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# Some congruences for modulus 13 related to partition generating function

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**Abstract** In this study, we obtain some congruences modulo 13 related to the partition generating function with the help of a congruence between $q$-series. We use an elementary method which appears in the work of Kolberg.

**Keywords** Partition · Partition generating function · $q$-Equivalence

**Mathematics Subject Classification (2010)** 11P83

## 1 Introduction

A partition of positive integer of $n$ is a non-increasing sequence of positive integers whose sum is $n$. The number of partition of $n$ is denoted $p(n)$. Euler gave a generating function for $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{r=1}^{\infty} \frac{1}{1-q^r}.$$

Ramanujan [8] discovered, and later proved, three famous congruences

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{2}$$

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$$p(11n+6) \equiv 0 \pmod{11}. \tag{3}$$

Obviously, congruences (1) and (2) follow from Ramanujan's identities:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{r=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}, \tag{4}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{r=1}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{r=1}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8}. \tag{5}$$

Zuckerman [9] gave the following analogue of Eqs. (4) and (5):

$$\begin{aligned}\sum_{n=0}^{\infty} p(13n+6)q^n = {} & 11\prod_{r=1}^{\infty} \frac{(1-q^{13n})}{(1-q^n)^2} + 468q\prod_{r=1}^{\infty} \frac{(1-q^{13n})^3}{(1-q^n)^4} \\ & + 6422q^2\prod_{r=1}^{\infty} \frac{(1-q^{13n})^5}{(1-q^n)^6} + 43940q^3\prod_{r=1}^{\infty} \frac{(1-q^{13n})^7}{(1-q^n)^8} \\ & + 171366q^4\prod_{r=1}^{\infty} \frac{(1-q^{13n})^9}{(1-q^n)^{10}} + 371293q^5\prod_{r=1}^{\infty} \frac{(1-q^{13n})^{11}}{(1-q^n)^{12}} \\ & + 371293q^6\prod_{r=1}^{\infty} \frac{(1-q^{13n})^{13}}{(1-q^n)^{14}}.\end{aligned} \tag{6}$$

Defining the rank of a partition, Dyson [3] discovered a combinatorial method of dividing the partitions of $5n+4$ and $7n+5$ into 5 and 7 equal classes, respectively. Atkin and Swinnerton-Dyer [1] developed a remarkable method for proving Dyson's conjectures. They regarded any power series in $q$ as a polynomial of degree $m-1$ in $q$, whose coefficients are power series in $y$ where $y := q^m$. We also use their method.

Throughout this paper, $m > 1$ always denotes a positive integer prime to 6, and the variables $y$ and $q$ are always related to $y = q^m$ ($|q| < 1$). We define

$$\mathrm{F} := \sum_{n=0}^{\infty} p(n)q^n.$$

For $k = 1, 2, \ldots, m-1$, we define the $k$th component of F as follows:

$$\mathrm{F}^{(k,m)} := q^k \sum_{n=0}^{\infty} p(mn+k)y^n,$$

then we have

$$\mathrm{F} = \sum_{n=0}^{\infty} p(n)q^n = \sum_{k=0}^{m-1} \mathrm{F}^{(k,m)}.$$

We prefer the following notation of Atkin and Swinnerton-Dyer:

$$P(a) = \left(y^a; y^m\right)_\infty \left(y^{m-a}; y^m\right)_\infty,$$
$$P(0) = \left(y^m; y^m\right)_\infty,$$

where

$$(z; q)_\infty = \prod_{r=1}^{\infty} \left(1 - zq^{r-1}\right)$$

and $a$ is not a multiple of $m$. It should be noted that $P(0)$ is not the expression that would be obtained by writing 0 instead of $a$ in the definition of $P(a)$ and $P(a)$ satisfies

$$P(m-a) = P(a), \qquad P(-a) = P(m+a) = -y^{-a} P(a).$$

Atkin and Swinnerton-Dyer gave some congruences for $\mathrm{F}^{(k,m)} \pmod{m}$ where $m = 5$, 7 and 11. They obtained the congruences without components. They needed $\prod_{r=1}^{\infty}(1-q^r)^3$ and $\prod_{r=1}^{\infty}(1-q^r)$ as a polynomial in $q$:

**Lemma 1** (Atkin and Swinnerton-Dyer) *We have*

$$\prod_{r=1}^{\infty} \left(1-q^r\right)^3 \equiv P(0) \sum_{c=0}^{(m-3)/2} (-1)^c (2c+1) q^{\frac{1}{2}c(c+1)} P\left(\frac{m-1}{2} - c\right) \pmod{m}.$$

**Lemma 2** (Atkin and Swinnerton-Dyer) *Let $m$ be a positive integer coprime to* 6 *and write $m = 6\lambda + \mu$ ($\mu = \pm 1$). We have*

$$\prod_{r=1}^{\infty} \left(1-q^r\right) = (-1)^{\lambda} q^{\frac{1}{2}\lambda(3\lambda+\mu)} P(0) \left[1 + \sum_{c=1}^{(m-1)/2} (-1)^c q^{\frac{1}{2}c(3c-m)} \frac{P(2c)}{P(c)}\right].$$

To calculate congruences for $\mathrm{F}^{(k,m)} \pmod{m}$ where $m = 5$, 7 and 11, they used Lemmas 1 and 2 in the congruences

$$\sum_{k=0}^{4} \mathrm{F}^{(k,5)} = (q;q)_\infty^{-1} \equiv \left\{\prod_{r=1}^{\infty}\left(1-q^r\right)^3\right\}^3 \Big/ \prod_{r=1}^{\infty}\left(1-y^r\right)^2 \pmod{5},$$
$$\sum_{k=0}^{6} \mathrm{F}^{(k,7)} = (q;q)_\infty^{-1} \equiv \left\{\prod_{r=1}^{\infty}\left(1-q^r\right)^3\right\}^2 \Big/ \prod_{r=1}^{\infty}\left(1-y^r\right) \pmod{7},$$

and

$$\sum_{k=0}^{10} \mathrm{F}^{(k,11)} q^k = (q;q)_\infty^{-1} \equiv \left\{\prod_{r=1}^{\infty}\left(1-q^r\right)^3\right\}^3 \prod_{r=1}^{\infty}\left(1-q^r\right) \Big/ \prod_{r=1}^{\infty}\left(1-y^r\right) \pmod{11}.$$

For simplifications, they gave the following lemma:

**Lemma 3** *Suppose that none of* $b, c, d, b \pm c, c \pm d, b \pm d$ *is divisible by* $m$. *Then we have*

$$\begin{aligned}&P^2(b)P(c+d)P(c-d) - P^2(c)P(b+d)P(b-d)\\&\quad + y^{c-d}P^2(d)P(b+c)P(b-c) = 0.\end{aligned}$$

In [5], using Winquist's identity, Hirschorn gave $\prod_{r=1}^{\infty}(1-q^r)^{10}$ as a polynomial of degree 11 in $q$ as follows:

$$\begin{aligned}\prod_{r=1}^{\infty}\left(1-q^r\right)^{10} \equiv P^2(0)\big\{&P(2)P(3)P(4)P(5) + qP^2(5)P(1)P(4)\\&+ 2q^2P^2(3)P(2)P(5) + 3q^3P^2(2)P(4)P(5)\\&+ 5q^4P(1)P(3)P(4)P(5) + 7q^5P^2(4)P(1)P(3)\\&+ 4q^7P(1)P(2)P(4)P(5) + 6yq^8P^2(1)P(2)P(3)\\&+ 8q^9P(1)P(2)P(3)P(5) + 9q^{10}P(1)P(2)P(3)P(4)\big\} \pmod{11}\end{aligned}$$

and this polynomial was used in the proof of the congruences for $\mathrm{F}^{(k,11)}$ (mod 11). He divided $\prod_{r=1}^{\infty}(1-q^r)^{10}$ by $(y;y)_\infty$ and re-proved the congruences for $\mathrm{F}^{(k,11)}$ (mod 11) given by Atkin and Swinnerton-Dyer.

We calculate the components for $m = 13$ and give some congruences for $\mathrm{F}^{(k,13)}$ (mod 13) by using these components. Our main result is the following theorem:

**Theorem 1** *For* $m = 13$, *we have*

$$\begin{aligned}
\mathrm{F}^{(0,13)} &\equiv y\frac{P(2)}{P(4)}\left\{\frac{P(0)P(6)}{yP(1)P(3)} - 4(y;y)_\infty^{11}\right\} \pmod{13},\\
\mathrm{F}^{(1,13)} &\equiv q\frac{P(5)}{P(3)}\left\{-3\frac{P(0)P(2)}{P(1)P(4)} + 4(y;y)_\infty^{11}\right\} \pmod{13},\\
\mathrm{F}^{(2,13)} &\equiv q^2y\frac{P(2)}{P(5)}\left\{3y\frac{P(0)P(1)}{P(2)P(6)} + 2\frac{P(0)P(6)}{yP(1)P(3)} + 5(y;y)_\infty^{11}\right\} \pmod{13},\\
\mathrm{F}^{(3,13)} &\equiv q^3\frac{P(4)}{P(3)}\left\{-5\frac{P(0)P(2)}{P(1)P(4)} - y\frac{P(0)P(1)}{P(2)P(6)} - 5(y;y)_\infty^{11}\right\} \pmod{13},\\
\mathrm{F}^{(4,13)} &\equiv q^4\frac{P(6)}{yP(1)}\left\{4\frac{P(0)P(5)}{P(3)P(4)} - 4(y;y)_\infty^{11}\right\} \pmod{13},\\
\mathrm{F}^{(5,13)} &\equiv q^5y\frac{P(1)}{P(4)}\left\{-6\frac{P(0)P(6)}{yP(1)P(3)} - 4y\frac{P(0)P(3)}{P(5)P(6)} - 5(y;y)_\infty^{11}\right\} \pmod{13},\\
\mathrm{F}^{(6,13)} &\equiv 11q^6(y;y)_\infty^{11} \pmod{13},\\
\mathrm{F}^{(7,13)} &\equiv q^7\frac{P(5)}{P(6)}\left\{4\frac{P(0)P(4)}{P(2)P(5)} + 6\frac{P(0)P(2)}{P(1)P(4)} + 5(y;y)_\infty^{11}\right\} \pmod{13},
\end{aligned}$$

$$\mathrm{F}^{(8,13)} \equiv q^8 \frac{P(4)}{P(5)} \left\{ -6y \frac{P(0)P(1)}{P(2)P(6)} - 4(y;y)_\infty^{11} \right\} \pmod{13},$$

$$\mathrm{F}^{(9,13)} \equiv q^9 \frac{P(6)}{yP(2)} \left\{ y \frac{P(0)P(3)}{P(5)P(6)} + 5 \frac{P(0)P(5)}{P(3)P(4)} - 5(y;y)_\infty^{11} \right\} \pmod{13},$$

$$\mathrm{F}^{(10,13)} \equiv q^{10} \frac{P(3)}{yP(1)} \left\{ 2 \frac{P(0)P(5)}{P(3)P(4)} + 3 \frac{P(0)P(4)}{P(2)P(5)} - 5(y;y)_\infty^{11} \right\} \pmod{13},$$

$$\mathrm{F}^{(11,13)} \equiv q^{11} \frac{P(1)}{P(2)} \left\{ -2y \frac{P(0)P(3)}{P(5)P(6)} + 4(y;y)_\infty^{11} \right\} \pmod{13},$$

$$\mathrm{F}^{(12,13)} \equiv q^{12} \frac{P(3)}{P(6)} \left\{ -5 \frac{P(0)P(4)}{P(2)P(5)} + 4(y;y)_\infty^{11} \right\} \pmod{13}.$$

## 2 Preparation

Kolberg [6] gave the components $\mathrm{F}^{(k,5)}$ and $\mathrm{F}^{(k,7)}$. A simpler form of $\mathrm{F}^{(k,7)}$ can be found in [4]. Here, using Kolberg's method, we obtain the components $\mathrm{F}^{(k,13)}$. The methods we employ in this paper are elementary.

We define, for $s = 0, 1, \ldots, m-1$

$$g_s := \sum_{\frac{1}{2}n(3n+1) \equiv s \pmod{m}} (-1)^n q^{\frac{1}{2}n(3n+1)}$$

and

$$h_s := \sum_{\substack{\frac{1}{2}n(n+1) \equiv s \pmod{m}, \\ n \geq 0}} (-1)^n (2n+1) q^{\frac{1}{2}n(n+1)}.$$

These are defined to divide the following well-known identity into $m$ parts:

$$\prod_{r=1}^{\infty} (1 - q^r) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(3n+1)},$$

and

$$\prod_{r=1}^{\infty} (1 - q^r)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{1}{2}n(n+1)}.$$

Kolberg gave the following.

**Lemma 4** *We have*

$$g_s = \begin{cases} 0, & \text{if } 24s+1 \text{ is a quadratic non-residue mod } m, \\ (-1)^{[\frac{m+1}{6}]} q^{\frac{m^2-1}{24}} P(0), & \text{if } 24s+1 \equiv 0 \pmod{m} \end{cases}$$

*and*

$$h_s = \begin{cases} 0, & \text{if } 8s+1 \text{ is a quad. non-res. mod } m, \\ (-1)^{[\frac{m-1}{2}]} m q^{\frac{m^2-1}{8}} P^3(0), & \text{if } 8s+1 \equiv 0 \pmod{m}. \end{cases}$$

From the definitions of $g_s$ and $h_s$ we have

$$\prod_{r=1}^{\infty}(1-q^r) = \sum_{s=0}^{m-1} g_s$$

and

$$\prod_{r=1}^{\infty}(1-q^r)^3 = \sum_{s=0}^{m-1} h_s.$$

By these equations, it can be seen that

$$(g_0 + g_1 + \cdots + g_{m-1})^3 = h_0 + h_1 + \cdots + h_{m-1}. \tag{7}$$

All $g_s$ can be determined in terms of $P(a)$.

**Lemma 5** *Let* $24s+1$ *be a quadratic residue mod* $m$ *and* $m = 6\lambda + \mu$ *where* $\lambda$ *is a positive integer and* $\mu = \pm 1$. *Then we have*

$$g_s = (-1)^{c+\lambda} q^{\frac{1}{2}(3c^2 - mc + 3\lambda^2 + \mu\lambda)} \frac{P(2c)}{P(c)},$$

*where* $c$ *is a solution of the congruence* $x^2 \equiv (4s - \mu\lambda)/6 \pmod{m}$.

*Proof* If $\mu$ is equal to 1, it is clear that $6^{-1} \equiv -\lambda \pmod{m}$. If $\mu$ is $-1$, then $6^{-1} \equiv \lambda \pmod{m}$. In both cases $(4s - \mu\lambda)/6 \equiv (24s+1)/36 \pmod{m}$ and there is a solution of the quadratic congruence $x^2 \equiv (4s - \mu\lambda)/6 \pmod{m}$; that is, $(4s - \mu\lambda)/6$ is a quadratic residue mod $m$. If we consider Lemma 2 and the definition of $g_s$, we obtain the lemma. □

Kolberg also showed that

**Lemma 6** *For* $s = 0, 1, \ldots, m-1$

$$\mathrm{F}^{(s,m)} = (-1)^{(m-1)s} \frac{P(0)}{(y;y)_\infty^{m+1}} D_s,$$

*where* $D_s$ *is the determinant of following matrix*:

$$\begin{bmatrix} g_{-s} & g_{-s+1} & \cdots & g_{-s+m-2} \\ g_{-s-1} & g_{-s} & \cdots & g_{-s+m-3} \\ \cdots & \cdots & \cdots & \cdots \\ g_{-s-m+2} & g_{-s-m+3} & \cdots & g_{-s} \end{bmatrix}. \tag{8}$$

*We put* $g_r = g_s$ *when* $r \equiv s \pmod{m}$ *in* (8).

We define

$$A_s := g_k^{1-m} D_s,$$

where $24k + 1 \equiv 0 \pmod{m}$. Therefore, we obtain

$$\mathrm{F}^{(s,m)} = q^{\frac{1}{24}(m^3-m^2-m+1)} \frac{P^m(0)}{(y;\, y)_\infty^{m+1}} A_s. \tag{9}$$

For the denominator of Eq. (9), it is easy to see that

$$(y;\, y)_\infty = \prod_{r=1}^{\infty} \big(1 - y^r\big) = P(0)P(1)P(2)\cdots P\big((m-1)/2\big). \tag{10}$$

After calculating the components, it is possible to obtain the congruence properties of the components by the following lemma.

**Lemma 7** *If m is a prime integer, then*

$$\left[\prod_{r=1}^{\infty} \big(1 - q^r\big)^3\right]^m \equiv P^{m+2}(0)P^{m+3}(1)P^{m+3}(2)\cdots P^{m+3}\big((m-1)/2\big) \pmod{m}.$$

*Proof* Since $m$ is a prime it is clear that

$$\big(1 - q^r\big)^m \equiv 1 - y^r \pmod{m} \quad \text{and} \quad \big(1 - y^r\big)^m \equiv 1 - y^{mr} \pmod{m}. \tag{11}$$

We write

$$\left[\prod_{r=1}^{\infty} \big(1 - q^r\big)^3\right]^m \equiv \prod_{r=1}^{\infty} \big(1 - y^r\big)^3 \pmod{m},$$

and with the help of Eq. (10), this congruence becomes

$$\left[\prod_{r=1}^{\infty} \big(1 - q^r\big)^3\right]^m \equiv P^3(0)P^3(1)P^3(2)\cdots P^3\big((m-1)/2\big) \pmod{m}. \tag{12}$$

By means of the second congruence in (11), we write

$$P^m(0)P^m(1)P^m(2)\cdots P^m\big((m-1)/2\big) \equiv P(0) \pmod{m}$$

and we have

$$P^{m-1}(0)P^m(1)P^m(2)\cdots P^m\big((m-1)/2\big) \equiv 1 \pmod{m}. \tag{13}$$

Finally, multiplying (12) by (13) proves the lemma. □

## 3 Components and congruences for $m = 13$

For $m = 13$, Lemma 4 gives

$$g_3 = g_4 = g_6 = g_8 = g_{10} = g_{11} = 0, \qquad g_7 = q^7 P(0) \tag{14}$$

and

$$h_4 = h_5 = h_7 = h_9 = h_{11} = h_{12} = 0, \qquad h_8 = 13q^{21} P^3(0).$$

From Eq. (7), we have

$$g_7^3 + 6(g_2g_7g_{12} + g_5g_7g_9 + g_0g_1g_7 + g_0g_9g_{12} + g_1g_2g_5) = h_8 = 13g_7^3,$$

and this equation gives

$$g_2g_7g_{12} + g_5g_7g_9 + g_0g_1g_7 + g_0g_9g_{12} + g_1g_2g_5 = 2g_7^3 \tag{15}$$

Now we set

$$\begin{aligned} \alpha &:= g_7^{-1}g_0, & \beta &:= g_7^{-1}g_1, & \gamma &:= g_7^{-1}g_2, \\ \theta &:= g_7^{-1}g_5, & \delta &:= g_7^{-1}g_9, & \zeta &:= g_7^{-1}g_{12}. \end{aligned} \tag{16}$$

and Eq. (15) becomes

$$\alpha\beta + \gamma\zeta + \theta\delta + \alpha\delta\zeta + \beta\gamma\theta = 2. \tag{17}$$

Lemma 5 gives

$$\begin{aligned} &g_0 = \frac{P(0)P(4)}{P(2)}, \qquad g_1 = -q\frac{P(0)P(6)}{P(3)}, \qquad g_2 = -q^2\frac{P(0)P(2)}{P(1)}, \\ &g_5 = q^5\frac{P(0)P(5)}{P(4)}, \qquad g_7 = q^7P(0), \qquad g_9 = q^{22}\frac{P(0)P(1)}{P(6)}, \\ &g_{12} = -q^{12}\frac{P(0)P(3)}{P(5)}, \end{aligned}$$

and from Eqs. (16), we have

$$\begin{aligned} \alpha &= \frac{1}{q^7}\frac{P(4)}{P(2)}, & \beta &= -\frac{1}{q^6}\frac{P(6)}{P(3)}, & \gamma &= -\frac{1}{q^5}\frac{P(2)}{P(1)}, \\ \theta &= \frac{1}{q^2}\frac{P(5)}{P(4)}, & \delta &= q^{15}\frac{P(1)}{P(6)}, & \zeta &= -q^5\frac{P(3)}{P(5)}. \end{aligned} \tag{18}$$

Multiplying all the terms in Eqs. (18) gives

$$\alpha\beta\gamma\theta\delta\zeta = -1. \tag{19}$$

Now, we set

$$x_1 := \alpha\beta, \qquad x_2 := \gamma\zeta, \qquad x_3 := \theta\delta, \qquad x_4 := \alpha\delta\zeta, \qquad x_5 := \beta\gamma\theta. \tag{20}$$

From Eq. (19), we have

$$x_1x_2x_3 = -1, \qquad x_4x_5 = -1. \tag{21}$$

and Eq. (17) becomes

$$x_1 + x_2 + x_3 + x_4 + x_5 = 2. \tag{22}$$

The following lemma can be obtained from the theta function identity ($\mathrm{LVII}_2$) in [7, p. 160] easily.

**Lemma 8** *Suppose that none of* $a \pm b, a \pm c, a \pm d, b \pm c, b \pm d, c \pm d$ *is divisible by* $m$. *Then we have*

$$\begin{aligned}&P(a+d)P(a-d)P(b+c)P(b-c) - P(a+c)P(a-c)P(b+d)P(b-d)\\&\quad + y^{b-c}P(a+b)P(a-b)P(c+d)P(c-e) = 0.\end{aligned}$$

This lemma gives the same identities with Lemma 3 for $m$ =5, 7 and 11. For $m = 13$, Lemma 8 gives a different identity from Lemma 3. For $(a, b, c, d) = (5, 3, 2, 1)$, we have

$$yP(1)P(2)P(3)P(5) - P(2)P(3)P(4)P(6) + P(1)P(4)P(5)P(6) = 0. \tag{23}$$

Equation (23) gives the following lemma.

**Lemma 9** *We have*

$$\begin{aligned}x_1x_2 &= x_1 - 1,\\ x_2x_3 &= x_2 - 1,\\ x_1x_3 &= x_3 - 1.\end{aligned}$$

*Proof* After dividing Eq. (23) by $yP(1)P(2)P(3)P(5)$, Eqs. (16) and (20) give the first relation in lemma. The other ones can be found similarly. □

To calculate all the components $\mathrm{F}^{(k,13)}$, we evaluate the determinants of matrix (8) at first. The size of each matrix is 12 × 12. We can facilitate evaluations of these determinants by using Eqs. (14). After evaluation, we should simplify all $\mathrm{F}^{(k,13)}$. If we use Eqs. (16), $\mathrm{F}^{(k,13)}$ turns to the form (9) where all $A_s$ is a linear combination of $\alpha^{k_1}\beta^{k_2}\gamma^{k_3}\theta^{k_4}\delta^{k_5}\zeta^{k_6}$ and each $k_i$ is a non-negative integer. We should do another transformation to use the relations in Lemma 9, but all $A_s$ should be multiplied by an appropriate factor. These factors can be chosen in different ways. We define

$$B_s := tA_s,$$

where $t := \alpha,\ \zeta,\ \alpha\theta,\ \theta\zeta,\ \delta,\ \alpha\gamma,\ 1,\ \beta\zeta,\ \theta,\ \gamma\delta,\ \beta\delta,\ \gamma$ and $\beta$ for $s = 0, 1, \ldots, 12$, respectively.

After these multiplications, all $B_s$ can be written as a linear combination of $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}$ where each $a_i$ is a non-negative integer with the help of Eqs. (20).

Let $T$ be a term in any $B_s$, i.e., a linear combination of $\alpha^{k_1}\beta^{k_2}\gamma^{k_3}\theta^{k_4}\delta^{k_5}\zeta^{k_6}$ and each $k_i$ is a non-negative integer. To write $T$ as a linear combination of products of $x_i$'s, we substitute Eqs. (20) for $T$ at first. If $T$ does not turn a linear combination of $x_i$'s completely, we multiply $T$ by $-\alpha\beta\gamma\theta\delta\zeta$ and substitute Eqs. (20) for $T$ again. After applying these two operations a number of times, most of the terms in $B_s$ turn a linear combination of $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}$ except 12 terms which are given in the following lemma.

**Lemma 10** *We have*

$$\begin{aligned}\theta^3\alpha &= x_3x_4 - x_1 - x_3 - x_4.\\ \alpha^3\gamma &= x_1x_5 - x_1 - x_2 - x_5,\\ \beta^3\zeta &= x_1x_4 - x_1 - x_2 - x_4,\\ \delta^3\beta &= x_3x_5 - x_1 - x_3 - x_5,\\ \zeta^3\theta &= x_2x_5 - x_2 - x_3 - x_5,\\ \gamma^3\delta &= x_2x_4 - x_2 - x_3 - x_4,\end{aligned} \tag{24}$$

*and*

$$\begin{aligned}\alpha^2\theta^2\zeta &= -x_3x_4 + x_1 + x_4,\\ \alpha^2\gamma^2\theta &= -x_1x_5 + x_2 + x_5,\\ \beta^2\delta^2\gamma &= -x_3x_5 + x_1 + x_5,\\ \gamma^2\delta^2\alpha &= -x_2x_4 + x_3 + x_4,\\ \theta^2\zeta^2\beta &= -x_2x_5 + x_3 + x_5,\\ \beta^2\zeta^2\delta &= -x_1x_4 + x_2 + x_4.\end{aligned} \tag{25}$$

*Proof* We express the terms $\theta^3\alpha$ and $\alpha^2\theta^2\zeta$ in terms of $x_i$'s. For $(b,c,d) = (5,4,1)$, $(5,4,2)$ and $(6,3,2)$ Lemma 3 gives, respectively,

$$\begin{aligned}y^3P^3(1)P(4) - P^3(4)P(6) + P^3(5)P(3) &= 0,\\ y^2P^2(2)P(1)P(4) - P^2(4)P(3)P(6) + P^2(5)P(2)P(6) &= 0,\\ yP^2(2)P(3)P(4) - P^2(3)P(4)P(5) + P^2(6)P(1)P(5) &= 0.\end{aligned}$$

Lemma 3 gives 20 relations for $m = 13$ which are found in [2]. From Eqs. (18), last three identities become

$$\theta^3\alpha^2\gamma^2 + \alpha^2\gamma^2\beta - \beta\delta = 0, \tag{26}$$

$$\alpha^2\theta\zeta + \theta^2\alpha + \delta = 0, \tag{27}$$

$$\gamma^2\beta\delta - \alpha\gamma\theta - \beta^2\theta = 0. \tag{28}$$

Multiplying Eq. (26) by $\beta\theta\delta\zeta$, we have

$$\theta^3\alpha = -x_1 + \beta^3\delta^3\theta^2\zeta^2\alpha. \tag{29}$$

Similarly, by multiplying Eq. (28) by $\delta^3\zeta^2\alpha\beta\theta$, we obtain

$$\beta^3\delta^3\theta^2\zeta^2\alpha = x_3x_4 - \delta^3\beta\gamma\zeta.$$

Using the last equation, Eq. (29) becomes

$$\theta^3\alpha = x_3x_4 - x_1 - \delta^3\beta\gamma\zeta. \tag{30}$$

After multiplying Eq. (27) by $\delta^2\beta\gamma\zeta$, we obtain

$$\delta^3\beta\gamma\zeta = x_3 + x_4.$$

Finally, using the last equation in Eq. (30), we have

$$\theta^3\alpha = x_3x_4 - x_1 - x_3 - x_4.$$

For the term $\alpha^2\theta^2\zeta$, multiplying Eq. (27) by $\theta$, we find

$$\begin{aligned}\alpha^2\theta^2\zeta &= -\theta^3\alpha - x_3\\ &= -x_3x_4 + x_1 + x_4.\end{aligned}$$

The other ones can be proven similarly. □

Using the equations in Lemma 10 with Eqs. (20), we obtain all the $B_s$ as a linear combination of $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}$. To summarize how to obtain all $B_s$, we give an example. The term $\alpha^{13}$ appears in $B_0$ and it is the most complicated term to transform in $B_0$. We have

$$\begin{aligned}\alpha^{13} &= -\alpha^{14}\beta\gamma\theta\delta\zeta\\ &= -\alpha^{10}\zeta\left\{\alpha^3\gamma\right\}x_1x_3\\ &= \alpha^{11}\beta\gamma\theta\delta\zeta^2\left\{\alpha^3\gamma\right\}x_1x_3\\ &= \alpha^7\zeta^2\left\{\alpha^3\gamma\right\}^2x_1^2x_3^2\\ &= -\alpha^8\beta\gamma\theta\delta\zeta^3\left\{\alpha^3\gamma\right\}^2x_1^2x_3^2\\ &= -\alpha^4\delta\left\{\alpha^3\gamma\right\}^3\left\{\zeta^3\theta\right\}x_1^3x_3^2\\ &= \alpha^5\beta\gamma\theta\delta^2\zeta\left\{\alpha^3\gamma\right\}^3\left\{\zeta^3\theta\right\}x_1^3x_3^2\\ &= x_1^4x_3^2x_4(x_1x_5 - x_1 - x_2 - x_5)^4(x_2x_5 - x_2 - x_3 - x_5).\end{aligned}$$

Finally, after writing as a linear combination of products of $x_i$'s, we can simplify all $B_s$ by using Lemma 9. We give how to simplify all $B_s$ in an algorithm to avoid repetition. We use the following equations which can be found by Eq. (22) and

Lemma 9:

$$
\begin{aligned}
x_1^2 &= -x_1x_5 - x_1x_4 + x_1 - x_3 + 2,\\
x_2^2 &= -x_2x_5 - x_2x_4 + x_2 - x_1 + 2,\\
x_3^2 &= -x_3x_5 - x_3x_4 + x_3 - x_2 + 2,\\
x_4^2 &= -x_1x_4 - x_2x_4 - x_3x_4 + 2x_4 + 1,\\
x_5^2 &= -x_1x_5 - x_2x_5 - x_3x_5 + 2x_5 + 1.
\end{aligned}
\tag{31}
$$

**Algorithm 1** Let $K$ be a linear combination of $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}$ where each $a_i$ is a non-negative integer.

1. Equations (21) simplify the terms in the form $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ or $x_4^{a_4}x_5^{a_5}$ in $K$.
2. Substitute the equations in Lemma 9 for $K$. After this operation, the terms that remain take the forms of $x_i^a$ and $x_i^a x_j^b$ where $(i, j) = (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)$ in $K$.
3. If $b > 1$, then substitute $x_j^2$ in Eqs. (31) for the terms in step 2, if $a > 1$ and $b = 0$, substitute $x_i^2$ in Eqs. (31) for the terms in step 2.

With these three steps, $K$ becomes a linear combination of the terms in the forms of $x_i^a x_j$ and $x_i$.

After this process, we obtain the components $\mathrm{F}^{(s,m)}$ where $s = 0, 1, \ldots, 12$ as follows:

$$
\mathrm{F}^{(s,m)} = q^{84}\frac{P(0)}{(y;y)_\infty^{14}}B_s, \tag{32}
$$

where

$$
\begin{aligned}
B_0 = {} & x_1^6x_5 + 77x_1^5x_4 - 86x_1^5x_5 - 22x_2^5x_4 + 56x_3^5x_4 + x_3^5x_5 - 802x_1^4x_4\\
& + 1008x_1^4x_5 + 462x_2^4x_4 + 5x_2^4x_5 - 791x_3^4x_4 - 120x_3^4x_5 + 3260x_1^3x_4\\
& - 3235x_1^3x_5 - 2344x_2^3x_4 - 239x_2^3x_5 + 3620x_3^3x_4 + 1128x_3^3x_5 - 4318x_1^2x_4\\
& + 6327x_1^2x_5 + 4965x_2^2x_4 + 1785x_2^2x_5 - 9009x_3^2x_4 - 4526x_3^2x_5 + 4741x_1x_4\\
& - 4262x_1x_5 - 7364x_2x_4 - 4400x_2x_5 + 11199x_3x_4 + 8436x_3x_5 - 4024x_1\\
& + 5603x_2 + 21268x_3 + 3552x_4 + 7727x_5 - 20230,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
B_1 = {} & x_2^6x_5 + 77x_2^5x_4 - 86x_2^5x_5 - 22x_3^5x_4 + 56x_1^5x_4 + x_1^5x_5 - 802x_2^4x_4\\
& + 1008x_2^4x_5 + 462x_3^4x_4 + 5x_3^4x_5 - 791x_1^4x_4 - 120x_1^4x_5 + 3260x_2^3x_4\\
& - 3235x_2^3x_5 - 2344x_3^3x_4 - 239x_3^3x_5 + 3620x_1^3x_4 + 1128x_1^3x_5 - 4318x_2^2x_4\\
& + 6327x_2^2x_5 + 4965x_3^2x_4 + 1785x_3^2x_5 - 9009x_1^2x_4 - 4526x_1^2x_5 + 4741x_2x_4\\
& - 4262x_2x_5 - 7364x_3x_4 - 4400x_3x_5 + 11199x_1x_4 + 8436x_1x_5 - 4024x_2\\
& + 5603x_3 + 21268x_1 + 3552x_4 + 7727x_5 - 20230,
\end{aligned}
$$

$$\begin{aligned}
B_2 = {} & 2x_1^6x_5 - 7x_3^6x_4 - 146x_1^5x_5 + 3x_2^5x_4 + 287x_3^5x_4 - 15x_3^5x_5 - 40x_1^4x_4 \\
& + 1252x_1^4x_5 - 186x_2^4x_4 - 30x_2^4x_5 - 1956x_3^4x_4 + 395x_3^4x_5 + 441x_1^3x_4 \\
& - 3859x_1^3x_5 + 1361x_2^3x_4 + 555x_2^3x_5 + 6546x_3^3x_4 - 1770x_3^3x_5 - 1253x_1^2x_4 \\
& + 7098x_1^2x_5 - 4394x_2^2x_4 - 2643x_2^2x_5 - 10436x_3^2x_4 + 3872x_3^2x_5 + 824x_1x_4 \\
& - 4507x_1x_5 + 6440x_2x_4 + 6638x_2x_5 + 13307x_3x_4 - 1615x_3x_5 + 2452x_1 \\
& + 16662x_2 + 4881x_3 - 1048x_4 + 7862x_5 - 30365,
\end{aligned} \tag{34}$$

$$\begin{aligned}
B_3 = {} & 2x_3^6x_4 - 7x_2^6x_5 - 146x_3^5x_4 + 3x_1^5x_5 + 287x_2^5x_5 - 15x_2^5x_4 - 40x_3^4x_5 \\
& + 1252x_3^4x_4 - 186x_1^4x_5 - 30x_1^4x_4 - 1956x_2^4x_5 + 395x_2^4x_4 + 441x_3^3x_5 \\
& - 3859x_3^3x_4 + 1361x_1^3x_5 + 555x_1^3x_4 + 6546x_2^3x_5 - 1770x_2^3x_4 - 1253x_3^2x_5 \\
& + 7098x_3^2x_4 - 4394x_1^2x_5 - 2643x_1^2x_4 - 10436x_2^2x_5 + 3872x_2^2x_4 + 824x_3x_5 \\
& - 4507x_3x_4 + 6440x_1x_5 + 6638x_1x_4 + 13307x_2x_5 - 1615x_2x_4 \\
& + 2452x_3 + 16662x_1 + 4881x_2 - 1048x_5 + 7862x_4 - 30365,
\end{aligned}$$

$$\begin{aligned}
B_4 = {} & x_3^6x_5 + 77x_3^5x_4 - 86x_3^5x_5 - 22x_1^5x_4 + 56x_2^5x_4 + x_2^5x_5 - 802x_3^4x_4 \\
& + 1008x_3^4x_5 + 462x_1^4x_4 + 5x_1^4x_5 - 791x_2^4x_4 - 120x_2^4x_5 + 3260x_3^3x_4 \\
& - 3235x_3^3x_5 - 2344x_1^3x_4 - 239x_1^3x_5 + 3620x_2^3x_4 + 1128x_2^3x_5 - 4318x_3^2x_4 \\
& + 6327x_3^2x_5 + 4965x_1^2x_4 + 1785x_1^2x_5 - 9009x_2^2x_4 - 4526x_2^2x_5 + 4741x_3x_4 \\
& - 4262x_3x_5 - 7364x_1x_4 - 4400x_1x_5 + 11199x_2x_4 + 8436x_2x_5 - 4024x_3 \\
& + 5603x_1 + 21268x_2 + 3552x_4 + 7727x_5 - 20230,
\end{aligned}$$

$$\begin{aligned}
B_5 = {} & 2x_2^6x_4 - 7x_1^6x_5 - 146x_2^5x_4 + 3x_3^5x_5 + 287x_1^5x_5 - 15x_1^5x_4 - 40x_2^4x_5 \\
& + 1252x_2^4x_4 - 186x_3^4x_5 - 30x_3^4x_4 - 1956x_1^4x_5 + 395x_1^4x_4 + 441x_2^3x_5 \\
& - 3859x_2^3x_4 + 1361x_3^3x_5 + 555x_3^3x_4 + 6546x_1^3x_5 - 1770x_1^3x_4 - 1253x_2^2x_5 \\
& + 7098x_2^2x_4 - 4394x_3^2x_5 - 2643x_3^2x_4 - 10436x_1^2x_5 + 3872x_1^2x_4 + 824x_2x_5 \\
& - 4507x_2x_4 + 6440x_3x_5 + 6638x_3x_4 + 13307x_1x_5 - 1615x_1x_4 + 2452x_2 \\
& + 16662x_3 + 4881x_1 - 1048x_5 + 7862x_4 - 30365,
\end{aligned}$$

$$\begin{aligned}
B_6 = {} & -11x_1^5x_4 - 11x_1^5x_5 - 11x_2^5x_4 - 11x_2^5x_5 - 11x_3^5x_4 - 11x_3^5x_5 + 325x_1^4x_4 \\
& + 325x_1^4x_5 + 325x_2^4x_4 + 325x_2^4x_5 + 325x_3^4x_4 + 325x_3^4x_5 - 1967x_1^3x_4 \\
& - 1967x_1^3x_5 - 1967x_2^3x_4 - 1967x_2^3x_5 - 1967x_3^3x_4 - 1967x_3^3x_5 + 4902x_1^2x_4 \\
& + 4902x_1^2x_5 + 4902x_2^2x_4 + 4902x_2^2x_5 + 4902x_3^2x_4 + 4902x_3^2x_5
\end{aligned}$$

$$
\begin{aligned}
&-8830x_1x_4-8830x_1x_5-8830x_2x_4-8830x_2x_5-8830x_3x_4-8830x_3x_5\\
&+2241x_1+2241x_2+2241x_3+13706x_4+13706x_5+23323, \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
B_7={}&2x_2^6x_5-7x_1^6x_4-146x_2^5x_5+3x_3^5x_4+287x_1^5x_4-15x_1^5x_5-40x_2^4x_4\\
&+1252x_2^4x_5-186x_3^4x_4-30x_3^4x_5-1956x_1^4x_4+395x_1^4x_5+441x_2^3x_4\\
&-3859x_2^3x_5+1361x_3^3x_4+555x_3^3x_5+6546x_1^3x_4-1770x_1^3x_5-1253x_2^2x_4\\
&+7098x_2^2x_5-4394x_3^2x_4-2643x_3^2x_5-10436x_1^2x_4+3872x_1^2x_5+824x_2x_4\\
&-4507x_2x_5+6440x_3x_4+6638x_3x_5+13307x_1x_4-1615x_1x_5+2452x_2\\
&+16662x_3+4881x_1-1048x_4+7862x_5-30365,
\end{aligned}
$$

$$
\begin{aligned}
B_8={}&x_3^6x_4+77x_3^5x_5-86x_3^5x_4-22x_1^5x_5+56x_2^5x_5+x_2^5x_4-802x_3^4x_5\\
&+1008x_3^4x_4+462x_1^4x_5+5x_1^4x_4-791x_2^4x_5-120x_2^4x_4+3260x_3^3x_5\\
&-3235x_3^3x_4-2344x_1^3x_5-239x_1^3x_4+3620x_2^3x_5+1128x_2^3x_4-4318x_3^2x_5\\
&+6327x_3^2x_4+4965x_1^2x_5+1785x_1^2x_4-9009x_2^2x_5-4526x_2^2x_4+4741x_3x_5\\
&-4262x_3x_4-7364x_1x_5-4400x_1x_4+11199x_2x_5+8436x_2x_4-4024x_3\\
&+5603x_1+21268x_2+3552x_5+7727x_4-20230,
\end{aligned}
$$

$$
\begin{aligned}
B_9={}&2x_3^6x_5-7x_2^6x_4-146x_3^5x_5+3x_1^5x_4+287x_2^5x_4-15x_2^5x_5-40x_3^4x_4\\
&+1252x_3^4x_5-186x_1^4x_4-30x_1^4x_5-1956x_2^4x_4+395x_2^4x_5+441x_3^3x_4\\
&-3859x_3^3x_5+1361x_1^3x_4+555x_1^3x_5+6546x_2^3x_4-1770x_2^3x_5-1253x_3^2x_4\\
&+7098x_3^2x_5-4394x_1^2x_4-2643x_1^2x_5-10436x_2^2x_4+3872x_2^2x_5+824x_3x_4\\
&-4507x_3x_5+6440x_1x_4+6638x_1x_5+13307x_2x_4-1615x_2x_5+2452x_3\\
&+16662x_1+4881x_2-1048x_4+7862x_5-30365,
\end{aligned}
$$

$$
\begin{aligned}
B_{10}={}&2x_1^6x_4-7x_3^6x_5-146x_1^5x_4+3x_2^5x_5+287x_3^5x_5-15x_3^5x_4-40x_1^4x_5\\
&+1252x_1^4x_4-186x_2^4x_5-30x_2^4x_4-1956x_3^4x_5+395x_3^4x_4+441x_1^3x_5\\
&-3859x_1^3x_4+1361x_2^3x_5+555x_2^3x_4+6546x_3^3x_5-1770x_3^3x_4-1253x_1^2x_5\\
&+7098x_1^2x_4-4394x_2^2x_5-2643x_2^2x_4-10436x_3^2x_5+3872x_3^2x_4+824x_1x_5\\
&-4507x_1x_4+6440x_2x_5+6638x_2x_4+13307x_3x_5-1615x_3x_4+2452x_1\\
&+16662x_2+4881x_3-1048x_5+7862x_4-30365,
\end{aligned}
$$

$$
\begin{aligned}
B_{11}={}&x_2^6x_4+77x_2^5x_5-86x_2^5x_4-22x_3^5x_5+56x_1^5x_5+x_1^5x_4-802x_2^4x_5\\
&+1008x_2^4x_4+462x_3^4x_5+5x_3^4x_4-791x_1^4x_5-120x_1^4x_4+3260x_2^3x_5\\
&-3235x_2^3x_4-2344x_3^3x_5-239x_3^3x_4+3620x_1^3x_5+1128x_1^3x_4-4318x_2^2x_5
\end{aligned}
$$

$$
\begin{aligned}
&+6327x_2^2x_4+4965x_3^2x_5+1785x_3^2x_4-9009x_1^2x_5-4526x_1^2x_4+4741x_2x_5\\
&-4262x_2x_4-7364x_3x_5-4400x_3x_4+11199x_1x_5+8436x_1x_4-4024x_2\\
&+5603x_3+21268x_1+3552x_5+7727x_4-20230,
\end{aligned}
$$

$$
\begin{aligned}
B_{12}=\,&x_1^6x_4+77x_1^5x_5-86x_1^5x_4-22x_2^5x_5+56x_3^5x_5+x_3^5x_4-802x_1^4x_5\\
&+1008x_1^4x_4+462x_2^4x_5+5x_2^4x_4-791x_3^4x_5-120x_3^4x_4+3260x_1^3x_5\\
&-3235x_1^3x_4-2344x_2^3x_5-239x_2^3x_4+3620x_3^3x_5+1128x_3^3x_4-4318x_1^2x_5\\
&+6327x_1^2x_4+4965x_2^2x_5+1785x_2^2x_4-9009x_3^2x_5-4526x_3^2x_4+4741x_1x_5\\
&-4262x_1x_4-7364x_2x_5-4400x_2x_4+11199x_3x_5+8436x_3x_4-4024x_1\\
&+5603x_2+21268x_3+3552x_5+7727x_4-20230.
\end{aligned}
$$

The following lemma was given by Kolberg.

**Lemma 11** *We have*

$$D=\frac{(y;y)_\infty^{m+1}}{(y^m;y^m)_\infty},$$

*where $D$ is the determinant of following matrix*:

$$\begin{bmatrix} g_0 & g_1 & \cdots & g_{m-1}\\ g_{m-1} & g_0 & \cdots & g_{m-2}\\ \cdots & \cdots & \cdots & \cdots\\ g_1 & g_2 & \cdots & g_0 \end{bmatrix}. \tag{36}$$

We define

$$A:=g_7^{-13}D. \tag{37}$$

From Lemma 11, we have

$$A=\frac{(y;y)_\infty^{14}}{y^7P^{14}(0)}. \tag{38}$$

Following the same process with simplifying the components and applying Algorithm 1 to Eq. (37), we obtain

$$
\begin{aligned}
A=\,&x_1^6x_4+x_1^6x_5+x_2^6x_4+x_2^6x_5+x_3^6x_4+x_3^6x_5+15x_1^5x_4+15x_1^5x_5+15x_2^5x_4\\
&+15x_2^5x_5+15x_3^5x_4+15x_3^5x_5+87x_1^4x_4+87x_1^4x_5+87x_2^4x_4+87x_2^4x_5\\
&+87x_3^4x_4+87x_3^4x_5+223x_1^3x_4+223x_1^3x_5+223x_2^3x_4+223x_2^3x_5+223x_3^3x_4\\
&+223x_3^3x_5+126x_1^2x_4+126x_1^2x_5+126x_2^2x_4+126x_2^2x_5+126x_3^2x_4\\
&+126x_3^2x_5-480x_1x_4-480x_1x_5-480x_2x_4-480x_2x_5-480x_3x_4\\
&-480x_3x_5+47935x_1+47935x_2+47935x_3+47543x_4+47543x_5-98057.
\end{aligned}
$$

We consider $(x_4+x_5-3)^7$. After expanding it and using Algorithm 1, we have

$$A=(x_4+x_5-3)^7,$$

and from Eq. (38), we find

$$x_4+x_5-3=\frac{(y;y)_\infty^2}{yP^2(0)}. \tag{39}$$

We can obtain $B_6$ by using Eq. (39). Applying Algorithm 1 to $(x_4+x_5-3)^6$, $(x_4+x_5-3)^5$, $(x_4+x_5-3)^4$, $(x_4+x_5-3)^3$ and $(x_4+x_5-3)^2$, it is easy to see that

$$\begin{aligned}B_6=\,&11(x_4+x_5-3)^6+468(x_4+x_5-3)^5+6422(x_4+x_5-3)^4\\&+43940(x_4+x_5-3)^3+171366(x_4+x_5-3)^2+371293(x_4+x_5-3)\\&+371293.\end{aligned} \tag{40}$$

The last equation and Eq. (32) give Eq. (6), and we have

$$\mathrm{F}^{(6,13)}\equiv 11q^6\frac{P(0)}{(y;y)_\infty^2}\pmod{13}.$$

Multiplying the right-hand side of Eq. (41) by Eq. (13), we obtain the following well-known result:

$$\mathrm{F}^{(6,13)}\equiv 11q^6(y;y)_\infty^{11}\pmod{13}. \tag{41}$$

Now we can obtain the other components in congruence form for modulus 13. For $m=13$, Lemma 7 gives

$$\begin{aligned}&P^{13}(6)+8yP^{13}(5)+5y^3P^{13}(4)+6y^6P^{13}(3)+9y^{10}P^{13}(2)+2y^{10}P^{13}(1)\\&\quad\equiv P^2(0)P^{16}(1)P^{16}(2)P^{16}(3)P^{16}(4)P^{16}(5)P^{16}(6)\pmod{13}.\end{aligned} \tag{42}$$

Now we define

$$\begin{aligned}E_1:=\,&2x_3^6x_4+5x_1^5x_4+7x_2^5x_4+2x_3^5x_4+3x_3^5x_5+10x_1^4x_4+x_1^4x_5+7x_2^4x_4\\&+4x_2^4x_5+x_3^4x_4+8x_3^4x_5+6x_1^3x_4+8x_1^3x_5+10x_2^3x_4+2x_2^3x_5+7x_3^3x_4\\&+10x_3^3x_5+4x_1^2x_4+7x_1^2x_5+5x_2^2x_4+9x_2^2x_5+x_3^2x_4+5x_3^2x_5+10x_2x_4\\&+11x_2x_5+3x_3x_4+4x_3x_5+4x_1+4x_2+3x_3+5x_4+x_5+9,\end{aligned}$$

$$\begin{aligned}E_2:=\,&7x_2^6x_4+5x_1^5x_4+7x_2^5x_4+4x_2^5x_5+11x_3^5x_4+5x_1^4x_4+x_1^4x_5+10x_2^4x_4\\&+2x_2^4x_5+9x_3^4x_4+10x_3^4x_5+9x_1^3x_4+7x_1^3x_5+5x_2^3x_4+9x_2^3x_5+8x_3^3x_4\\&+2x_3^3x_5+11x_1^2x_4+12x_1^2x_5+10x_2^2x_4+11x_2^2x_5+x_3^2x_4+5x_3^2x_5+9x_1x_4\\&+6x_1x_5+4x_2x_4+x_2x_5+11x_1+x_2+11x_3+8x_4+7x_5+5,\end{aligned}$$

$$\begin{aligned}
E_3 :=\ & 10x_3^6x_5 + 12x_1^5x_5 + 9x_2^5x_5 + 2x_3^5x_4 + 10x_3^5x_5 + 5x_1^4x_4 + 11x_1^4x_5 \\
& + 7x_2^4x_4 + 9x_2^4x_5 + x_3^4x_4 + 5x_3^4x_5 + x_1^3x_4 + 4x_1^3x_5 + 10x_2^3x_4 + 11x_2^3x_5 \\
& + 11x_3^3x_4 + 9x_3^3x_5 + 9x_1^2x_4 + 7x_1^2x_5 + 6x_2^2x_4 + 12x_2^2x_5 + 12x_3^2x_4 + 5x_3^2x_5 \\
& + 3x_2x_4 + 11x_2x_5 + 7x_3x_4 + 2x_3x_5 + 6x_1 + 6x_2 + x_3 \\
& + 4x_4 + 11x_5 + 8,
\end{aligned} \tag{43}$$

$$\begin{aligned}
E_4 :=\ & 8x_1^6x_4 + 8x_1^5x_4 + 12x_1^5x_5 + 7x_2^5x_4 + 2x_3^5x_4 + 4x_1^4x_4 + 6x_1^4x_5 + x_2^4x_4 \\
& + 4x_2^4x_5 + 2x_3^4x_4 + 3x_3^4x_5 + 2x_1^3x_4 + x_1^3x_5 + 11x_2^3x_4 + 6x_2^3x_5 + x_3^3x_4 \\
& + 8x_3^3x_5 + 4x_1^2x_4 + 7x_1^2x_5 + 3x_2^2x_4 + 2x_2^2x_5 + 7x_3^2x_4 + 10x_3^2x_5 + 12x_1x_4 \\
& + 3x_1x_5 + x_3x_4 + 5x_3x_5 + 9x_1 + 4x_4 + x_5 + 3, \\
E_5 :=\ & 9x_2^6x_5 + 12x_1^5x_5 + 7x_2^5x_4 + 9x_2^5x_5 + 3x_3^5x_5 + 5x_1^4x_4 + 12x_1^4x_5 \\
& + 10x_2^4x_4 + 11x_2^4x_5 + 11x_3^4x_4 + 6x_3^4x_5 + 9x_1^3x_4 + 6x_1^3x_5 + 6x_2^3x_4 + 12x_2^3x_5 \\
& + 10x_3^3x_4 + x_3^3x_5 + 8x_1^2x_4 + 3x_1^2x_5 + 3x_2^2x_4 + 11x_2^2x_5 + 12x_3^2x_4 + 5x_3^2x_5 \\
& + 4x_1x_4 + 6x_1x_5 + 5x_2x_4 + 7x_2x_5 + 2x_2 + 6x_4 + 11x_5 + 5, \\
E_6 :=\ & x_1^6x_5 + 8x_1^5x_4 + x_1^5x_5 + 9x_2^5x_5 + 10x_3^5x_5 + 4x_1^4x_4 + 7x_1^4x_5 + 7x_2^4x_4 \\
& + 5x_2^4x_5 + 2x_3^4x_4 + 10x_3^4x_5 + 5x_1^3x_4 + 10x_1^3x_5 + 4x_2^3x_4 + 3x_2^3x_5 + x_3^3x_4 \\
& + 5x_3^3x_5 + 9x_1^2x_4 + 7x_1^2x_5 + 10x_2^2x_4 + 2x_2^2x_5 + 11x_3^2x_4 + 9x_3^2x_5 + 2x_1x_4 \\
& + 8x_1x_5 + 12x_3x_4 + 5x_3x_5 + 6x_1 + 5x_4 + 7x_5 + 2.
\end{aligned} \tag{44}$$

After dividing both sides of Eq. (42) by $y^5P(1)P^3(2)P^2(3)P^2(4)P^2(5)P^3(6)$, $y^5P^2(1)P^2(2)P(3)P^2(4)P^3(5)P^3(6)$, $y^6P^2(1)P^2(2)P^3(3)P^3(4)P(5)P^2(6)$, $y^6P^2(1)P^3(2)P^2(3)P(4)P^3(5)P^2(6)$, $y^6P^3(1)P(2)P^2(3)P^3(4)P^2(5)P^2(6)$ and $y^7P^3(1)P^2(2)P^3(3)P^2(4)P^2(5)P(6)$, respectively, using Eqs. (16), (20), and Algorithm 1, we obtain

$$\begin{aligned}
E_1 &\equiv y^{-5}P^2(0)P^{15}(1)P^{13}(2)P^{14}(3)P^{14}(4)P^{14}(5)P^{13}(6) \pmod{13}, \\
E_2 &\equiv y^{-5}P^2(0)P^{14}(1)P^{14}(2)P^{15}(3)P^{14}(4)P^{13}(5)P^{13}(6) \pmod{13}, \\
E_3 &\equiv y^{-6}P^2(0)P^{14}(1)P^{14}(2)P^{13}(3)P^{13}(4)P^{15}(5)P^{14}(6) \pmod{13}, \\
E_4 &\equiv y^{-6}P^2(0)P^{14}(1)P^{13}(2)P^{14}(3)P^{15}(4)P^{13}(5)P^{14}(6) \pmod{13}, \\
E_5 &\equiv y^{-6}P^2(0)P^{13}(1)P^{15}(2)P^{14}(3)P^{13}(4)P^{14}(5)P^{14}(6) \pmod{13}, \\
E_6 &\equiv y^{-7}P^2(0)P^{13}(1)P^{14}(2)P^{13}(3)P^{14}(4)P^{14}(5)P^{15}(6) \pmod{13}.
\end{aligned} \tag{45}$$

From Eqs. (33), (35), and (44) we can see that

$$E_6 + 2B_6 \equiv \alpha B_0 \pmod{13}. \tag{46}$$

Eqs. (41) and (45) give

$$B_0 \equiv y^{-7} P^2(0) P^{13}(1) P^{14}(2) P^{13}(3) P^{14}(4) P^{14}(5) P^{15}(6) + 9(y;y)_\infty^{11} \pmod{13}.$$

Finally, from the definition of $\alpha$ in Eqs. (18) and Eq. (32), we have

$$\mathrm{F}^{(0,13)} \equiv y \frac{P(2)}{P(4)} \left\{ \frac{P(0)P(6)}{yP(1)P(3)} - 4(y;y)_\infty^{11} \right\} \pmod{13}.$$

Similarly, we have

$$\begin{aligned}
B_1 &\equiv 3E_5 + 2B_6 \pmod{13},\\
B_2 &\equiv 3E_1 + 2E_6 + 4B_6 \pmod{13},\\
B_3 &\equiv E_1 + 5E_5 + 4B_6 \pmod{13},\\
B_4 &\equiv 4E_3 + 2B_6 \pmod{13},\\
B_5 &\equiv 4E_2 + 6E_6 + 4B_6 \pmod{13},\\
B_7 &\equiv 4E_4 + 6E_5 + 4B_6 \pmod{13},\\
B_8 &\equiv -6E_1 + 2B_6 \pmod{13},\\
B_9 &\equiv -E_2 - 5E_3 + 4B_6 \pmod{13},\\
B_{10} &\equiv -2E_3 - 3E_4 + 4B_6 \pmod{13},\\
B_{11} &\equiv 2E_2 + 2B_6 \pmod{13},\\
B_{12} &\equiv 5E_4 + 2B_6 \pmod{13},
\end{aligned}$$

and obtain Theorem 1.

## 4 Proof of Theorem 1

In this section we give a direct proof of Theorem 1. To do this, it is sufficient to prove that

$$\left\{ \prod_{r=1}^{\infty} (1-q^r) \right\} \left\{ \prod_{r=1}^{\infty} \frac{1}{1-q^r} \right\} \equiv 1 \pmod{13}. \tag{47}$$

Lemma 2 gives the expression in the first curly bracket as a polynomial, namely

$$\begin{aligned}
\prod_{r=1}^{\infty} (1-q^r) &= g_0 + g_1 + g_2 + g_5 + g_7 + g_9 + g_{12}\\
&= P(0) \left\{ \frac{P(4)}{P(2)} - q\frac{P(6)}{P(3)} - q^2 \frac{P(2)}{P(1)} + q^5 \frac{P(5)}{P(4)} + q^7 \right.\\
&\qquad \left. + q^9 y \frac{P(1)}{P(6)} - q^{12} \frac{P(3)}{P(5)} \right\}.
\end{aligned} \tag{48}$$

If we consider the congruences $\mathrm{F}^{(k,13)}$ given in Theorem 1, we have the expression in the second curly bracket in Eq. (47) as a polynomial, namely

$$\prod_{r=1}^{\infty}\frac{1}{1-q^r}\equiv\sum_{k=0}^{12}\mathrm{F}^{(k,13)}q^k \pmod{13}.$$

We need the following for the proof. Lemma 1 gives

$$\prod_{r=1}^{\infty}(1-q^r)^3\equiv P(0)\{P(6)-3P(5)q-11P(1)yq^2+5P(4)q^3-7P(3)q^6 +9P(2)q^{10}\} \pmod{13}, \tag{49}$$

and Eqs. (20) and (39) give

$$(y;y)_\infty^2=P^2(0)\left\{-y^2\frac{P(1)P(3)P(4)}{P(2)P(5)P(6)}+\frac{P(2)P(5)P(6)}{P(1)P(3)P(4)}-3y\right\}. \tag{50}$$

We have

$$\begin{aligned}\prod_{r=1}^{\infty}(1-q^r)^{10}\equiv P^2(0)\{&-12P(2)P(4)P(5)P(6)-10P^2(3)P(4)P(6)q\\ &-4P(1)P(5)P(6)^2q^2-4P(2)P(3)P(5)P(6)q^3\\ &+12P(2)P(3)P^2(5)q^4+[-2P(2)P(3)P(4)P(6)\\ &+6P(1)P(4)P(5)P(6)-8yP(1)P(2)P(3)P(5)]q^5\\ &-12y^2P^2(1)P(2)P(3)q^6-4P(1)P(2)P(3)P(4)q^7\\ &+4P(1)P^2(4)P(5)q^8+10P^2(2)P(4)P(6)q^9\\ &-12P(1)P(3)P(4)P(6)q^{10}+10P(1)P(3)P(4)P(5)q^{11}\\ &-10P(1)P(2)P(5)P(6)q^{12}\} \pmod{13}.\end{aligned} \tag{51}$$

We obtain Eq. (51) by taking the same steps with Hirschhorn in [5]. These steps are very elementary and simple. Now, we consider the following congruence:

$$\begin{aligned}\left\{\prod(1-q^r)^{10}\right\}\left\{\prod(1-q^r)^3\right\}&\equiv\prod(1-q^r)^{13}\equiv\prod(1-y^r)\\ &\equiv P(0)P(1)P(2)P(3)P(4)P(5)P(6) \pmod{13}.\end{aligned}$$

This congruence gives

$$\begin{aligned}P^3(0)\{&-6y^2P^2(1)P(3)P(4)P(5)+3yP^2(2)P(3)P(5)P(6)\\ &+2y^2P^2(3)P(1)P(2)P(4)+5yP^2(4)P(1)P(3)P(6)\\ &+4yP^2(5)P(1)P(2)P(6)+P^2(6)P(2)P(4)P(5)\}\end{aligned}$$

$$\equiv P(0)P(1)P(2)P(3)P(4)P(5)P(6) \pmod{13}. \tag{52}$$

The congruence (52) can be written in the following form:

$$P^2(0)\left\{-6y^2\frac{P(1)}{P(2)P(6)}+3y\frac{P(2)}{P(1)P(4)}+2y^2\frac{P(3)}{P(5)P(6)}+5y\frac{P(4)}{P(2)P(5)}\right.$$
$$\left.+4y\frac{P(5)}{P(3)P(4)}+\frac{P(6)}{P(1)P(3)}\right\}\equiv 1 \pmod{13}. \tag{53}$$

Finally, we are ready to prove Theorem 1. The coefficient of $q^0$ on the left-hand side of Eq. (47) is

$$P^2(0)\{-6y^2P^2(1)P(3)P(4)P(5)+3yP^2(2)P(3)P(5)P(6)$$
$$+2y^2P^2(3)P(1)P(2)P(4)+5yP^2(4)P(1)P(3)P(6)+4yP^2(5)P(1)P(2)P(6)$$
$$+P^2(6)P(2)P(4)P(5)\}/P(1)P(2)P(3)P(4)P(5)P(6) \pmod{13}.$$

From Eq. (52), this coefficient is congruent to 1 modulo 13. It remains to show that the other coefficients of $q^k$ on the left-hand side of Eq. (47) are congruent to 0 modulo 13. These 12 coefficients fall into two equal classes, one being given by the coefficients of $q$, $q^3$, $q^4$, $q^9$, $q^{10}$ and $q^{12}$, and the other by those of $q^2$, $q^5$, $q^6$, $q^7$, $q^8$ and $q^{11}$. We shall prove that the coefficient of $q^2$ is congruent to 0 modulo 13, the proofs of the other ones being exactly similar.

The coefficients of $q^2$ on the left-hand side of Eq. (47) is

$$C_1+C_2,$$

where

$$C_1=P^2(0)\left\{-3\frac{P^2(2)P(6)}{P^2(1)P(3)P(4)}+2\frac{P(4)P(6)}{P(1)P(3)P(5)}+3\frac{P(2)P(5)P(6)}{P^2(3)P(1)P(4)}\right.$$
$$-2y^2\frac{P(1)P(4)}{P(2)P(5)P(6)}+5y\frac{P(2)}{P(1)P(5)}+2\frac{P^2(5)}{P^2(4)P(1)}$$
$$\left.+3\frac{P(3)}{P(1)P(2)}\right\} \pmod{13}$$

and

$$C_2=P(0)(y;y)_\infty^{11}\left\{6y\frac{P(4)}{P(5)}+4y\frac{P^2(2)}{P(1)P(4)}-4\frac{P(5)P(6)}{P^2(3)}\right.$$
$$\left.-5\frac{P(3)P(5)}{P(1)P(4)}-2y^2\frac{P(1)}{P(6)}\right\} \pmod{13}.$$

Equation (13) gives

$$P^{12}(0)P^{13}(1)P^{13}(2)P^{13}(3)P^{13}(4)P^{13}(5)P^{13}(6)\equiv 1 \pmod{13}. \tag{54}$$

After multiplying $C_1$ by the expression on the left-hand side of (54) and $C_2$ by the expression on the left-hand side of (53), the coefficients of $q^2$ becomes

$$
\begin{aligned}
&(y;y)_\infty^{11}\Bigg\{P(0)(y;y)_\infty^2\Bigg[-3\frac{P^2(2)P(6)}{P^2(1)P(3)P(4)}+2\frac{P(4)P(6)}{P(1)P(3)P(5)}\\
&\quad+3\frac{P(2)P(5)P(6)}{P^2(3)P(1)P(4)}-2y^2\frac{P(1)P(4)}{P(2)P(5)P(6)}+5y\frac{P(2)}{P(1)P(5)}+2\frac{P^2(5)}{P^2(4)P(1)}\\
&\quad+3\frac{P(3)}{P(1)P(2)}\Bigg]+P^3(0)\Bigg[-6y^2\frac{P(1)}{P(2)P(6)}+3y\frac{P(2)}{P(1)P(4)}+2y^2\frac{P(3)}{P(5)P(6)}\\
&\quad+5y\frac{P(4)}{P(2)P(5)}+4y\frac{P(5)}{P(3)P(4)}+\frac{P(6)}{P(1)P(3)}+6y\frac{P(4)}{P(5)}+4y\frac{P^2(2)}{P(1)P(4)}\\
&\quad-4\frac{P(5)P(6)}{P^2(3)}-5\frac{P(3)P(5)}{P(1)P(4)}-2y^2\frac{P(1)}{P(6)}\Bigg]\Bigg\} \pmod{13}.
\end{aligned}
$$

Using Eq. (50) and the definitions in Eqs. (18) and (20), we have

$$
\begin{aligned}
&\frac{y^2P^3(0)(y;y)_\infty^{11}}{P(3)}\big\{-2x_1x_2x_4^{10}x_5^{11}-3x_1x_3x_4^9x_5^{10}-3x_4^{13}x_5^{15}-x_2^2x_3x_4^{10}x_5^{11}\\
&\quad-2x_4^{11}x_5^9-x_4^{10}x_5^9-x_2x_4^{10}x_5^{11}+4x_4^9x_5^9-5x_3x_4^9x_5^9-5x_4^{11}x_5^{12}-6x_1x_4^{10}x_5^{10}\\
&\quad+3x_2x_3x_4^{10}x_5^{11}-2x_2x_4^{10}x_5^{10}+4x_1x_4^{10}x_5^{11}+4x_2x_3x_4^9x_5^9+6x_2x_4^{11}x_5^{10}\\
&\quad-2x_1x_3x_4^9x_5^9-x_3x_4^{10}x_5^9-5x_2^2x_3x_4^{10}x_5^{10}-x_2x_4^{12}x_5^{14}-4x_2x_3x_4^{10}x_5^9\\
&\quad-2x_1x_2x_4^{10}x_5^9+\alpha^2\theta^2\zeta\big[3x_5^9x_4^{10}x_2-2x_5^{10}x_4^9x_2+5x_5^9x_4^9x_2\big]\\
&\quad+\theta^3\alpha\big[2x_5^9x_4^{10}x_2+3x_5^9x_4^{10}x_2^2+2x_2x_4^{10}x_5^{1}1-2x_4^{10}x_2^2x_5^{10}\big]\big\} \pmod{13}. \qquad (55)
\end{aligned}
$$

After substituting Eqs. (24) and (25) into (55), and using Algorithm 1, the expression between the curly brackets in (55), therefore, the coefficient of $q^2$ on the left-hand side of Eq. (47) is congruent to 0 modulo 13. Similarly, it can be shown that the other coefficients of $q$ are congruent to 0 modulo 13. So, the proof is completed.

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