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The crank of partitions mod 13

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Abstract Let $M(r,m,n)$ be the number of unrestricted partitions of n with crank congruent to r modulo m. Here the relations between the numbers $M(r,13,13n+k)$ when $0 \leq r < 13$ are given. All of the results are proved by elementary methods.

Keywords Partition · Rank · Crank

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1 Introduction

Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan [9] discovered and later proved that

$$p(5n+4) \equiv 0 \pmod 5, \tag{1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{3}$$

For combinatorial interpretations of (1) and (2), Dyson defined the "rank" of a partition as the largest part minus the number of parts [4]. Let $N(r,m,n)$ be the number of partitions of n with rank congruent to r modulo m. Dyson conjectured that

$$N(0,5,5n+4) = \cdots = N(4,5,5n+4) = \frac{1}{5}p(5n+4) \tag{4}$$

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and

$$N(0,7,7n+5)=\cdots=N(6,7,7n+5)=\frac{1}{7}p(7n+5). \tag{5}$$

Dyson also conjectured a number of equalities between the $N(r,m,n)$ when $m=5$ or 7. Later, Atkin and Swinnerton-Dyer [3] proved all Dyson's conjectures. The statement corresponding to (4) and (5) for the partitions of $11n+6$ is false. For combinatorial interpretations of (3), Garvan found "crank" [6]. First, he defined certain "vector" partitions, $\vec{\pi}=(\pi_1,\pi_2,\pi_3)$, where π_1 is a partition into distinct parts, and π_2, π_3 are unrestricted partitions. Let $\#(\pi_i)$ be the number of parts of π_i, and $\sigma(\pi_j)$ be the sum of the parts of π_j. He defined for $\vec{\pi}$ the sum of parts, s, a weight ω, and a crank r by

$$s(\vec{\pi})=\sigma(\pi_1)+\sigma(\pi_2)+\sigma(\pi_3), \tag{6}$$

$$\omega(\vec{\pi})=(-1)^{\#(\pi_1)}, \tag{7}$$

$$r(\vec{\pi})=\#(\pi_2)-\#(\pi_3). \tag{8}$$

He then defined

$$N_V(m,n)=\sum_{\substack{s(\vec{\pi})=m,\\ r(\vec{\pi})=n}}\omega(\vec{\pi}) \tag{9}$$

and noted that

$$\sum_{n\geq 0}\sum_{m}N_V(m,n)z^mq^n=\frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{10}$$

Then he showed that the numbers $N_V(k,t,n)=\sum_m N_V(mt+k,n)$ satisfy the Ramanujan congruences (1), (2), and (3) with the following results:

$$N_V(k,5,5n+4)=\frac{1}{5}p(5n+4)\quad (0\leq k\leq 4),$$

$$N_V(k,7,7n+5)=\frac{1}{7}p(7n+5)\quad (0\leq k\leq 6),$$

$$N_V(k,11,11n+6)=\frac{1}{11}p(11n+6)\quad (0\leq k\leq 10).$$

Finally, Andrews and Garvan [1] constructed the genuine crank. For a partition π, let $\lambda(\pi)$ denote the largest part, $\nu(\pi)$ denote the number of ones in π, and $\mu(\pi)$ denote the number of parts of π larger than $\nu(\pi)$. Then

$$\operatorname{crank}(\pi)=\begin{cases}\lambda(\pi) & \text{if } \nu(\pi)=0,\\ \mu(\pi)-\nu(\pi) & \text{if } \nu(\pi)>0.\end{cases} \tag{11}$$

Let $M(m,n)$ be the number of partitions of n with crank m. Andrews and Garvan showed that

$$M(m,n) = N_V(m,n) \quad \text{for all } n > 1.$$

We change the definition of $M(m,n)$ a little by setting $M(0,1) = -1$, $M(-1,1) = M(1,1) = 1$, and $M(0,0) = 1$.

Atkin and Hussain [2] gave the equalities between $N(r,m,n)$ when $m = 11$, and O'Brien gave the equalities between $N(r,m,n)$ when $m = 13$ in his doctoral thesis [8], but O'Brien has never published his results.

Let $M(r,m,n)$ be the number of partitions of n with crank congruent to r modulo m. All the equalities between $M(r,m,n)$ when $m = 5$, 7, or 11 were given by Ekin [5].

In this paper the equalities between $M(r,m,n)$ when $m = 13$ are given. All of the results are proved by elementary methods. For the proof, we use the method and variables of O'Brien.

2 Preparation

Since O'Brien used the method of Atkin and Swinnerton-Dyer, we adopt their notation. Throughout this paper, m will take the value 13, and the variables q and y are always related by

$$y = q^m.$$

We write

$$(z;q)_\infty := \prod_{r=1}^{\infty} \left(1 - zq^{r-1}\right)$$

and

$$P(z,q) := (z;q)_\infty \left(z^{-1}q;q\right)_\infty. \tag{12}$$

$P(z,q)$ satisfies

$$P\left(z^{-1}q,q\right) = P(z,q), \qquad P(zq,q) = -z^{-1}P(z,q). \tag{13}$$

We also write

$$P(a) := P\left(y^a, y^m\right) = \prod_{r=1}^{\infty} \left(1 - y^{m(r-1)+a}\right)\left(1 - y^{mr-a}\right),$$

$$P(0) := \prod_{r=1}^{\infty} \left(1 - y^{mr}\right),$$

where a is not a multiple of m. By Eqs. (13) we have

$$P(m-a) = P(a), \qquad P(-a) = P(m+a) = -y^{-a}P(a).$$

We define

$$S(a) := \sum_n{}' (-1)^n \frac{q^{n(n+1)/2+an}}{1-q^{mn}}. \tag{14}$$

Writing $-n$ for n in (14), we obtain

$$S(a) = -S(m-1-a). \tag{15}$$

This equation gives

$$S\left(\frac{m-1}{2}\right) = 0. \tag{16}$$

Ekin gave

$$\begin{aligned} S\left(\frac{m-1}{2}\right) &= (-1)^b q^{b(m-b)/2} T\left(q^{mb}, 1, q^{m^2}\right) \\ &\quad + T^*\left(q^{-mb}, q^{m^2}\right) + q^{mb} T\left(q^{2mb}, q^{mb}, q^{m^2}\right) \\ &\quad + \sum_{\substack{a=1 \\ a \not\equiv \pm b \pmod m}}^{(m-1)/2} q^{(a-b)(a+b-m)/2} \\ &\quad \times \left\{q^{ma} T\left(q^{m(b+a)}, q^{ma}, q^{m^2}\right) + T\left(q^{m(b-a)}, q^{-ma}, q^{m^2}\right)\right\}, \end{aligned}$$

where

$$T(z, \zeta, q) = \sum_n (-1)^n \frac{\zeta^n q^{n(n+1)/2}}{1 - zq^n}$$

and

$$T^*(\zeta, q) = \sum_n{}' (-1)^n \frac{\zeta^n q^{n(n+1)/2}}{1-q^n}.$$

The following is Eq. (2.14) in [5]:

$$\begin{aligned} &S\left(\frac{m-1}{2} - b\right) \\ &\quad = h\left(q^{mb}; q^{m^2}\right) + (-1)^b q^{b(m-b)/2} \frac{P^2(0)}{P(b)} \\ &\qquad + \sum_{\substack{a=1 \\ a \not\equiv \pm b \pmod m}}^{(m-1)/2} (-1)^{a+b} q^{(a-b)(a+b-m)/2} \frac{P(b)P(2a)P^2(0)}{P(b-a)P(b+a)P(a)}, \end{aligned} \tag{17}$$

where

$$h(z;q)=T^*\big(z^{-1},q\big)+zT\big(z^2,z,q\big).$$

We define

$$h(a):=h\big(y^a;y^m\big).$$

The following was given by Ekin:

$$h(a)-h(m+a)=1, \tag{18}$$

$$h(a)+h(m-a)=-1, \tag{19}$$

$$3h(a)-h(3a)=\frac{P^3(2a)P^2(0)}{P^3(a)P(3a)}-\frac{P^3(4a)P^2(0)}{P^3(2a)P(6a)}. \tag{20}$$

Now we write

$$\begin{aligned}
M_b &:= \sum_{n=0}^{\infty} M(b,13,n)q^n,\\
M_{bc} &:= M_b - M_c,
\end{aligned}$$

and

$$c_{ij}(k):=\sum_{n\geq 0}\big(M(i,13,13n+k)-M(j,13,13n+k)\big)y^n.$$

It is clear that

$$M_{bc}=\sum_{d=0}^{12}c_{bc}(d)q^d. \tag{21}$$

From Eqs. (2.5) in [5] we have

$$\begin{aligned}
M_0 &= \big(S(0)-S(m-1)+1\big)(q;q)_\infty^{-1},\\
M_r &= \big(S(r)-S(r-1)\big)(q;q)_\infty^{-1},
\end{aligned} \tag{22}$$

and for $m=13$, these equations give

$$\begin{aligned}
(q;q)_\infty M_{01} &= 3S(0)-S(1)+1,\\
(q;q)_\infty M_{12} &= 2S(1)-S(0)-S(2),\\
(q;q)_\infty M_{23} &= 2S(2)-S(1)-S(3),\\
(q;q)_\infty M_{34} &= 2S(3)-S(2)-S(4),\\
(q;q)_\infty M_{45} &= 2S(4)-S(3)-S(5),\\
(q;q)_\infty M_{56} &= 2S(5)-S(4).
\end{aligned} \tag{23}$$

3 Some results for cranks modulo 13

For $m = 13$, using Eq. (17), we have

$$\begin{aligned}
S(0) &= h(6) + P^2(0)\Big\{-q\frac{P(3)P(6)}{P(1)P(2)P(5)} - q^2y\frac{P(2)}{P(1)P(5)} + q^3\frac{P(5)P(6)}{P(2)P(3)P(4)} \\
&\quad - q^6\frac{P^2(6)}{P^2(3)P(4)} + q^8\frac{y}{P(6)} + q^{10}\frac{P(6)}{P(2)P(5)}\Big\}, \\
S(1) &= h(5) + P^2(0)\Big\{qy\frac{P(2)P(5)}{P(1)P(4)P(6)} - q^2\frac{P^2(5)}{P^2(4)P(1)} + q^5\frac{P(6)}{P(2)P(3)} \\
&\quad - q^7\frac{y}{P(5)} - q^9\frac{P(4)P(5)}{P(2)P(3)P(6)} + q^{12}\frac{P(5)}{P(2)P(6)}\Big\}, \\
S(2) &= h(4) + P^2(0)\Big\{-q^3\frac{P(4)}{P(1)P(3)} + q^4\frac{y}{P(4)} + q^7\frac{P^2(4)}{P^2(2)P(6)} \\
&\quad - q^{10}y\frac{P(1)P(4)}{P(2)P(3)P(6)} + q^{11}\frac{3}{P(1)P(5)} - q^{12}\frac{P(2)P(4)}{P(1)P(3)P(5)}\Big\}, \\
S(3) &= h(3) + P^2(0)\Big\{-q^2\frac{y}{P(3)} - q^4\frac{P(3)P(4)}{P(1)P(2)P(5)} + q^7y^2\frac{P(1)}{P(4)P(6)} \\
&\quad - q^8y\frac{P^2(3)}{P^2(5)P(2)} + q^9\frac{P(3)}{P(1)P(4)} + q^{10}\frac{P(3)P(5)}{P(1)P(4)P(6)}\Big\}, \\
S(4) &= h(2) + P^2(0)\Big\{-q^3y^3\frac{P(1)P(2)}{P(4)P(5)P(6)} + q^4y^2\frac{P(2)}{P(5)P(6)} - q^5\frac{P^2(2)}{P^2(1)P(3)} \\
&\quad - q^6y\frac{P(5)}{P(4)P(6)} + q^9\frac{P(2)P(6)}{P(1)P(3)P(5)} + \frac{q^{11}}{P(2)}\Big\}, \\
S(5) &= h(1) + P^2(0)\Big\{qy^2\frac{P(1)}{P(3)P(4)} - q^4y\frac{P(1)P(6)}{P(2)P(3)P(4)} - \frac{q^6}{P(1)} \\
&\quad + q^8\frac{P(4)}{P(2)P(3)} + q^{11}y^3\frac{P^2(1)}{P^2(6)P(5)} + q^{12}y^2\frac{P(1)P(3)}{P(4)P(5)P(6)}\Big\}.
\end{aligned} \tag{24}$$

Now we need some variables in [8, p. 14]. We write

$$\begin{aligned}
l &:= y^2\frac{P(3)}{P(5)P(6)}, & m &:= y\frac{P(4)}{P(2)P(5)}, & n &:= -y^2\frac{P(1)}{P(2)P(6)}, \\
l' &:= y\frac{P(2)}{P(1)P(4)}, & m' &:= \frac{P(6)}{P(1)P(3)}, & n' &:= -y\frac{P(5)}{P(3)P(4)}.
\end{aligned} \tag{25}$$

We obtain all $h(a)$ in terms of the variables l, m, n, l', m', and n'.

Lemma 1 *We have*

$$h(1)=-\frac{1}{13}+\frac{1}{13}\big(2l+5m+6n+3l'+m'-4n'\big)P^2(0),$$

$$h(2)=-\frac{2}{13}+\frac{1}{13}\big(4l-3m-n+6l'+2m'+5n'\big)P^2(0),$$

$$h(3)=-\frac{3}{13}+\frac{1}{13}\big(6l+2m+5n-4l'+3m'+n'\big)P^2(0),$$

$$h(4)=-\frac{4}{13}+\frac{1}{13}\big(-5l-6m-2n-l'+4m'-3n'\big)P^2(0),$$

$$h(5)=-\frac{5}{13}+\frac{1}{13}\big(-3l-m+4n+2l'+5m'+6n'\big)P^2(0),$$

$$h(6)=-\frac{6}{13}+\frac{1}{13}\big(-l+4m-3n+5l'+6m'+2n'\big)P^2(0).$$

Proof Putting $a=1,2,3,4,5,$ and 6 into (18)–(20), we obtain

$$\begin{aligned}3h(1)-h(3)&=B_1,\\3h(2)-h(6)&=B_2,\\3h(3)+h(4)&=B_3-1,\\3h(4)+h(1)&=B_4-1,\\3h(5)-h(2)&=B_5-1,\\3h(6)-h(5)&=B_6-1,\end{aligned}\tag{26}$$

where

$$B_i=\frac{P^3(2i)P^2(0)}{P^3(i)P(3i)}-\frac{P^3(4i)P^2(0)}{P^3(2i)P(6i)}.\tag{27}$$

The solution of the equation system (26) is

$$\begin{aligned}h(1)&=-\frac{1}{13}+\frac{1}{26}(9B_1+3B_3-B_4),\\h(2)&=-\frac{2}{13}+\frac{1}{26}(9B_2+B_5-3B_6),\\h(3)&=-\frac{3}{13}+\frac{1}{26}(B_1+9B_3-3B_4),\\h(4)&=-\frac{4}{13}+\frac{1}{26}(-3B_1-B_3+9B_4),\\h(5)&=-\frac{5}{13}+\frac{1}{26}(3B_2+9B_5+B_6),\\h(6)&=-\frac{6}{13}+\frac{1}{26}(B_2+3B_5+9B_6).\end{aligned}\tag{28}$$

Here we need the following:

$$\begin{aligned} &P^2(b)P(c+d)P(c-d)-P^2(c)P(b+d)P(b-d) \\ &\quad + y^{c-d}P^2(d)P(b+c)P(b-c)=0, \end{aligned} \tag{29}$$

where none of $b, c, d, b \pm c, b \pm d, c \pm d$ is divisible by m. This relation is Lemma 3 in [3].

For $m = 13$, by taking $(b, c, d) = (6, 5, 1), (5, 4, 1), (6, 4, 2), (3, 2, 1), (5, 3, 2), (6, 4, 3), (6, 4, 1), (5, 3, 1), (6, 3, 1), (5, 4, 2), (6, 5, 3), (4, 2, 1), (5, 2, 1), (6, 2, 1), (4, 3, 2), (6, 3, 2), (5, 4, 3)$, and $(6, 5, 4)$, Eq. (29) gives, respectively,

$$P^3(6)P(4)-P^3(5)P(6)+y^4P^3(1)P(2)=0, \tag{30}$$

$$P^3(5)P(3)-P^3(4)P(6)+y^3P^3(1)P(4)=0, \tag{31}$$

$$P^3(6)P(2)-P^3(4)P(5)+y^2P^3(2)P(3)=0, \tag{32}$$

$$P^3(3)P(1)-P^3(2)P(4)+yP^3(1)P(5)=0, \tag{33}$$

$$P^3(5)P(1)-P^3(3)P(6)+yP^3(2)P(5)=0, \tag{34}$$

$$P^3(6)P(1)-P^3(4)P(3)+yP^3(3)P(2)=0, \tag{35}$$

$$P^2(6)P(5)P(3)-P^2(4)P(6)P(5)+y^3P^2(1)P(3)P(2)=0, \tag{36}$$

$$P^2(5)P(4)P(2)-P^2(3)P(6)P(4)+y^2P^2(1)P(5)P(2)=0, \tag{37}$$

$$P^2(6)P(4)P(2)-P^2(3)P(6)P(5)+y^2P^2(1)P(4)P(3)=0, \tag{38}$$

$$P^2(5)P(6)P(2)-P^2(4)P(6)P(3)+y^2P^2(2)P(4)P(1)=0, \tag{39}$$

$$P^2(6)P(5)P(2)-P^2(5)P(4)P(3)+y^2P^2(3)P(2)P(1)=0, \tag{40}$$

$$P^2(4)P(3)P(1)-P^2(2)P(5)P(3)+yP^2(1)P(6)P(2)=0, \tag{41}$$

$$P^2(5)P(3)P(1)-P^2(2)P(6)P(4)+yP^2(1)P(6)P(3)=0, \tag{42}$$

$$P^2(6)P(3)P(1)-P^2(2)P(6)P(5)+yP^2(1)P(5)P(4)=0, \tag{43}$$

$$P^2(4)P(5)P(1)-P^2(3)P(6)P(2)+yP^2(2)P(6)P(1)=0, \tag{44}$$

$$P^2(6)P(5)P(1)-P^2(3)P(5)P(4)+yP^2(2)P(4)P(3)=0, \tag{45}$$

$$P^2(5)P(6)P(1)-P^2(4)P(5)P(2)+yP^2(3)P(4)P(1)=0, \tag{46}$$

$$P^2(6)P(4)P(1)-P^2(5)P(3)P(2)+yP^2(4)P(2)P(1)=0. \tag{47}$$

From (27) with $i = 1$ we have

$$B_1 = P^2(0)\left\{\frac{P^3(2)}{P^3(1)P(3)} - \frac{P^3(4)}{P^3(2)P(6)}\right\}. \tag{48}$$

Dividing (32) and (33) by $P^3(2)P(5)P(6)$ and $P^3(1)P(3)P(4)$, respectively, we obtain

$$\begin{aligned} \frac{P^3(4)}{P^3(2)P(6)} &= y^2\frac{P(3)}{P(5)P(6)} + \frac{P^2(6)}{P^2(2)P(5)}, \\ \frac{P^3(2)}{P^3(1)P(3)} &= y\frac{P(5)}{P(3)P(4)} + \frac{P^2(3)}{P^2(1)P(4)}. \end{aligned} \tag{49}$$

By substituting Eqs. (49) into (48) we have

$$B_1 = P^2(0)\left\{y\frac{P(5)}{P(3)P(4)} - y^2\frac{P(3)}{P(5)P(6)} + \frac{P^2(3)}{P^2(1)P(4)} - \frac{P^2(6)}{P^2(2)P(5)}\right\}. \tag{50}$$

Dividing (44) and (47) by $P^2(1)P(2)P(4)P(6)$ and $P^2(2)P(1)P(4)P(5)$, respectively, we get

$$\begin{aligned} \frac{P^2(3)}{P^2(1)P(4)} &= y\frac{P(2)}{P(1)P(4)} + \frac{P(4)P(5)}{P(1)P(2)P(6)}, \\ \frac{P^2(6)}{P^2(2)P(5)} &= -y\frac{P(4)}{P(2)P(5)} + \frac{P(3)P(5)}{P(1)P(2)P(4)}. \end{aligned} \tag{51}$$

Substituting Eqs. (51) into (50), we obtain

$$\begin{aligned} B_1 = P^2(0)\bigg\{& y\frac{P(5)}{P(3)P(4)} - y^2\frac{P(3)}{P(5)P(6)} + y\frac{P(4)}{P(2)P(5)} + y\frac{P(2)}{P(1)P(4)} \\ & - \frac{P(3)P(5)}{P(1)P(2)P(4)} + \frac{P(4)P(5)}{P(1)P(2)P(6)}\bigg\}. \end{aligned} \tag{52}$$

Dividing (38) and (40) by $P(1)P(2)P(3)P(4)P(6)$ and $P(1)P(2)P(3)P(5)P(6)$, respectively, we find

$$\begin{aligned} \frac{P(3)P(5)}{P(1)P(2)P(4)} &= \frac{P(6)}{P(1)P(3)} + y^2\frac{P(1)}{P(2)P(6)}, \\ \frac{P(4)P(5)}{P(1)P(2)P(6)} &= \frac{P(6)}{P(1)P(3)} + y^2\frac{P(3)}{P(5)P(6)}. \end{aligned} \tag{53}$$

Substituting Eqs. (53) into (52), we have

$$B_1 = P^2(0)\left\{y\frac{P(4)}{P(2)P(5)} - y^2\frac{P(1)}{P(2)P(6)} + y\frac{P(2)}{P(1)P(4)} + y\frac{P(5)}{P(3)P(4)}\right\},$$

and from (25) we obtain

$$B_1 = \big(m + n + l' - n'\big)P^2(0). \tag{54}$$

For the others, dividing (39), (41), (43), and (46) by

$$\begin{aligned} & y^{-1}P(2)P(3)P(4)P(5)P(6), \qquad && y^{-1}P(1)P(2)P(3)P(4)P(5), \\ & y^{-1}P(1)P(2)P(4)P(5)P(6), \qquad && y^{-1}P(1)P(3)P(4)P(5)P(6), \end{aligned}$$

respectively, dividing (36), (37), (42), and (45) by

$$P^2(3)P(1)P(5)P(6), \qquad y^{-1}P^2(4)P(2)P(3)P(5),$$
$$y^{-1}P^2(6)P(1)P(2)P(3), \qquad y^{-2}P^2(5)P(3)P(4)P(6),$$

respectively, and dividing (30), (31), (34), and (35) by

$$y^{-1}P^3(6)P(2)P(5), \qquad P^3(4)P(1)P(3),$$
$$y^{-2}P^3(5)P(2)P(6), \qquad P^3(3)P(1)P(4),$$

respectively, by similar steps we obtain

$$\begin{aligned} B_2 &= \left(l - m + l' + n'\right)P^2(0), \\ B_3 &= \left(l + n - l' + m'\right)P^2(0), \\ B_4 &= \left(-l - m + m' - n'\right)P^2(0), \\ B_5 &= \left(-l + n + m' + n'\right)P^2(0), \\ B_6 &= \left(m - n + l' + m'\right)P^2(0). \end{aligned} \tag{55}$$

Equations (54), (55), and (28) give the lemma. □

We define the "normalized forms" of $c_{ij}(k)$ following O'Brien:

$$\begin{aligned} C_{ij}(0) &:= P(3)c_{ij}(0)/P(6), \\ C_{ij}(1) &:= P(1)c_{ij}(1)/P(2), \\ C_{ij}(2) &:= -P(2)c_{ij}(2)/P(3), \\ C_{ij}(3) &:= P(4)c_{ij}(3)/P(6), \\ C_{ij}(4) &:= -P(4)c_{ij}(4)/P(5) \\ C_{ij}(5) &:= -yP(1)c_{ij}(5)/P(5), \\ C_{ij}(6) &:= c_{ij}(6), \\ C_{ij}(7) &:= y^{-1}P(5)c_{ij}(7)/P(1), \\ C_{ij}(8) &:= -y^{-1}P(6)c_{ij}(8)/P(1), \\ C_{ij}(9) &:= -P(6)c_{ij}(9)/P(4), \\ C_{ij}(10) &:= P(3)c_{ij}(10)/P(2), \\ C_{ij}(11) &:= P(5)c_{ij}(11)/P(3), \\ C_{ij}(12) &:= -yP(2)c_{ij}(12)/P(4), \end{aligned}$$

where i and j lie between 0 and 6, and for all remaining values of i and j, we use the relations

$$C_{ij}(b) + C_{jk}(b) = C_{ik}(b),$$
$$C_{ij}(b) = -C_{ij}(b).$$

Searching for representations for $C_{ij}(k)$ is inspired by O'Brien's work, which is similar to the method of Atkin and Swinnerton-Dyer. He obtained the infinite product $\prod_{r=1}^{\infty}(1-q^r)^{-1}$ as a polynomial in q whose coefficients are power series in y where $y=q^{13}$. For this purpose, there are two methods which can be used. The first is to use thirteenth roots of unity, and the second is the method of Kolberg in [7]. For $m=13$, Lemma 6 in [3] gives

$$(q;q)_\infty = P(0)\left\{\frac{P(4)}{P(2)} - q\frac{P(6)}{P(3)} - q^2\frac{P(2)}{P(1)} + q^5\frac{P(5)}{P(4)} + q^7 + q^9 y\frac{P(1)}{P(6)} - q^{12}\frac{P(3)}{P(5)}\right\}. \tag{56}$$

From (56), O'Brien defined the following [8, p. 1]:

$$\alpha := -q^{-5}\frac{P(2)}{P(1)}, \qquad \beta := -q^{-6}\frac{P(6)}{P(3)}, \qquad \gamma := q^{-2}\frac{P(5)}{P(4)},$$
$$\alpha' := -q^{5}\frac{P(3)}{P(5)}, \qquad \beta' := q^{-7}\frac{P(4)}{P(2)}, \qquad \gamma' := q^{15}\frac{P(1)}{P(6)}.$$

With these variables, (56) becomes

$$(q;q)_\infty = q^7 P(0)\left\{\alpha + \beta' + \gamma + \alpha' + \beta + \gamma' + 1\right\}. \tag{57}$$

We replace q by $\omega_r q$ in (57), where ω_r ($r=1$ to 13) are the thirteenth roots of unity, and from Lemma 3 in [7] we have

$$(q;q)_\infty^{-1} = \frac{P(0)}{(y;y)_\infty^{14}}(\omega_r q;\omega_r q)_\infty\left(\omega_r^2 q;\omega_r^2 q\right)_\infty\cdots\left(\omega_r^{12} q;\omega_r^{12} q\right)_\infty. \tag{58}$$

After expanding (58) and simplifying the roots of unity, we obtain $(q;q)_\infty^{-1}$ as a polynomial in q.

The next step in [8] is to discover the congruences properties of $(q;q)_\infty^{-1}$ for modulo 13. O'Brien simplified all the coefficients of q in $(q;q)_\infty^{-1}$ systematically and gave the congruences properties in Theorem 3.1 in [8, p. 8]. These congruence forms of $(q;q)_\infty^{-1}$ take us to the properties of rank and crank. O'Brien gave the steps of obtaining congruent forms of all the $R_{ij}(k)$ for the rank in [8, p. 24], which is defined similarly to the $C_{ij}(k)$.

After dividing both sides of the each equation in (23) by $(q;q)_\infty$, we determine each of M_{01} to M_{56} as a power series in q as far as q^{168}. This gives us every $c_{ij}(k)$ as a power series in y, as far as y^{12}, and it is easy to find the corresponding terminated power series for $C_{ij}(k)$.

The factor $P(0)/(y;y)_\infty^2$ in Theorem 3.1 in [8] and the factor $P^2(0)$ in (24) suggest that each $C_{ij}(k)$ will involve a factor $P^3(0)/(y;y)_\infty^2$. Theorem 4.1 in [8] suggests that each $C_{ij}(k)$ will involve a linear combination of l, m', n, l', m, n' and a further variable which appears in Theorem 1. Finally, we find each $C_{ij}(k)$ in Theorem 1 by comparing the coefficients of power series of y in the expansions of the appropriate quantities.

For example, from (23) we have

$$M_{01} = \{3S(0) - S(1) + 1\}(q;q)_\infty^{-1}. \tag{59}$$

We can obtain the right-hand side of (59) as a power series in q. The most difficult part is to determine the left-hand side of (59). We write

$$C_{01}(0) = U\{e_1 l + e_2 m + e_3 n + e_4 l' + e_5 m' + e_6 n' + e_7 Kl + e_8 Km + e_9 Kn\}, \tag{60}$$

where

$$U := \frac{P^3(0)}{(y;y)_\infty^2}, \qquad K := y\frac{P(1)P(3)P(4)}{P(2)P(5)P(6)}, \tag{61}$$

and e_i $(1 \le i \le 9)$ are integers. The variables U and K are taken from p. 29 and p. 3 in [8], respectively. If we consider the coefficients of q^0 in (59) with (60), we have a system of equations. Solving this system, we obtain

$$C_{01}(0) = U\{-5l - 2m - 2n + l' + m' + n' - 2Kl\}.$$

For the other $C_{ij}(k)$, we repeat similar steps.

We state the result, a complete set of conjectural values of the $C_{ij}(k)$ for $m = 13$ and then prove these results.

Theorem 1 *We have the following*:

$$C_{01}(0) = U\{-5l - 2m - 2n + l' + m' + n' - 2Kl\},$$

$$C_{01}(1) = U\{-l + 5m + 4n + l' - 2m' - 2n' - Kn\},$$

$$C_{01}(2) = U\{3m - 2l' + m' + n' + n'/K\},$$

$$C_{01}(3) = U\{-2l - m + 2n + m' + n'\},$$

$$C_{01}(4) = U\{-3l + 3m + 3n + l' - m' + 3Km\},$$

$$C_{01}(5) = U\{2l + m + n + n' - 2Km\},$$

$$C_{01}(6) = U\{l + m - 2n + 2n'\},$$

$$C_{01}(7) = U\{-2n + 3m' - n' - m'/K\},$$

$$C_{01}(8) = U\{-2m - n + l'\},$$

$$C_{01}(9) = U\{-3n - 4l' - m' + n' + 2l'/K\},$$

$$C_{01}(10) = U\{2l - 4m - n + m' + 2Kn\},$$

$$C_{01}(11) = U\{m + n + l' - 2m' - 2n' - n'/K\},$$

$$C_{01}(12) = U\{l + m - n + m' + n' - l'/K\},$$

$$C_{12}(0) = U\{4l + m + 3n - 2l' - n' + Kl\},$$

$$C_{12}(1) = U\{2l - 4m - n + m' + 2Kn\},$$

$$C_{12}(2) = U\{-l - m - l' + m' + n'\},$$

$$C_{12}(3) = U\{-l + m - 2n + n'\},$$

$$C_{12}(4) = U\{l - 3m - 2n + m' + n' - 2Km\},$$

$$C_{12}(5) = U\{-m - n + l' + n' + Km\},$$

$$C_{12}(6) = U\{-l + 2n + l'\},$$

$$C_{12}(7) = U\{-l + n - 3m' - n' + m'/K\},$$

$$C_{12}(8) = U\{m + n + l' - m'\},$$

$$C_{12}(9) = U\{3n + 3l' - m'\},$$

$$C_{12}(10) = U\{-l + 3m + n + l' - m' - 2Kn\},$$

$$C_{12}(11) = U\{-m - 2n - n'\},$$

$$C_{12}(12) = U\{n + m' - n' - l'/K\},$$

$$C_{23}(0) = U\{-l - m - 2n + l' - Kl\},$$

$$C_{23}(1) = U\{-2l + 2m + n + n' - 2Kn\},$$

$$C_{23}(2) = U\{-m + 2l' - 2m' - 2n' - n'/K\},$$

$$C_{23}(3) = U\{2l + m - m' - 2n' + Kl\},$$

$$C_{23}(4) = U\{2m - m' - n'\},$$

$$C_{23}(5) = U\{-l - m + 2n - 2n'\},$$

$$C_{23}(6) = U\{-m - 2n - n'\},$$

$$C_{23}(7) = U\{-n + m' + 2n'\},$$

$$C_{23}(8) = U\{m - n - l' + 3m' + n' - m'/K\},$$

$$C_{23}(9) = U\{-m - 2n - 2l' + m' + n'\},$$

$$C_{23}(10) = U\{2l - 2m - l' + m' + 2Kn\},$$

$$C_{23}(11) = U\{-l + m + 2n - l' + 2m' + 2n' + n'/K\},$$

$$C_{23}(12) = U\{-m - n - l' - m' - n' + l'/K\},$$

$$
\begin{aligned}
C_{34}(0) &= U\left\{-l+2n+l'\right\},\\
C_{34}(1) &= U\left\{2l-l'-n'+2Kn\right\},\\
C_{34}(2) &= U\left\{l+2m+3n'\right\},\\
C_{34}(3) &= U\left\{-2m+m'+n'\right\},\\
C_{34}(4) &= U\left\{-2l-3m+n+m'+Km\right\},\\
C_{34}(5) &= U\left\{2l+2m-3n-2l'+n'-Km\right\},\\
C_{34}(6) &= U\left\{2m+n-l'\right\},\\
C_{34}(7) &= U\left\{n+l'-n'\right\},\\
C_{34}(8) &= U\left\{-l-2m+n-l'-3m'-n'+m'/K\right\},\\
C_{34}(9) &= U\left\{2m+n-2n'-l'/K\right\},\\
C_{34}(10) &= U\{-2l+m-n-2Kn\},\\
C_{34}(11) &= U\left\{l-n+l'-m'+n'-n'/K\right\},\\
C_{34}(12) &= U\left\{m+n-m'+2n'+l'/K\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_{45}(0) &= U\left\{2l+m-2n-l'+n'+Kl\right\},\\
C_{45}(1) &= U\left\{-2l-m-2n+l'-2Kn\right\},\\
C_{45}(2) &= U\left\{-l-2m-l'+2m'-3n'+2n'/K\right\},\\
C_{45}(3) &= U\left\{-l+2m+n+n'-Kl\right\},\\
C_{45}(4) &= U\left\{3l+4m-n-l'-m'+n'-Km\right\},\\
C_{45}(5) &= U\left\{-3l+3n+l'+2Km\right\},\\
C_{45}(6) &= U\left\{-2m+m'+n'\right\},\\
C_{45}(7) &= U\left\{l-n-n'\right\},\\
C_{45}(8) &= U\left\{l+m-n-m'\right\},\\
C_{45}(9) &= U\left\{-2m-n+l'\right\},\\
C_{45}(10) &= U\{l-n+Kn\},\\
C_{45}(11) &= U\left\{-m+n+l'-m'-n'-n'/K\right\},\\
C_{45}(12) &= U\left\{-m+2l'+m'+n'-l'/K\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_{56}(0) &= U\left\{-2l-n'\right\},\\
C_{56}(1) &= U\left\{l+m+n+n'+Kn\right\},\\
C_{56}(2) &= U\left\{l+m-2m'+n'-2n'/K\right\},\\
C_{56}(3) &= U\left\{l-2m-2n'\right\},\\
C_{56}(4) &= U\left\{-l-2m+n+l'-n'+Km\right\},
\end{aligned}
$$

$$\begin{aligned}
C_{56}(5) &= U\{2l-2m-2n+l'-2Km\},\\
C_{56}(6) &= U\{l+m+l'-m'-n'\},\\
C_{56}(7) &= U\{n-l'+m'+n'\},\\
C_{56}(8) &= U\{n+l'+3m'+n'-m'/K\},\\
C_{56}(9) &= U\{m+n-m'+2n'+l'/K\},\\
C_{56}(10) &= U\{-l+2n+l'\},\\
C_{56}(11) &= U\{m-n-2l'+2m'-n'+2n'/K\},\\
C_{56}(12) &= U\{-l+m-n-3l'-3n'\}.
\end{aligned}$$

Proof To prove Theorem 1, we should show that all the coefficients of q^i on both sides of (23) are equal. For example, the first equation in (23) is

$$\begin{aligned}
&(q;q)_\infty\{c_{01}(0)+qc_{01}(1)+q^2c_{01}(2)+\cdots+q^{12}c_{01}(12)\}\\
&\quad=3S(0)-S(1)+1. \qquad (62)
\end{aligned}$$

After substituting $(q;q)_\infty$ in (56) and $c_{01}(k)$ ($k=0$ to 12) in Theorem 1 into (62), a difficulty arises. It is to simplify the expressions on the left-hand side of (62) systematically. We adopt the proof of O'Brien at this point.

We should write the following as given in [8, p. 6]:

$$F := \frac{(y;y)_\infty^2}{y(y^{13};y^{13})_\infty^2}. \qquad (63)$$

We need relations amongst the variables l, m, n, l', m', n', F, and K. O'Brien gave the following relations:

$$F = 1/K - K - 3, \qquad (64)$$

$$l/l' = m/m' = n/n' = K, \qquad (65)$$

$$lm/n = -l-m, \qquad mn/l = -m-n, \qquad nl/m = -n-l, \qquad (66)$$

$$l^2/m = -Fl-3l+m-n, \qquad m^2/n = -Fm-3m+n-l, \qquad (67)$$

$$n^2/l = -Fn-3n+l-m, \qquad l^2/n = Fl+2l-m+n, \qquad (68)$$

$$m^2/l = Fm+2m-n+l, \qquad n^2/m = Fn+2n-l+m, \qquad (69)$$

$$\begin{aligned}
&Kl = -Fl-3l+l', \qquad Km = -Fm-3m+m',\\
&Kn = -Fn-3n+n',
\end{aligned} \qquad (70)$$

$$\begin{aligned}
&l'/K = Fl'+3l'+l, \qquad m'/K = Fm'+3m'+m,\\
&n'/K = Fn'+3n'+n,
\end{aligned} \qquad (71)$$

$$K^2 l = F(3l - Kl) + 10l - 3l', \qquad K^2 m = F(3m - Km) + 10m - 3m',$$
$$K^2 n = F(3n - Kn) + 10n - 3n', \tag{72}$$

$$\begin{aligned} l'/K^2 &= F\bigl(3l' + l'/K\bigr) + 10l' + 3l, \\ m'/K^2 &= F\bigl(3m' + m'/K\bigr) + 10m' + 3m, \\ n'/K^2 &= F\bigl(3n' + n'/K\bigr) + 10n' + 3n. \end{aligned} \tag{73}$$

Equation (64) is (1.17) in [8, p. 6], (65) is (2.10) in [8, p. 15], and the others are (4.8)–(4.28), respectively, in [8, p. 34].

We can add more relations. For example, dividing the first equation in (68) by K and using (65), we obtain

$$nn'/l = -Fn' - 3n' + l' - m'. \tag{74}$$

Similarly, the following can be calculated:

$$ll'/m = -Fl' - 3l' + m' - n', \tag{75}$$
$$mm'/n = -Fm' - 3m' + n' - l', \tag{76}$$
$$ll'/n = Fl' + 2l' - m' + n', \tag{77}$$
$$mm'/l = Fm' + 2m' - n' + l', \tag{78}$$
$$nn'/m = Fn' + 2n' - l' + m'. \tag{79}$$

All equations we need can be found by dividing or multiplying Eqs. (65)–(73) by K as above.

The proof of Theorem 1 is similar to those of Theorem 4.1 in [8], Theorems 4 and 5 in [3], Theorem 6 in [2], and Theorems 1, 2, and 3 in [5]. Now we write

$$\begin{aligned} M' := {} & \frac{P(6)}{P(3)} C_0 + q \frac{P(2)}{P(1)} C_1 - q^2 \frac{P(3)}{P(2)} C_2 + q^3 \frac{P(6)}{P(4)} C_3 - q^4 \frac{P(5)}{P(4)} C_4 \\ & - q^5 y^{-1} \frac{P(5)}{P(1)} C_5 + q^6 C_6 + q^7 y \frac{P(1)}{P(5)} C_7 - q^8 y \frac{P(1)}{P(6)} C_8 \\ & - q^9 \frac{P(4)}{P(6)} C_9 + q^{10} \frac{P(2)}{P(3)} C_{10} + q^{11} \frac{P(3)}{P(5)} C_{11} - q^{12} y^{-1} \frac{P(4)}{P(2)} C_{12}, \end{aligned} \tag{80}$$

where for convenience, $C_{ij}(k)$ with $j = i + 1$ is written as C_k. We write

$$(q;q)_\infty M'/P(0) = \sum_{k=0}^{12} t_k q^k. \tag{81}$$

Using (57) and (80), we can find each t_k as a linear combination of C_k in which each C_k is multiplied by some multiplicative combination of the $P(a)$. O'Brien defined the "normalized" form of t_k. These normalized forms of t_k can be easily found by substituting (57) and (80) into (81). We adopt the following to (4.37) in [8]:

$$
\begin{aligned}
T_0 &:= y^{-1}t_0 = m(C_0+C_{12})/l + l(C_1+C_{11})/n + n(C_4+C_8)/m + C_6,\\
T_1 &:= y^{-2}\frac{P(5)}{P(1)}t_1 = -m'C_0/l + \big(-l-m'\big)(C_1+C_{12})/n + (-m/n+K)C_2\\
&\quad + (-l/m-1/K)(C_5+C_9) + C_7,\\
T_2 &:= -y^{-2}\frac{P(6)}{P(1)}t_2 = -m'(C_0+C_1)/n + \big(m'/l - n/l + 1\big)C_2 + \big(-l'/n-1\big)C_3\\
&\quad - C_6 + C_8 - C_{10}/K,\\
T_3 &:= -y^{-1}\frac{P(6)}{P(4)}t_3 = -l'C_1/n + \big(-n-l'\big)(C_4+C_{11})/m + (-l/m+K)C_7\\
&\quad + (-n/l-1/K)(C_2+C_3) + C_9,\\
T_4 &:= y^{-1}\frac{P(3)}{P(2)}t_4 = nC_8/m' + \big(-m'+n\big)(C_4+C_{12})/l' + \big(-n'/l'-1/K\big)C_3\\
&\quad + \big(-m'/n'+K\big)(C_2+C_5) + C_{10},\\
T_5 &:= y^{-1}\frac{P(5)}{P(3)}t_5 = -n'(C_0+C_4)/l + \big(n'/m - l/m + 1\big)C_9 + \big(-m'/l-1\big)C_5\\
&\quad - C_6 + C_{11} - C_3/K,\\
T_6 &:= -\frac{P(2)}{P(4)}t_6 = -l'(C_1+C_4)/m + \big(l'/n - m/n + 1\big)C_7 + \big(-n'/m-1\big)C_{10}\\
&\quad - C_6 + C_{12} - C_5/K, \qquad\qquad (82)\\
T_7 &:= \frac{P(3)}{P(6)}t_7 = l(C_8+C_{11})/m' + \big(-l/n' - m'/n' + 1\big)C_5 + \big(n/m'-1\big)C_2\\
&\quad - C_6 + C_0 + KC_7,\\
T_8 &:= \frac{P(1)}{P(2)}t_8 = n(C_8+C_{12})/l' + \big(-n/m' - l'/m' + 1\big)C_3 + \big(m/l'-1\big)C_7\\
&\quad - C_6 + C_1 + KC_9,\\
T_9 &:= -\frac{P(2)}{P(3)}t_9 = -n'C_4/m + \big(-m-n'\big)(C_0+C_8)/l + (-n/l+K)C_9\\
&\quad + (-m/n-1/K)(C_7+C_{10}) + C_2,\\
T_{10} &:= \frac{P(4)}{P(6)}t_{10} = lC_{11}/n' + \big(-n'-l\big)(C_1+C_8)/m' + \big(-l'/m'-1/\big)C_5\\
&\quad + \big(n'/l'+K\big)(C_9+C_{10}) + C_3,\\
T_{11} &:= -\frac{P(4)}{P(5)}t_{11} = m(C_{11}+C_{12})/n' + \big(-m/l' - n'/l' + 1\big)C_{10} + \big(l/n'-1\big)C_9\\
&\quad - C_6 + C_4 + KC_2,\\
T_{12} &:= -y\frac{P(1)}{P(5)}t_{12} = mC_{12}/l' + \big(-l'+m\big)(C_0+C_{11})/n' + \big(-m'/n'-1/K\big)C_{10}\\
&\quad + \big(-l'/m'+K\big)(C_3+C_7) + C_5.
\end{aligned}
$$

The coefficient of q^0 in $(q;q)_\infty M'_{01}/P(0)$ is $t_{01}(0)$. If we substitute the values $C_{ij}(k)$ into the first equation of (82), then we have

$$\begin{aligned}
T_{01}(0) &= m\bigl(C_{01}(0)+C_{01}(12)\bigr)/l + l\bigl(C_{01}(1)+C_{01}(11)\bigr)/n \\
&\quad + n\bigl(C_{01}(4)+C_{01}(8)\bigr)/m + C_{01}(6) \\
&= U\bigl\{m/l\bigl(-4l-m-3n+l'+2m'+2n'-2Kl-l'/K\bigr) \\
&\quad + l/n\bigl(-l+6m+5n+2l'-4m'-4n'-Kn-n'/K\bigr) \\
&\quad + n/m\bigl(-3l+m+2n+2l'-m'+3Km\bigr)+l+m-2n+2n'\bigr\} \\
&= U\bigl\{-4m-m^2/l-3mn/l+ml'/l+2mm'/l+2mn'/l-2Km \\
&\quad - ml'/lK - l^2/n+6ml/n+5l+2ll'/n-4lm'/n-4ln'/n-Kl \\
&\quad - ln'/nK-3ln/m+n+2n^2/m+2nl'/m-nm'/m+3Kn+l \\
&\quad + m-2n+2n'\bigr\} \\
&= U\bigl\{6l-3m-n-4l'+m'+n'-Kl-2Km+3Kn-l'/K \\
&\quad - m'/K-l^2/n-m^2/l+2n^2/m-3mn/l+2mm'/l+2mn'/l \\
&\quad + 6lm/n+2ll'/n-4lm'/n-3ln/m+2nl'/m\bigr\}.
\end{aligned}$$

Substituting the convenient equations between (64) and (79), we find

$$\begin{aligned}
T_{01}(0) &= UF\bigl\{m-n+l'+m'\bigr\} \\
&= \frac{P(0)}{y}\bigl\{m-n+l'+m'\bigr\}.
\end{aligned}$$

Thus we have

$$yT_{01}(0) = P(0)\bigl(m-n+l'+m'\bigr). \tag{83}$$

We find alternative expressions for the T_k from the expression on the right-hand side of (23). From (23), (24), and (28) we have

$$t_{01}(0) = \bigl\{3h(6)-h(5)-1\bigr\}/P(0).$$

Since $T_{01}(0)=y^{-1}t_{01}(0)$, this equation becomes

$$yT_{01}(0) = P(0)\bigl(m-n+l'+m'\bigr), \tag{84}$$

and this value is equal to (83). This equality verifies $t_{01}(0)$. Since the values of i and j are related by $j=i+1$, we have $78T_{ij}(k)$ to determine ($T_{01}(k)$ to $T_{56}(k)$).

For other $yT_{ij}(k)$, we need more variables in [8, p. 2]. We write

$$\begin{aligned}
a &:= y^2\frac{P^2(1)}{P(4)P(5)}, & a' &:= \frac{P^2(5)}{yP(1)P(6)}, \\
b &:= -\frac{P^2(3)}{yP(1)P(2)}, & b' &:= -y\frac{P^2(2)}{P(3)P(5)}, \\
c &:= -\frac{P^2(4)}{P(3)P(6)}, & c' &:= \frac{P^2(6)}{yP(2)P(4)}.
\end{aligned} \tag{85}$$

O'Brien gave in [8, page 35]

$$\begin{aligned} &a = l/m - K, \qquad b = m/n - K, \qquad c = n/l - K,\\ &a' := l'/m' + 1/K, \qquad b' = m'/n' + 1/K, \qquad c' = n'/l' + 1/K. \end{aligned} \tag{86}$$

The coefficient of q in $(q;q)_\infty M'/P(0)$ is $t_{01}(1)$. From the second equation in (82) we have

$$\begin{aligned} T_{01}(1) &= -m'/lC_{01}(1) + \big(-l/n - m'/n\big)(C_{01}(1) + C_{01}(12)\\ &\quad + (-m/n + K)C_{01}(2) + (-l/m - 1/K)\big(C_{01}(5) + C_{01}(9)\big) + C_{01}(7)\\ &= U\big\{-m/l\big(-l + 5m + 4n + l' - 2m' - 2n' - Kn\big)\\ &\quad + \big(-l/n - m'/n\big)\big(6m + 3n + l' - m' - n' - l'/K\big)\\ &\quad + (-m/n + K)\big(3m - 2l' + m' + n' + n'/K\big)\\ &\quad + (-l/m - 1/K)\big(2l + m - 2n - 4l' - m' + 2n' - 2Km + 2l'/K\big)\big\}\\ &= U\big\{m - 5m^2/l - 4mn/l - m' + 2mm'/l + 2mn'/l + mn/l'\\ &\quad - 6lm/n - 3l - ll'/n + lm'/n + l' + l'^2/n - 6mm'/n - 3m'\\ &\quad - l'm'/n + m'^2/n + m'/K + lm'/n - 3m^2/n + 2ml'/n - mm'/n\\ &\quad - m' - m'/K + 3Km - 2l + m + n + n'/K^2 - 2l^2/m - l + 2ln/m\\ &\quad + 4ll'/m + l' - 2ln'/m + 2Kl - 2l'^2/m - 2l' - m' + 2n' + 4l'/K\\ &\quad + m'/K - 2n'/K + 2m - 2l'/K^2\big\}\\ &= U\big\{-6l + 4m + n - 6m' + 2n' + 2Kl + 3Km + 4l'/K + m'/K\\ &\quad - 2n'/K - 2l^2/m - 5m^2/l - 3m^2/n + 4mn/l - 6lm/n + 2ln/m\\ &\quad - 2l'/K^2 + n'/K^2 + 2mm'/l - ll'/n - 7mm'/n + 4ll'/m + l'^2/n\\ &\quad + m'^2/n - 2l'^2/m + 2mn'/l + mn/l' - l'm'/n - 2ln'/m\big\}. \end{aligned}$$

Using the relations between (64) and (79), we obtain

$$T_{01}(1) = \frac{P(0)}{y}\big\{-l - 3l' - m' + 4n' - 3m'/K\big\}$$

and

$$yT_{01}(1) = P(0)\big\{-l - 3l' - m' + 4n' - 3m'/K\big\}. \tag{87}$$

On the other hand, from (23), (24), and (28) we have

$$t_{01}(1) = P(0)\left\{-3\frac{P(3)P(6)}{P(1)P(2)P(5)} - y\frac{P(2)P(5)}{P(1)P(4)P(6)}\right\}.$$

Multiplying this equation by $P(5)/yP(1)$, we obtain

$$\begin{aligned}
yT_{01}(1) &= P(0)\left\{-3\frac{P(3)P(6)}{yP^2(1)P(2)} - \frac{P^2(5)P(2)}{P^2(1)P(4)P(6)}\right\} \\
&= P(0)\{3m'b - l'a'\} \\
&= P(0)\{3m'(m/n - K) - l'(l'/m' + 1/K)\} \\
&= P(0)\{3mm'/n - 3m'K - ll'/m - l'/K\} \\
&= P(0)\{-l - 3l' - m' + 4n' - 3m'/K\},
\end{aligned}$$

and the last equation is equal to (87). Thus, we showed that the coefficients of q^0 and q on both sides of the first equation in (23) are equal. The remaining $76T_{ij}(k)$ can be verified similarly. All alternative values of $yT_{ij}(k)$ are given at the end of the paper. These verifications prove the theorem.

There are certain linear congruence relations between the $c_{ij}(k)$. We have, modulo 13,

$$\begin{aligned}
9c_{12}(0) + 6c_{23}(0) + 9c_{34}(0) + 10c_{45}(0) + c_{56}(0) &\equiv 0 \pmod{13}, \\
5c_{01}(1) + 10c_{12}(1) + 4c_{34}(1) + 12c_{45}(1) + c_{56}(1) &\equiv 0 \pmod{13}, \\
9c_{01}(2) + 3c_{12}(2) + 3c_{23}(2) + 11c_{45}(2) + c_{56}(2) &\equiv 0 \pmod{13}, \\
4c_{01}(3) + 2c_{12}(3) + 9c_{23}(3) + 5c_{34}(3) + 9c_{45}(3) + c_{56}(3) &\equiv 0 \pmod{13}, \\
2c_{01}(4) + 12c_{12}(4) + c_{23}(4) + 7c_{34}(4) + 3c_{45}(4) + c_{56}(4) &\equiv 0 \pmod{13}, \\
9c_{01}(5) + 7c_{12}(5) + 10c_{23}(5) + 4c_{34}(5) + c_{45}(5) &\equiv 0 \pmod{13}, \\
10c_{01}(6) + 11c_{12}(6) + 7c_{23}(6) + 12c_{34}(6) + c_{45}(6) + c_{56}(6) &\equiv 0 \pmod{13}, \\
8c_{01}(7) + 8c_{12}(7) + 12c_{23}(7) + c_{34}(7) + 8c_{45}(7) + c_{56}(7) &\equiv 0 \pmod{13}, \\
3c_{01}(8) + 7c_{12}(8) + 5c_{23}(8) + 6c_{34}(8) + 6c_{45}(8) + c_{56}(8) &\equiv 0 \pmod{13}, \\
7c_{01}(9) + 8c_{23}(9) + 2c_{34}(9) + 5c_{45}(9) + c_{56}(9) &\equiv 0 \pmod{13}, \\
12c_{01}(10) + c_{12}(10) + 2c_{23}(10) + 10c_{34}(10) + 7c_{45}(10) + c_{56}(10) &\equiv 0 \pmod{13}, \\
11c_{01}(11) + 6c_{12}(11) + 11c_{23}(11) + 11c_{34}(11) + 4c_{45}(11) + c_{56}(11) &\equiv 0 \pmod{13}, \\
c_{01}(12) + 4c_{12}(12) + 10c_{23}(12) + 8c_{34}(12) + c_{56}(12) &\equiv 0 \pmod{13}.
\end{aligned}$$

These congruences can be verified by using the values $C_{ij}(k)$ in Theorem 1. □

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Appendix

See Table 1.

Table 1 All alternative values of $yT_{ij}(k)$

k	i, j					
	0, 1	1, 2	2, 3	3, 4	4, 5	5, 6
0	$m-n+l'+m'$	$n+n'$	$-l-m-n-n'$	$l+m+n-l'$	$-m-n+l'+n'$	$n+m-n'$
1	$-l-3l'-m'+4n'-3m'/K$	$l+m-l'+m'-m'/K$	$l+m'+n'$	0	n'	$-2n'$
2	$l'+m'+n'-m'/K$	$-l'-m'-2n'+2m'/K$	$-m'+n'-m'/K$	$2m'$	$-m'$	0
3	$3n+3l'-3m'-3n'/K$	$n+l'-2m'-n'/K$	$2m'$	$-n'-l'/K$	$2m'-2n'-2l'/K$	$-m'+n'+l'/K$
4	0	0	$l-n+m'$	$-3l+2n-2m'$	$4l+m-n+m'+Kn$	$-3l-2m-2Kn$
5	$-m'-n'$	$2m'+3n'$	$-m'-3n'$	$l'+n'-n'/K$	$-2l'+2n'/K$	$l'-n'/K$
6	$-3m'+3l'/K$	$m'-l'/K$	0	$-l'-n'$	$-l'+2n'$	$l'-n'$
7	l	$-2l-m-Kl$	$2l+2m+n+2Kl$	$-2l-m-2n-Kl$	$n+l$	0
8	$-3n$	n	$l+Kn$	$-2l-2Kn$	$l-m-n+Kn$	$2m+2n$
9	$m-l'+n'/K$	$-2m+2l'-2n'/K$	$m'+n'/K$	$-m-l'-n'$	$2m-l'+2n'$	$-m'+l'-n'$
10	$3m$	$l-2m-n'$	$-2l+m+n-2n'-Kl$	$l+m-2n-n'+2Kl$	$-m+n-Kl$	0
11	0	$l+m$	$-2l-2m$	$l+2m$	$n-2m+Km$	$m-2n-2Km$
12	$-n$	$-l+3n-m'$	$2l-3n+2m'$	$-l+n-m'$	$-m+n+l'$	$2m-2n-2l'$

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