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11-Dissection and Modulo 11 Congruences Properties for Partition Generating Function

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Abstract

In a recent paper, we give 13-dissection and some congruences for modulo 13 for the partition generating function $\prod(1-q^r)^{-1}$ by using a method of Kolberg. In this paper, by following similar course, we develop an algoritmic approach and give 11-dissection for the partition generating function $\prod(1-q^r)^{-1}$. Then we re-obtain the congruences given by Atkin and Swinnerton-Dyer.

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1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. The number of partitions of n is denoted $p(n)$ and $p(0)$ is assumed as 1. Euler gave the following generating function for the series $\{p(n)\}_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{r=1}^{\infty} \frac{1}{1-q^r}.$$

Throughout this paper, $m > 1$ always denotes a positive integer prime to 6, and the variables y and q are always related to $y = q^m$ ($|q| < 1$). We define

$$\mathrm{F} := \sum_{n=0}^{\infty} p(n)q^n.$$

For $k = 1, 2, ..., m-1$, we define k-th component of F as follows:

$$\mathrm{F}^{(k,m)} := q^k \sum_{n=0}^{\infty} p(mn+k)y^n,$$

then we have a dissection for partition generating function, namely

$$\mathrm{F} = \sum_{n=0}^{\infty} p(n)q^n = \sum_{k=0}^{m-1} \mathrm{F}^{(k,m)}.$$

Kolberg gave 5-dissection and 7-dissection for F. A simpler form of 7-dissection for F was given by Ekin in his doctoral thesis [3]. In a recent paper [2], the authors obtained 13-dissection for F and some congruences for the components $\mathrm{F}^{(k,13)}$ where $0 \leq k \leq 12$. In this paper, by following the same course in [2], we obtain 11-dissection for F. The 11-dissection for F may appear for the first time. After obtaining the components, using a q-equivalence given in [2], we obtain the congruences for the components $\mathrm{F}^{(k,11)}$ (mod 11) given by Atkin and Swinnerton-Dyer in [1].

We prefer the following notation:

$$\begin{aligned} P(a) &= (y^a; y^m)_\infty (y^{m-a}; y^m)_\infty \\ P(0) &= (y^m, y^m)_\infty \end{aligned}$$

where

$$(z; q)_\infty = \prod_{r=1}^{\infty} (1 - zq^{r-1})$$

and a is not a multiple of m. $P(a)$ satisfies

$$P(m-a) = P(a), \quad P(-a) = P(m+a) = -y^{-a}P(a).$$

For $m = 11$, Atkin and Swinnerton-Dyer gave

Theorem 1.1 *For $m = 11$, we have*

$$
\begin{aligned}
\mathrm{F}^{(0,11)} &\equiv \frac{P(0)}{P(1)} \pmod{11},\\
\mathrm{F}^{(1,11)} &\equiv q\frac{P(0)P(5)}{P(2)P(3)} \pmod{11},\\
\mathrm{F}^{(2,11)} &\equiv 2q^2\frac{P(0)P(3)}{P(1)P(4)} \pmod{11},\\
\mathrm{F}^{(3,11)} &\equiv 3q^3\frac{P(0)P(2)}{P(1)P(3)} \pmod{11},\\
\mathrm{F}^{(4,11)} &\equiv 5q^4\frac{P(0)}{P(2)} \pmod{11},\\
\mathrm{F}^{(5,11)} &\equiv 7q^5\frac{P(0)P(4)}{P(2)P(5)} \pmod{11},\\
\mathrm{F}^{(6,11)} &\equiv 0 \pmod{11},\\
\mathrm{F}^{(7,11)} &\equiv 4q^7\frac{P(0)}{P(3)} \pmod{11},\\
\mathrm{F}^{(8,11)} &\equiv 6yq^8\frac{P(0)P(1)}{P(4)P(5)} \pmod{11},\\
\mathrm{F}^{(9,11)} &\equiv 8q^9\frac{P(0)}{P(4)} \pmod{11},\\
\mathrm{F}^{(10,11)} &\equiv 9q^{10}\frac{P(0)}{P(5)} \pmod{11}.
\end{aligned}
$$

They use the following congruence to calculate Theorem 1.1:

$$
\sum_{k=0}^{10} \mathrm{F}^{(k,11)} q^k = (q;q)_\infty^{-1} \equiv \left\{\prod(1-q^r)^3\right\}^3 \prod(1-q^r)\Big/\prod(1-y^r) \pmod{11}. \tag{1}
$$

We calculate 11-dissection for the partition generating function at first. Then, we obtain the Theorem 1.1 by using the components with an algorithmic approach.

2 Preliminaries

Kolberg defines, for $s = 0, 1, ..., m-1$

$$
g_s := \sum_{\frac{1}{2}n(3n+1)\equiv s \pmod{m}} (-1)^n q^{\frac{1}{2}n(3n+1)}
$$

and

$$h_s := \sum_{\substack{\frac{1}{2}n(n+1)\equiv s \pmod m,\\ n\geq 0}} (-1)^n(2n+1)q^{\frac{1}{2}n(n+1)}.$$

These definitions give

$$\prod_{r=1}^{\infty}(1-q^r) = \sum_{s=0}^{m-1} g_s \quad \text{and} \quad \prod_{r=1}^{\infty}(1-q^r)^3 = \sum_{s=0}^{m-1} h_s.$$

By these equations, we conclude the following relation

$$(g_0+g_1+\cdots+g_{m-1})^3 = h_0+h_1+\cdots+h_{m-1}. \tag{2}$$

We have the following lemma from [4].

Lemma 2.1

$$g_s = \begin{cases} 0, & \text{if } 24s+1 \text{ is a quad. non-residue mod } m \\ (-1)^{\left[\frac{1}{6}(m+1)\right]}q^{\frac{1}{24}(m^2-1)}P(0), & \text{if } 24s+1\equiv 0 \pmod m \end{cases} \tag{3}$$

and

$$h_s = \begin{cases} 0, & \text{if } 8s+1 \text{ is a quad. non-res. mod } m \\ (-1)^{\left[\frac{1}{2}(m-1)\right]}mq^{\frac{1}{8}(m^2-1)}P^3(0), & \text{if } 8s+1\equiv 0 \pmod m. \end{cases} \tag{4}$$

Using the following lemma which is given by the authors in [2], we can determine the g_s in terms of $P(a)$.

Lemma 2.2 *Let $24s+1$ is a quadratic residue mod m and $m=6\lambda+\mu$ where λ is a positive integer and $\mu=\pm 1$. Then we have*

$$g_s = (-1)^{c+\lambda}q^{\frac{1}{2}(3c^2-mc+3\lambda^2+\mu\lambda)}\frac{P(2c)}{P(c)} \tag{5}$$

where c is a solution of the congruence $x^2\equiv(4s-\mu\lambda)/6 \pmod m$.

Kolberg also gives

Lemma 2.3 *For $s=0,1,...,m-1$*

$$\mathrm{F}^{(s,m)} = (-1)^{(m-1)s}\frac{P(0)}{(y;y)_{\infty}^{m+1}}D_s \tag{6}$$

where D_s is the following determinant;

$$\begin{vmatrix} g_{-s} & g_{-s+1} & \cdots & g_{-s+m-2} \\ g_{-s-1} & g_{-s} & \cdots & g_{-s+m-3} \\ \cdots & \cdots & \cdots & \cdots \\ g_{-s-m+2} & g_{-s-m+3} & \cdots & g_{-s} \end{vmatrix} \tag{7}$$

we put $g_r=g_s$ when $r\equiv s \pmod m$ in (7).

We define

$$A_s := g_k^{1-m} D_s$$

where m is prime and $24k+1 \equiv 0 \pmod{m}$. So we have

$$\mathrm{F}^{(s,m)} = q^{\frac{1}{24}(m^3-m^2-m+1)} \frac{P^m(0)}{(y;y)_\infty^{m+1}} A_s. \tag{8}$$

For the denominator of (8), we have

$$(y;y)_\infty = \prod_{r=1}^{\infty}(1-y^r) = P(0)P(1)P(2)\cdots P((m-1)/2). \tag{9}$$

We use the following lemma which is given by the authors in [2] to obtain the congruence properties of components.

Lemma 2.4 *If $m \in \mathrm{Z}^+$ is a prime, then*

$$\left[\prod_{r=1}^{\infty}(1-q^r)^3\right]^m \equiv P^{m+2}(0)P^{m+3}(1)P^{m+3}(2)\cdots P^{m+3}((m-1)/2) \pmod{m}. \tag{10}$$

For the left-hand side on Eq.(10), we need the following lemma which is Lemma 3 in [1]:

Lemma 2.5 *We have*

$$\prod_{r=1}^{\infty}(1-q^r)^3 \equiv P(0) \sum_{c=0}^{(m-3)/2} (-1)^c(2c+1)q^{\frac{1}{2}c(c+1)} P\left(\frac{m-1}{2} - c\right) \pmod{m}. \tag{11}$$

3 Components and Congruences for $m = 11$

In this section we give the components $\mathrm{F}^{(k,11)}$ and find the congruences given by Atkin and Swinnerton-Dyer. For $m = 11$, from (3) and (4) we have

$$g_3 = g_6 = g_8 = g_9 = g_{10} = 0, \qquad g_5 = q^5 P(0)$$

and

$$h_2 = h_5 = h_7 = h_8 = h_9 = 0, \qquad h_4 = -11q^{15}P^3(0).$$

We set

$$\alpha := g_0 g_5^{-1}, \quad \beta := g_1 g_5^{-1}, \quad \gamma := g_2 g_5^{-1}, \quad \theta := g_4 g_5^{-1}, \quad \delta := g_7 g_5^{-1}. \tag{12}$$

From (2) we find

$$\begin{aligned} 3(2g_2g_4g_7+2g_1g_5g_7+g_0g_1^2+g_4^2g_5+g_0^2g_2) &= h_2=0,\\ 3(2g_0g_1g_4+2g_4g_5g_7+g_2g_7^2+g_1g_2^2+g_0^2g_5) &= h_5=0,\\ 3(2g_0g_2g_5+2g_1g_2g_4+g_4g_7^2+g_0^2g_7+g_1^2g_5) &= h_7=0,\\ 3(2g_0g_1g_7+2g_1g_2g_5+g_2^2g_4+g_5g_7^2+g_0g_4^2) &= h_8=0,\\ 3(2g_0g_4g_5+2g_0g_2g_7+g_2^2g_5+g_1^2g_7+g_1g_4^2) &= h_9=0,\\ 3(g_1g_7^2+g_0^2g_4+g_0g_2^2+g_4^2g_7+g_1^2g_2)+g_5^3 &= h_4=-11g_5^3. \end{aligned}$$

By the help of (12), these equations become, respectively

$$2\gamma\theta\delta+\alpha\beta^2+\alpha^2\gamma+\theta^2+2\beta\delta=0, \tag{13}$$
$$2\alpha\beta\theta+\gamma\delta^2+\beta\gamma^2+\alpha^2+2\theta\delta=0, \tag{14}$$
$$2\beta\gamma\theta+\theta\delta^2+\alpha^2\delta+\beta^2+2\alpha\gamma=0, \tag{15}$$
$$2\alpha\beta\delta+\alpha\theta^2+\gamma^2\theta+\delta^2+2\beta\gamma=0, \tag{16}$$
$$2\alpha\gamma\delta+\beta\theta^2+\beta^2\delta+\gamma^2+2\alpha\theta=0, \tag{17}$$
$$\alpha^2\theta+\beta^2\gamma+\theta^2\delta+\gamma^2\alpha+\delta^2\beta=-4. \tag{18}$$

Now we put

$$x_1:=\alpha^2\theta,\quad x_2:=\beta^2\gamma,\quad x_3:=\theta^2\delta,\quad x_4:=\gamma^2\alpha,\quad x_5:=\delta^2\beta. \tag{19}$$

Thus (18) becomes

$$x_1+x_2+x_3+x_4+x_5=-4. \tag{20}$$

After multiplying A_s $(s=0,1,..,10)$ by α, α^{-1}, β^{-1}, γ^{-1}, δ, θ^{-1}, 1, θ, δ^{-1}, γ and β respectively, A_s can be written in terms of x_i.

From (3) and (5), we get

$$g_0=\frac{P(0)P(4)}{P(2)},\ g_1=-q\frac{P(0)P(2)}{P(1)},\ g_2=-q^2\frac{P(0)P(5)}{P(3)},$$
$$g_4=-q^4y\frac{P(0)P(1)}{P(5)},\ g_5=q^5P(0),\ g_7=q^7\frac{P(0)P(3)}{P(4)}.$$

These equations give

$$\alpha\beta\gamma\theta\delta=-1$$

and

$$x_1x_2x_3x_4x_5=-1. \tag{21}$$

Lemma 3.1 *We have*

$$x_1x_2 = x_4+1, \tag{22}$$
$$x_2x_3 = x_5+1, \tag{23}$$
$$x_3x_4 = x_1+1, \tag{24}$$
$$x_4x_5 = x_2+1, \tag{25}$$
$$x_5x_1 = x_3+1. \tag{26}$$

Proof. We define $A := x_3x_4x_5$, $B := x_1x_2x_5$, $C := x_1x_4x_5$, $D := x_1x_2x_3$ and $E := x_2x_3x_4$. Multiplying equations (13), (14), (15), (16) and (17) by δ, θ, γ, β and α respectively give us

$$\begin{aligned} 2A+B+C &= x_3+2x_5, \\ 2D+A+E &= x_1+2x_3, \\ 2E+A+C &= x_2+2x_4, \\ 2B+D+E &= x_5+2x_2, \\ 2C+D+B &= x_4+2x_1. \end{aligned}$$

The solution of this equations system is

$$\begin{aligned} A = x_3x_4x_5 &= \frac{1}{4}(-x_1-x_2-x_4+3x_3+3x_5), \\ B = x_1x_2x_5 &= \frac{1}{4}(-x_1+3x_2-x_4-x_3+3x_5), \\ C = x_1x_4x_5 &= \frac{1}{4}(3x_1-x_2+3x_4-x_3-x_5), \\ D = x_1x_2x_3 &= \frac{1}{4}(3x_1-x_2-x_4+3x_3-x_5), \\ E = x_2x_3x_4 &= \frac{1}{4}(-x_1+3x_2+3x_4-x_3-x_5). \end{aligned}$$

Using equation (20), we get

$$\begin{aligned} x_3x_4x_5 &= x_3+x_5+1, && (27) \\ x_1x_2x_5 &= x_2+x_5+1, && (28) \\ x_1x_4x_5 &= x_1+x_4+1, && (29) \\ x_1x_2x_3 &= x_1+x_3+1, && (30) \\ x_2x_3x_4 &= x_2+x_4+1. && (31) \end{aligned}$$

Multiplying both sides of (27) by x_1x_2 and using (21), we find

$$-1 = x_1x_2x_3 + x_1x_2x_5 + x_1x_2. \tag{32}$$

Substituting for (28), (30) and (20) into the equation (32), we obtain

$$x_1x_2 = x_4 + 1 \tag{33}$$

and we find the others similarly.

Algorithm 1. Let U be a linear combination of $x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}x_5^{i_5}$ where each i_r is a non-negative integer.

1. Substitute for the equations (22)-(26) into U. This step turns all terms into the form x_i^a or $x_i^a x_j^b$ where (i,j)=(1,4), (2,5), (4,2), (3,1), (5,3).

2. If $b>1$ then substitute for x_j^2, if $a>1$ and b=0 substitute for x_i^2 into U. This step writes U as sums of $x_i^a x_j$ and x_i. To evaluate this step we use the following equations which can be easily found by (20):

$$\begin{aligned}
x_1^2 &= -x_1x_4 - x_3x_1 - 4x_1 - x_4 - x_3 - 2, && (34)\\
x_2^2 &= -x_4x_2 - x_2x_5 - 4x_2 - x_4 - x_5 - 2, && (35)\\
x_3^2 &= -x_3x_1 - x_5x_3 - 4x_3 - x_5 - x_1 - 2, && (36)\\
x_4^2 &= -x_1x_4 - x_4x_2 - 4x_4 - x_1 - x_2 - 2, && (37)\\
x_5^2 &= -x_5x_3 - x_2x_5 - 4x_5 - x_2 - x_3 - 2. && (38)
\end{aligned}$$

By using Algorithm 1, we get the A_s in simple forms:

$$\begin{aligned}
\alpha A_0 &= -x_4^4x_2 + 31x_1^3x_4 + 42x_3^3x_1 - 42x_4^3x_2 - 5x_5^3x_3 + 172x_1^2x_4 + 277x_3^2x_1\\
&\quad +10x_2^2x_5 - 296x_4^2x_2 - 64x_5^2x_3 + 508x_1x_4 + 724x_3x_1 - 332x_4x_2\\
&\quad +199x_2x_5 - 429x_5x_3 - 362x_1 - 1132x_2 - 1464x_3 - 116x_4\\
&\quad -1147x_5 - 3970, \qquad (39)\\
\alpha^{-1}A_1 &= 7x_2^4x_5 - 8x_4^3x_2 + 3x_1^3x_4 - 22x_5^3x_3 + 140x_2^3x_5 + 2x_3^3x_1 + 28x_4^2x_2\\
&\quad +85x_1^2x_4 - 180x_5^2x_3 + 433x_2^2x_5 + 74x_3^2x_1 - 113x_4x_2 + 234x_4x_1\\
&\quad +1004x_5x_2 - 524x_5x_3 + 176x_1x_3 + 562x_4 + 705x_5 + 578x_2 + 345x_1\\
&\quad -158x_3 + 1224,\\
A_6 &= 11[-x_1^3x_4 - x_2^3x_5 - x_4^3x_2 - x_3^3x_1 - x_5^3x_3 - 14x_1^2x_4 - 14x_2^2x_5\\
&\quad -14x_4^2x_2 - 14x_3^2x_1 - 14x_5^2x_3 - 29x_1x_4 - 29x_2x_5 - 29x_2x_4\\
&\quad -29x_1x_3 - 29x_3x_5 + 106].
\end{aligned}$$

The last equation proves the famous congruence $p(11n+6)\equiv 0 \pmod{11}$ of Ramanujan.

If we observe the indices of the terms in A_s, having αA_0 and $\alpha^{-1}A_1$ is enough to obtain other components via the permutation (12345). This permutation gives the following relations

$$\alpha A_0 \to \beta A_{10} \to \theta A_7 \to \gamma A_9 \to \delta A_4 \to \alpha A_0 \qquad (40)$$

$$\alpha^{-1}A_1 \to \beta^{-1}A_2 \to \theta^{-1}A_5 \to \gamma^{-1}A_3 \to \delta^{-1}A_8 \to \alpha^{-1}A_1 \qquad (41)$$

$$A_6 \to A_6. \qquad (42)$$

Using these relations, that is, changing indices in convenient order and making the same operations, we get the other components. For $m=11$, with the help of Eq.(10) and Lemma 2.5, we have

$$\begin{aligned}
&P^{11}(5) + 8yP^{11}(4) + 5y^3P^{11}(3) + 4y^6P^{11}(2) + 9y^{10}P^{11}(1) \equiv\\
&\qquad P^2(0)P^{14}(1)P^{14}(2)P^{14}(3)P^{14}(4)P^{14}(5) \pmod{11}. \qquad (43)
\end{aligned}$$

For abbreviation, we define

$$(a_1, a_2, \cdots, a_{(m+1)/2}) := y^{a_1} P^{a_2}(1) P^{a_3}(2) P^{a_{(m+1)/2}}((m-1)/2).$$

We write (43) in terms of x_i in 20 different ways by dividing (43) by (a, b, c, d, e, f) where $1 \le b, c, d, e, f \le 5$ and $b + c + d + e + f = 11$. Five of them are useful for us and the remaining 15 of them are linearly independent on these five. These can be found dividing (43) by $(3, 1, 2, 3, 2, 3)$, $(4, 2, 2, 3, 3, 1)$, $(4, 2, 3, 1, 3, 2)$, $(4, 3, 1, 2, 2, 3)$ and $(5, 3, 3, 2, 1, 2)$:

$$\begin{aligned}
7x_2^6x_3^5x_1x_4 + 2x_3^6x_4^4x_2 + 3x_1^5x_2^4x_3 + 10x_4^5x_5^3x_3 + 6x_5^4x_1^3& \\
\equiv (y;y)_\infty^2(-3, 13, 12, 11, 12, 11) \pmod{11}&, \\
9x_3^6x_4^5x_2x_5 + x_4^6x_5^4x_3 + 7x_2^5x_3^4x_4 + 5x_5^5x_1^3x_4 + 3x_1^4x_2^3& \\
\equiv (y;y)_\infty^2(-4, 12, 12, 11, 11, 13) \pmod{11}&, \\
x_4^6x_5^5x_1x_3 + 5x_5^6x_1^4x_4 + 2x_3^5x_4^4x_5 + 3x_1^5x_2^3x_5 + 4x_2^4x_3^3& \\
\equiv (y;y)_\infty^2(-4, 12, 11, 13, 11, 12) \pmod{11}&, \\
8x_1^6x_2^5x_3x_5 + 7x_2^6x_3^4x_1 + 5x_5^5x_1^4x_2 + 10x_4^4x_5^3 + 2x_3^5x_4^3x_2& \\
\equiv (y;y)_\infty^2(-4, 11, 13, 12, 12, 11) \pmod{11}&, \\
5x_5^6x_1^5x_2x_4 + 3x_1^6x_2^4x_5 + 10x_4^5x_5^4x_1 + 4x_2^5x_3^3x_1 + 9x_3^4x_4^3& \\
\equiv (y;y)_\infty^2(-5, 11, 11, 12, 13, 12) \pmod{11}&.
\end{aligned}$$

By using Algorithm 1 to the left hand side of these congruences, we obtain

$$\begin{aligned}
&3x_1^4x_4 + 5x_1^3x_4 + 6x_1x_3^3 + 4x_2^3x_5 + 6x_3x_5^3 + 8x_1^2x_4 + x_1x_3^2 + 5x_2^2x_5 \\
&+3x_2x_4^2 + 5x_3x_5^2 + 6x_1x_4 + 5x_1x_3 + 8x_2x_4 + 6x_3x_5 + 6x_1 + 8x_2 + 7x_4 \\
&+7x_3 + 2x_5 + 4 \qquad \equiv (y;y)_\infty^2(-3, 13, 12, 11, 12, 11) \pmod{11}, \qquad (44) \\
&7x_2^4x_5 + 3x_1^3x_4 + 2x_1x_3^3 + 8x_2^3x_5 + 3x_2x_4^3 + 8x_1^2x_4 + 8x_1x_3^2 + 4x_2^2x_5 \\
&+6x_2x_4^2 + 7x_3x_5^2 + 3x_1x_4 + 8x_2x_4 + 3x_2x_5 + 4x_3x_5 + x_1 + 3x_2 + 9x_4 \\
&+4x_3 + 9x_5 + 2 \qquad \equiv (y;y)_\infty^2(-4, 12, 12, 11, 11, 13) \pmod{11}, \qquad (45) \\
&2x_1x_3^4 + 7x_1x_3^3 + 4x_2^3x_5 + 10x_2x_4^3 + 4x_3x_5^3 + 2x_1^2x_4 + 9x_1x_3^2 + 7x_2^2x_5 \\
&+7x_2x_4^2 + 8x_3x_5^2 + 9x_1x_4 + 4x_1x_3 + 4x_2x_5 + 7x_3x_5 + x_1 + 5x_2 + 9x_4 \\
&+4x_3 + x_5 + 10 \qquad \equiv (y;y)_\infty^2(-4, 12, 11, 13, 11, 12) \pmod{11}, \qquad (46) \\
&5x_3x_5^4 + 3x_1^3x_4 + 10x_2^3x_5 + 10x_2x_4^3 + x_3x_5^3 + x_1^2x_4 + 5x_1x_3^2 + 9x_2^2x_5 + x_2x_4^2 \\
&+6x_3x_5^2 + 6x_1x_3 + 10x_2x_4 + x_2x_5 + 10x_3x_5 + 6x_1 + 8x_2 + 7x_4 + 8x_3 \\
&+10x_5 + 3 \qquad \equiv (y;y)_\infty^2(-4, 11, 13, 12, 12, 11) \pmod{11}, \qquad (47) \\
&10x_2x_4^4 + 9x_1^3x_4 + 9x_1x_3^3 + 2x_2x_4^3 + 6x_3x_5^3 + 7x_1^2x_4 + 2x_1x_3^2 + 10x_2^2x_5 \\
&+x_2x_4^2 + 2x_3x_5^2 + 2x_1x_4 + 9x_1x_3 + 9x_2x_4 + x_2x_5 + 5x_1 + 5x_2 + 9x_4 \\
&+3x_3 + x_5 + 6 \qquad \equiv (y;y)_\infty^2(-5, 11, 11, 12, 13, 12) \pmod{11}. \qquad (48)
\end{aligned}$$

Now we can obtain the congruences given by Atkin and Swinnerton-Dyer. From (39), we have

$$\alpha A_0 \equiv 10x_2x_4^4 + 9x_1^3x_4 + 9x_1x_3^3 + 2x_2x_4^3 + 6x_3x_5^3 + 7x_1^2x_4 + 2x_1x_3^2 + 10x_2^2x_5$$

$$+x_2x_4^2 + 2x_3x_5^2 + 2x_1x_4 + 9x_1x_3 + 9x_2x_4 + x_2x_5 + x_1 + x_2 + 5x_4 + 10x_3$$
$$+8x_5 + 1 \pmod{11}$$

From (48), we find

$$A_0 \equiv \alpha^{-1}(y;y)_\infty^2(-5, 11, 11, 12, 13, 12) \pmod{11}. \tag{49}$$

Then the equations (8) and (49) give us

$$\mathrm{F}^{(0,11)} \equiv \frac{P(0)}{P(1)} \pmod{11}.$$

The other congruences in Theorem 1.1 can be obtained similarly.

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