^{'}(0)\right) = \vartheta_1^{'}(0)$. If we take $p = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ in the equation

$$\left(l_p\right)_*\left(\gamma_1^{'}(0)\right) = \vartheta_1^{'}(0)$$

then we get

$$\vartheta_1^{'}(0) = pe_1 = -\lambda_1\lambda_2 a_1 + a_0 e_1 + \lambda_2 a_3 e_2 - \lambda_1 a_2 e_3 = X_1.$$

Similarly, we have

$$\vartheta_2^{'}(0) = pe_2 = -\lambda_1\lambda_3 a_2 - \lambda_3 a_3 e_1 + a_0 e_2 + \lambda_1 a_1 e_3 = X_2$$

and

$$\vartheta_3^{'}(0) = pe_3 = -\lambda_2\lambda_3 a_2 + \lambda_3 a_2 e_1 - \lambda_2 a_1 e_2 + a_0 e_3 = X_3,$$

where $\vartheta_1, \vartheta_2 : I \to G$. Thus a base of $\mathcal{X}_l(S_{\mathbb{K}})$ is $\{X_1, X_2, X_3\}$. The base vectors are written as a linear combination of base vectors of $\mathcal{X}_l\left(E^4\right)$:

$$\begin{aligned} X_1 &= -\lambda_1\lambda_2 a_1 \frac{\partial}{\partial x_0} + a_0 \frac{\partial}{\partial x_1} + \lambda_2 a_3 \frac{\partial}{\partial x_2} - \lambda_1 a_2 \frac{\partial}{\partial x_3}, \\ X_2 &= -\lambda_1\lambda_3 a_2 \frac{\partial}{\partial x_0} - \lambda_3 a_3 \frac{\partial}{\partial x_1} + a_0 \frac{\partial}{\partial x_2} + \lambda_1 a_1 \frac{\partial}{\partial x_3}, \\ X_3 &= -\lambda_2\lambda_3 a_3 \frac{\partial}{\partial x_0} + \lambda_3 a_2 \frac{\partial}{\partial x_1} - \lambda_2 a_1 \frac{\partial}{\partial x_2} + a_0 \frac{\partial}{\partial x_3}. \end{aligned}$$

Now let us define the Bracket operator on $\mathcal{X}_l(S_{\mathbb{K}})$. For this purpose, giving the multiplication rule of the bases of the set $\mathcal{X}_l(S_{\mathbb{K}})$ will be enough:

$$\begin{aligned} D_{X_1}X_2 &= (-\lambda_1\lambda_2\lambda_3 a_3, \lambda_1\lambda_3 a_2, -\lambda_1\lambda_2 a_1, \lambda_1 a_0) \\ D_{X_2}X_1 &= (\lambda_1\lambda_2\lambda_3 a_3, -\lambda_1\lambda_3 a_2, \lambda_1\lambda_2 a_1, -\lambda_1 a_0) \end{aligned}$$

is found. If we use

$$[X_1, X_2] = D_{X_1}X_2 - D_{X_2}X_1$$

then we obtain

$$[X_1, X_2] = (-2\lambda_1\lambda_2\lambda_3 a_3, 2\lambda_1\lambda_3 a_2, -2\lambda_1\lambda_2 a_1, 2\lambda_1 a_0) = 2\lambda_1 X_3.$$

In the same way

$$\begin{aligned} D_{X_2}X_3 &= (-\lambda_1\lambda_2\lambda_3 a_1, \lambda_3 a_0, \lambda_2\lambda_3 a_3, -\lambda_1\lambda_3 a_2), \\ D_{X_3}X_2 &= (\lambda_1\lambda_2\lambda_3 a_1, -\lambda_3 a_0, -\lambda_2\lambda_3 a_3, \lambda_1\lambda_3 a_2) \end{aligned}$$

are obtained. Also from these equations

$$[X_2, X_3] = 2\lambda_3 X_1$$

is found. Finally, from the equations

$$\begin{aligned} D_{X_3}X_1 &= (-\lambda_1\lambda_2\lambda_3 a_2, -\lambda_2\lambda_3 a_3, \lambda_2 a_0, \lambda_1\lambda_2 a_1) \\ D_{X_1}X_3 &= (\lambda_1\lambda_2\lambda_3 a_2, \lambda_2\lambda_3 a_3, -\lambda_2 a_0, -\lambda_1\lambda_2 a_1) \end{aligned}$$

we get

$$[X_3, X_1] = 2\lambda_2 X_2.$$

Since $\mathcal{X}_l(S_{\mathbb{K}}) \cong T_{S_{\mathbb{K}}}(e)$, we can give the Bracket multiplication rule on $T_{S_{\mathbb{K}}}(e)$ here provided that

$$e_i = \left(\frac{\partial}{\partial x_i}\right)\bigg|_e, \quad i = 1, 2, 3,$$

respectively

$$[X_1, X_2]\mid_e = 2\lambda_1 (X_3)\mid_e$$

and

$$[(X_1)\mid_e, (X_2)\mid_e] = 2\lambda_1 e_3$$

is found. Therefore

$$[e_1, e_2] = 2\lambda_1 e_3$$

is obtained. In a similar way the following equations are obtained:

$$[e_2, e_3] = 2\lambda_3 e_1 \quad \text{and} \quad [e_3, e_1] = 2\lambda_2 e_2.$$

5.2.3 Matrix Representation for the Lie Algebra of $S_{\mathbb{K}}$

Let $X \in T_{S_{\mathbb{K}}}(e)$ and define the mapping

$$\begin{aligned} Ad_X &: T_{S_{\mathbb{K}}}(e) \to T_{S_{\mathbb{K}}}(e) \\ Y &\to Ad_X(Y) = [X, Y]. \end{aligned}$$

According to the mapping, the matrix that corresponds to the linear mapping Ad_X is the matrix of the Lie algebra $S_{\mathbb{K}}$.

Theorem 5.6 *Let* $X = x_1e_1 + x_2e_2 + x_3e_3$. *Then*

$$Ad_X = \begin{bmatrix} 0 & -2\lambda_3 x_3 & 2\lambda_3 x_2 \\ 2\lambda_2 x_3 & 0 & -2\lambda_2 x_1 \\ -2\lambda_1 x_2 & 2\lambda_1 x_1 & 0 \end{bmatrix}.$$

Proof Let $X = x_1e_1 + x_2e_2 + x_3e_3$. Let us find the matrix corresponding to the linear mapping. If we write

$$Ad_X(e_1) = [X, e_1] = [x_1e_1 + x_2e_2 + x_3e_3, e_1]$$

then since

$$[e_1, e_2] = 2\lambda_1 e_3, [e_2, e_3] = 2\lambda_3 e_1 \text{ ve } [e_3, e_1] = 2\lambda_2 e_2$$

and

$$[e_1, e_1] = [e_2, e_2] = [e_3, e_3] = 0$$

and also the Lie multiplication is linear, we obtain

$$[X, e_1] = 2\lambda_2 x_3 e_2 - 2\lambda_1 x_2 e_3.$$

Similarly,

$$\begin{aligned} Ad_X(e_2) &= [X, e_2] = [x_1e_1 + x_2e_2 + x_3e_3, e_2] = -2\lambda_3 x_3 e_1 + 2\lambda_1 x_1 e_3, \\ Ad_X(e_3) &= [X, e_3] = [x_1e_1 + x_2e_2 + x_3e_3, e_3] = -2\lambda_3 x_2 e_1 - 2\lambda_2 x_1 e_2. \end{aligned}$$

Therefore the matrix that corresponds to the linear map Ad_X is the matrix of the Lie algebra $S_{\mathbb{K}}$. And this matrix is

$$Ad_X = \begin{bmatrix} 0 & -2\lambda_3 x_3 & 2\lambda_3 x_2 \\ 2\lambda_2 x_3 & 0 & -2\lambda_2 x_1 \\ -2\lambda_1 x_2 & 2\lambda_1 x_1 & 0 \end{bmatrix}.$$

5.2.4 Killing Bi-linear Form

It is well known that the Killing form is a specific bi-linear form on a finite-dimensional Lie algebra, defined by W. Killing. The following mapping is called the Killing bi-linear form of the Lie group $S_{\mathbb{K}}$:

$$\begin{gathered} \mathcal{K} : T_{S_{\mathbb{K}}}(e) \times T_{S_{\mathbb{K}}}(e) \to T_{S_{\mathbb{K}}}(e) \\ (X, Y) \to \mathcal{K}(X, Y) = \operatorname{tr}(Ad_X Ad_Y). \end{gathered}$$

The mapping $\mathcal{K}$ has the following properties:

i. $\mathcal{K}$ is bi-linear,
ii. $\mathcal{K}(X, Y) = \mathcal{K}(Y, X)$,
iii. $\mathcal{K}(X, Y) = \mathcal{K}(Ad_X, Ad_Y)$.

Theorem 5.7 *If*

$$\begin{aligned} &f : \operatorname{Im}(\mathbb{K}) \times \operatorname{Im}(\mathbb{K}) \to \mathbb{R} \\ &(X, Y) \to f(X, Y) = \lambda_1\lambda_2 x_1 y_1 + \lambda_1\lambda_3 x_2 y_2 + \lambda_2\lambda_3 x_3 y_3 \end{aligned},$$

then

$$\mathcal{K}(X, Y) = -8f(X, Y).$$

Proof Let $X = x_1e_1 + x_2e_2 + x_3e_3$ and $Y = y_1e_1 + y_2e_2 + y_3e_3$. Since

$$Ad_X = \begin{bmatrix} 0 & -2\lambda_3 x_3 & 2\lambda_3 x_2 \\ 2\lambda_2 x_3 & 0 & -2\lambda_2 x_1 \\ -2\lambda_1 x_2 & 2\lambda_1 x_1 & 0 \end{bmatrix}$$

and

$$Ad_Y = \begin{bmatrix} 0 & -2\lambda_3 y_3 & 2\lambda_3 y_2 \\ 2\lambda_2 y_3 & 0 & -2\lambda_2 y_1 \\ -2\lambda_1 y_2 & 2\lambda_1 y_1 & 0 \end{bmatrix}$$

we obtain $Ad_X Ad_Y$ as

$$\begin{bmatrix} -4\lambda_1\lambda_3 x_2 y_2 - 4\lambda_2\lambda_3 x_3 y_3 & 4\lambda_1\lambda_3 x_2 y_1 & 4\lambda_2\lambda_3 x_3 y_1 \\ 4\lambda_1\lambda_2 x_1 y_2 & -4\lambda_1\lambda_2 x_1 y_1 - 4\lambda_2\lambda_3 x_3 y_3 & 4\lambda_2\lambda_3 x_3 y_2 \\ 4\lambda_1\lambda_2 x_1 y_3 & 4\lambda_1\lambda_3 x_2 y_3 & -4\lambda_1\lambda_2 x_1 y_1 - 4\lambda_1\lambda_3 x_2 y_2 \end{bmatrix}.$$

The sum of the diagonal elements of the matrix $Ad_X Ad_Y$ is

$$tr\left(Ad_X Ad_Y\right) = -8\left(\lambda_1\lambda_2 x_1 y_1 + \lambda_1\lambda_3 x_2 y_2 + \lambda_2\lambda_3 x_3 y_3\right).$$

Thus

$$\mathrm{tr}\left(Ad_X Ad_Y\right) = -8f(X, Y)$$

is obtained.

Theorem 5.8 *Let* $i \in \{1, 2, 3\}$ $\lambda_i > 0$. *Then* $S_{\mathbb{K}} = \left\{p \in \mathbb{K} : N_p = 1\right\}$ *is compact.*

Proof If $\mathcal{K}(X, X) < 0$, then the Lie group is compact. Since $i \in \{1, 2, 3\}$, $\lambda_i > 0$, we obtain $f(X, X) > 0$, $\mathcal{K}(X, X) < 0$. This gives us the desired result.

Theorem 5.9 *Let K be the matrix that corresponds to the Killing bi-linear form of the Lie group $S_{\mathbb{K}}$ and*

$$\varepsilon = \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{bmatrix}.$$

Then $K = -8\varepsilon$.

Proof The following mapping corresponds to the Killing bi-linear form of the Lie group $S_{\mathbb{K}}$:

$$\begin{gathered}\mathcal{K} : T_{S_{\mathbb{K}}}(e) \times T_{S_{\mathbb{K}}}(e) \to T_{S_{\mathbb{K}}}(e) \\ (X, Y) \to \mathcal{K}(X, Y) = -8f(X, Y)\end{gathered}$$

and since $T_{S_{\mathbb{K}}}(e) \cong Sp\{e_1, e_2, e_3\}$

$$K = \begin{bmatrix} \mathcal{K}(e_1, e_1) & \mathcal{K}(e_1, e_2) & \mathcal{K}(e_1, e_3) \\ \mathcal{K}(e_2, e_1) & \mathcal{K}(e_2, e_2) & \mathcal{K}(e_2, e_3) \\ \mathcal{K}(e_3, e_1) & \mathcal{K}(e_3, e_2) & \mathcal{K}(e_3, e_3) \end{bmatrix}$$

is achieved. Therefore

$$K = \begin{bmatrix} -8\lambda_1\lambda_2 & 0 & 0 \\ 0 & -8\lambda_1\lambda_3 & 0 \\ 0 & 0 & -8\lambda_2\lambda_3 \end{bmatrix} = -8\varepsilon$$

is obtained.

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