Results in Applied Mathematics 23 (2024) 100472

Contents lists available at ScienceDirect

Results in Applied Mathematics

journal homepage: www.elsevier.com/locate/results-in-applied-mathematics

Oscillatory properties for Emden–Fowler type difference equations with oscillating coefficients ☆

Yaşar Bolat [a], Murat Gevgeşoğlu [a], George E. Chatzarakis [b,*]

[a] Faculty of Science and Arts, Kastamonu University, 037100 Kastamonu, Turkey
[b] School of Pedagogical & Technological Education (ASPETE), Marousi 15122, Athens, Greece

ARTICLE INFO

MSC:
39A10
34C10
39A21

Keywords:
Oscillation
Emden–Fowler type difference equation
Neutral difference equation

ABSTRACT

In this paper, we give new criteria on the oscillation of the fourth-order Emden–Fowler type delay difference equation with oscillating coefficients of the form

$$\Delta W_n + r_n y^{\beta}_{n-\tau} = 0, n \geq n_0,$$

where $W_n = p_n\left(\Delta^3 v_n\right)^{\alpha}$ and $v_n = y_n + q_n y_{n-\sigma}$. For this we use the Riccati transformation method and the comparison method. Also we give some examples to illustrate our results.

1. Introduction

It is known that neutral differential equations with delay and neutral difference equations with delay have wide applications. Emden–Fowler equations have a special place; see, e.g., the papers [1–3] for more details. In dynamical models, delay and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [4]. Therefore, we were interested in examining such an equation. We could not find difference equations regarding the oscillatory solutions of the fourth-order Emden–Fowler equation in the literature, but there are a scarce of studies on the fourth order Emden–Fowler differential equation; see, e.g., the papers [5,6].

There are very few studies in the literature on the oscillation of fourth-order difference equations. Researchers can see these in references [6–9]. In this work, we investigate the oscillatory behavior of solutions of Emden–Fowler type fourth-order neutral difference equation with oscillating coefficients of the form

$$\Delta W_n + r_n y^{\beta}_{n-\tau} = 0,\ n \in \mathbb{N}, \tag{1.1}$$

where $W_n = p_n\left(\Delta^3 v_n\right)^{\alpha}$ and $v_n = y_n + q_n y_{n-\sigma}$.

We will assume that the following assumptions hold throughout the study:

H_1 : $q_n : \mathbb{N} \to \mathbb{R}$ and it is an oscillating function with $\lim_{n\to\infty} q_n = 0$, and there exists $q_0 > 0$ such that $q_n \leq q_0$,

H_2 : τ and σ are positive integers such that $\tau < \sigma$, and $\sigma \leq n$, $\tau \leq n$,

H_3 : $r_n > 0$ and $0 < p_n \leq p_0$, $\Delta p_n \geq 0$,

H_4 : α, β are ratios of odd positive integers.

By a solution of Eq. (1.1), we mean any function $y_n : \mathbb{Z} \to \mathbb{R}$, which is defined for all $n \geq \min_{i\geq 0}\{i-\sigma, i-\tau\}$, and satisfies Eq. (1.1) for sufficiently large n. We consider only such solutions which are nontrivial, for all large n. As it is customary, a solution $\{y_n\}$ is

☆ This paper is in final form and no version of it will be submitted for publication elsewhere.

* Corresponding author.
E-mail addresses: ybolat@kastamonu.edu.tr (Y. Bolat), mgevgesoglu@kastamonu.edu.tr (M. Gevgeşoğlu), geaxatz@otenet.gr (G.E. Chatzarakis).

https://doi.org/10.1016/j.rinam.2024.100472
Received 27 April 2024; Received in revised form 14 June 2024; Accepted 18 June 2024
Available online 5 July 2024

said to be oscillatory if the terms y_n of the sequence are not eventually positive or not eventually negative. Otherwise, the solution is called nonoscillatory. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory. In this paper, we restrict our attention to real-valued solutions y_n.

2. Auxiliary lemmas

To obtain our main results, we need the following lemmas. The first of these is the discrete analog of the well-known Kiguradze's lemma.

Lemma 1 ([10]). *Let x_n be defined for $n \geq n_0 \in \mathbb{N}$, and $x_n > 0$ with $\Delta^m x_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer k, $0 \leq k \leq m$ with $(m+k)$ odd for $\Delta^m x_n \leq 0$ and $(m+k)$ even for $\Delta^m x_n \geq 0$ such that*

(i) $k \leq m-1$ *implies* $(-1)^{m+i}\Delta^i x_n > 0$ *for all* $n \geq n_0$, $k \leq i \leq m-1$,
(ii) $k \geq 1$ *implies* $\Delta^i x_n > 0$ *for all large* $n \geq n_0$, $1 \leq i \leq k-1$.

Lemma 2 ([10]). *Let x_n be defined for $n \geq n_0$, and $x_n > 0$ with $\Delta^m x_n \leq 0$ for $n \geq n_0$ and not identically zero. Then, there exists a large integer $n_1 \geq n_0$ such that*

$$x_n \geq \frac{1}{(m-1)!}(n-n_1)^{m-1}\Delta^{m-1}x_{2^{m-k-1}n}, \quad n \geq n_1$$

where k is defined as in Lemma 1. Further, if x_n is increasing, then

$$x_n \geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x_n, \quad n \geq 2^{m-1}n_1.$$

Lemma 3 ([11, Lemmas 1 and 2]). *Let A, $B \geq 0$. Then* $(A+B)^\eta \leq \begin{cases} 2^{\eta-1}(A^\eta + B^\eta) & \text{for } \eta \geq 1 \\ A^\eta + B^\eta & \text{for } 0 \leq \eta \leq 1. \end{cases}$

3. Main results

Theorem 1. *Assume that α is the ratio of odd integers. If the following inequalities (c_1) $\beta \leq 1$ is the ratio of odd integers and*

$$\Delta\omega_n + \frac{\left(\frac{\mu}{6}n^3\right)^\beta \rho_n}{\left(p_0 + q_0^\beta p_0\right)^{\frac{\beta}{\alpha}}}\omega_{n-k}^{\frac{\beta}{\alpha}} \leq 0,$$

or (c_2) $\beta > 1$ is the ratio of odd integers and

$$\Delta\omega_n + \frac{\left(\frac{\mu}{12}n^3\right)^\beta \rho_n}{2\left(p_0 + q_0^\beta p_0\right)^{\frac{\beta}{\alpha}}}\omega_{n-k}^{\frac{\beta}{\alpha}} \leq 0$$

where $\rho_n = \min\{r_{n-\sigma+\tau}, r_{n+\tau}\}$ and $k = \tau - \sigma \leq n$ ($k \in \mathbb{N}$), are oscillatory, then every bounded solution of Eq. (1.1) is oscillatory.

Proof. Let y_n be a nonoscillatory solution of (1.1) on $\mathbb{N}$. Without loss of generality, assume that y_n is eventually positive (the proof is similar when y_n is eventually negative). Then there exists an $n_1 \geq n_0$ such that $y_n > 0$, $y_{n-\sigma} > 0$, and $y_{n-\tau} > 0$ for $n \geq n_1$. Further, we assume that y_n does not tend to zero as $n \to \infty$. From (1.1) we have

$$\Delta W_n = -r_n y_{n-\tau}^\beta \leq 0. \tag{3.1}$$

That is, $\Delta W_n \leq 0$. It follows that W_n is strictly monotone and eventually of constant sign. Since q_n is an oscillating sequence and $\lim_{n\to\infty} q_n = 0$, there exists an $n_2 \geq n_1$ such that we have $\upsilon_n > 0$ for $n \geq n_2$. From (3.1) we have $\Delta(p_n\left(\Delta^3\upsilon_n\right)^\alpha) \leq 0$. Hence since $\Delta p_n > 0$, we have

$$\Delta\upsilon_n > 0,\ \Delta^2\upsilon_n > 0,\ \Delta^3\upsilon_n > 0,\ \Delta^4\upsilon_n < 0 \text{ and } \Delta\left(p_n\left(\Delta^3\upsilon_n\right)^\alpha\right) \leq 0 \text{ for } n \geq n_2. \tag{3.2}$$

We will make the proofs for both cases separately. From(1.1) we get

$$\Delta\left(p_{n+\tau}\left(\Delta^3\upsilon_{n+\tau}\right)^\alpha\right) + r_{n+\tau}y_n^\beta = 0. \tag{3.1*}$$

(c_1) By Lemma 3 and the definition of υ we obtain

$$\begin{aligned} \upsilon_n^\beta &= \left(y_n + q_n y_{n-\sigma}\right)^\beta \\ &\leq y_n^\beta + q_0^\beta y_{n-\sigma}^\beta. \end{aligned} \tag{3.3}$$

From (3.1*) and (3.3) we obtain

$$
\begin{aligned}
0 &= \Delta\left(p_{n+\tau}\left(\Delta^3 v_{n+\tau}\right)^{\alpha}\right) + r_{n+\tau} y_n^{\beta} + q_0^{\beta}\left(\Delta\left(p_{n-\sigma+\tau}\left(\Delta^3 v_{n-\sigma+\tau}\right)^{\alpha}\right) + r_{n-\sigma+\tau} y_{n-\sigma}^{\beta}\right) \\
&= \Delta\left(p_{n+\tau}\left(\Delta^3 v_{n+\tau}\right)^{\alpha}\right) + q_0^{\beta}\Delta\left(p_{n-\sigma+\tau}\left(\Delta^3 v_{n-\sigma+\tau}\right)^{\alpha}\right) + q_0^{\beta} r_{n-\sigma+\tau} y_{n-\sigma}^{\beta} + r_{n+\tau} y_n^{\beta} \\
&\geq \Delta\left(p_{n+\tau}\left(\Delta^3 v_{n+\tau}\right)^{\alpha}\right) + q_0^{\beta}\Delta\left(p_{n-\sigma+\tau}\left(\Delta^3 v_{n-\sigma+\tau}\right)^{\alpha}\right) + \rho_n v_n^{\beta}
\end{aligned}
$$

which gives

$$\Delta\left(p_{n+\tau}\left(\Delta^3 v_{n+\tau}\right)^{\alpha}\right) + q_0^{\beta}\Delta\left(p_{n-\sigma+\tau}\left(\Delta^3 v_{n-\sigma+\tau}\right)^{\alpha}\right) + \rho_n v_n^{\beta} \leq 0. \tag{3.4}$$

Since $\Delta v_n > 0$, we find $\lim_{n\to\infty} v_n \neq 0$, and by Lemma 2 we obtain

$$v_n \geq \frac{\mu}{3!} n^3 \Delta^3 v_n, \tag{3.5}$$

where $\mu \in (0,1)$. Combining (3.4) and (3.5), we see that

$$\Delta\left(p_{n+\tau}\left(\Delta^3 v_{n+\tau}\right)^{\alpha}\right) + q_0^{\beta}\Delta\left(p_{n-\sigma+\tau}\left(\Delta^3 v_{n-\sigma+\tau}\right)^{\alpha}\right) + \rho_n\left(\frac{\mu}{6} n^3\right)^{\beta}\left(\Delta^3 v_n\right)^{\beta} \leq 0. \tag{3.6}$$

Setting

$$\omega_n = p_{n+\tau}\left(\Delta^3 v_{n+\tau}\right)^{\alpha} + q_0^{\beta} p_{n-\sigma+\tau}\left(\Delta^3 v_{n-\sigma+\tau}\right)^{\alpha},$$

we easily see that

$$\omega_{n-\tau+\sigma} \leq \left(p_0 + q_0^{\beta} p_0\right)\left(\Delta^3 v_n\right)^{\alpha}.$$

From (3.6) we find

$$\Delta\omega_n + \frac{\left(\frac{\mu}{6} n^3\right)^{\beta}\rho_n}{\left(p_0 + q_0^{\beta} p_0\right)^{\frac{\beta}{\alpha}}}\omega_{n-k}^{\frac{\beta}{\alpha}} \leq 0,$$

which has positive solutions. This contradicts (c_1).
(c_2) By Lemma 3 and the definition of v we obtain

$$
\begin{aligned}
v_n^{\beta} &= \left(y_n + q_n y_{n-\sigma}\right)^{\beta} \\
&\leq 2^{\beta-1}\left(y_n^{\beta} + q_0^{\beta} y_{n-\sigma}^{\beta}\right).
\end{aligned} \tag{3.7}
$$

We get with similar operations (3.3)–(3.6) the following result

$$\Delta\omega_n + \frac{\left(\frac{\mu}{12} n^3\right)^{\beta}\rho_n}{2\left(p_0 + q_0^{\beta} p_0\right)^{\frac{\beta}{\alpha}}}\omega_{n-k}^{\frac{\beta}{\alpha}} \leq 0$$

which has positive solutions. This contradicts (c_2).

Corollary 1. *Assume that $\alpha = 1$. If*

(c_1') *$\beta \leq 1$ is the ratio of odd integers and*

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} i^3\rho_i\right] > \frac{k^k}{(k+1)^{k+1}}\frac{\left(p_0 + q_0^{\beta} p_0\right)^{\frac{\beta}{\alpha}}}{\left(\frac{\mu}{6}\right)^{\beta}}$$

or

(c_2') $\beta \geq 1$ is the ratio of odd integers and

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} i^3\rho_i\right] > \frac{k^k}{(k+1)^{k+1}}\frac{2\left(p_0 + q_0^{\beta} p_0\right)^{\frac{\beta}{\alpha}}}{\left(\frac{\mu}{12}\right)^{\beta}}$$

then, according to Theorem 2 [4, Section 3], every bounded solution of Eq. (1.1) is oscillatory.

Theorem 2. *Let $q_0 < 1$ and $\alpha > \beta$. If for some $\mu \in (0,1)$,*

$$\Delta\varphi_n + (1-q_0)^{\beta}\left(\frac{\mu(n-\sigma)^3}{6p_{n-\sigma}^{\frac{1}{\alpha}}}\right)^{\beta} r_n\varphi_{n-\sigma}^{\frac{\beta}{\alpha}} \leq 0 \tag{c_3}$$

is oscillatory, then (1.1) is oscillatory.

Proof. By the definition of v we find

$$y_n \geq v_n - q_0 y_{n-\sigma} \geq v_n - q_0 v_{n-\sigma} \geq (1-q_0)v_n$$

which, consider together with (1.1), gives

$$\Delta\left(p_n\left(\Delta^3 v_n\right)^\alpha\right) + r_n(1-q_0)^\beta v_{n-\sigma}^\beta \leq 0. \tag{3.8}$$

From Lemma 2 we obtain

$$v_n \geq \frac{\mu}{6} n^3 \Delta^3 v_n. \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\Delta\left(p_n\left(\Delta^3 v_n\right)^\alpha\right) + r_n(1-q_0)^\beta \left(\frac{\mu(n-\sigma)^3}{6}\right)^\beta \left(\Delta^3 v_{n-\sigma}\right)^\beta \leq 0. \tag{3.10}$$

If we set $\varphi_n = p_n\left(\Delta^3 v_n\right)^\alpha$ in (3.10), from (3.10) we obtain

$$\Delta\varphi_n + (1-q_0)^\beta \left(\frac{\mu(n-\sigma)^3}{6p_{n-\sigma}^{\frac{1}{\alpha}}}\right)^\beta r_n \varphi_{n-\sigma}^{\frac{\beta}{\alpha}} \leq 0,$$

which has positive solutions. This contradicts (c_3).

Corollary 2. *Let $q_0 < 1$ and $\alpha = \beta$. If*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} \frac{(s-\sigma)^{3\beta}}{p_{s-\sigma}^{\frac{\beta}{\alpha}}} r_s > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}} \frac{6^\beta}{\mu^\beta(1-q_0)^\beta},$$

then, according to Theorem 2 [4, Section 3], every bounded solution of Eq. (1.1) is oscillatory.

Example 1. Consider the difference equation

$$\Delta(p_n\left(\Delta^3\left(y_n + q_n y_{n-\sigma}\right)\right)^\alpha) + r_n y_{n-\tau}^\beta = 0 \tag{3.11}$$

where $p_n = 3 - \frac{1}{n}$, $q_n = \left(-\frac{1}{2}\right)^n$, $\tau = 3$, $\sigma = 1$, $\alpha = 1$, $\beta = \frac{1}{3}$, $q_0 = \frac{1}{4}$, $p_0 = 3$,

$r_n = \left(3 - \frac{1}{n}\right)\left(8 + (-1)^{n-1}\left(\frac{1}{2}\right)^{n+3}\right) - \left(3 - \frac{1}{n+1}\right)\left(-8 + (-1)^{n-1}\left(\frac{1}{2}\right)^{n+4}\right)$. All conditions of Theorem 1 are satisfied.

Moreover, $\frac{k^k}{(k+1)^{k+1}} \frac{\left(p_0 + q_0^\beta p_0\right)^{\frac{1}{\alpha}}}{\left(\frac{\mu}{6}\right)^\beta} = \frac{1}{4} \frac{\left(3 + \left(\frac{1}{4}\right)^{\frac{1}{3}} 3\right)}{\left(\frac{1}{12}\right)^{\frac{1}{3}}} \approx 2.798\,8$ and

$$\begin{aligned}\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} i^3 \rho_i\right] &= \liminf_{n\to\infty}\left[\sum_{i=n-1}^{n-1} i^3 \rho_i\right] \\ &= \liminf_{n\to\infty}\left[(n-1)^3\left(3 - \frac{1}{n-1}\right)\right] = \infty\end{aligned}$$

$\infty > 2.798\,8$.

Hence conditions of Corollary 1 are satisfied. Therefore all solutions of (3.11) are oscillatory. One of these solutions is $y_n = (-1)^n$.

Example 2. Consider the difference equation

$$\Delta(p_n\left(\Delta^3\left(y_n + q_n y_{n-\sigma}\right)\right)^\alpha) + r_n y_{n-\tau}^\beta = 0 \tag{3.12}$$

where $p_n = 2 - \frac{1}{n}$, $q_n = \left(-\frac{1}{2}\right)^n$, $\tau = 3$, $\sigma = 1$, $\alpha = 3$, $\beta = 3$, $q_0 = \frac{1}{4}$

$$r_n = \left(2 - \frac{1}{n}\right)\left(8 + (-1)^{n-1}\left(\frac{1}{2}\right)^{n+3}\right)^3 - \left(2 - \frac{1}{n+1}\right)\left(-8 + (-1)^{n-1}\left(\frac{1}{2}\right)^{n+4}\right)^3.$$

All conditions of Theorem 1 are satisfied.

Moreover $\frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}} \frac{6^\beta}{\mu^\beta(1-q_0)^\beta} = \frac{1}{4} \frac{6^3}{\left(\frac{1}{2}\right)^3\left(\frac{3}{4}\right)^3} = 1024$,

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} \frac{(s-\sigma)^{3\beta}}{p_{s-\sigma}^{\frac{\beta}{\alpha}}} r_s = \liminf_{n\to\infty} \sum_{s=n-1}^{n-1} \frac{(s-1)^9}{2 - \frac{1}{s-1}} r_s$$

$$= \liminf_{n\to\infty} \frac{(n-2)^9}{2-\frac{1}{n-2}} r_{n-1} = \infty$$

$\infty > 1024$.

Hence conditions of Corollary 1 are satisfied. Therefore all solutions of (3.12) are oscillatory. One of these solutions is $y_n = (-1)^n$.

4. Conclusion

In this paper, we consider the oscillation and asymptotic behavior of a class of fourth-order Emden–Fowler type neutral difference equations. Using the Riccati transformation and comparison method, we establish new oscillation conditions for the solutions of fourth-order neutral type difference equations. Our results unify and extend some known results for difference equations. In a future work, we will discuss the oscillatory behavior of these equations by using comparison technique with second-order equations.

Declaration of competing interest

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data availability

No data was used for the research described in the article.

Acknowledgment

The third author was supported by the special Account for Research of ASPETE through the funding program "Strengthening research of ASPETE faculty members".

References

[1] Džurina J, Grace SR, Jadlovská I, Li T. Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. Math Nachr 2020;293(5):910–22.
[2] Li T, Rogovchenko YV. Oscillation of second-order neutral differential equations. Math Nachr 2015;288(10):1150–62.
[3] Li T, Rogovchenko YV. Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations. Monatsh Math 2017;184(3):489–500.
[4] Angew Z. Combining effects ensuring boundedness in an attraction–repulsion chemotaxis model with production and consumption. Math Phys 2023;74(3):1–21, Art. 109.
[5] Li T, Rogovchenko YV. On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations. Appl Math Lett 2017;67:53–9.
[6] Tripathy K. Oscillation of fourth order nonlinear neutral difference equations I. Math Slovaca 2008;58(2):221–40. http://dx.doi.org/10.2478/s12175-008-0069-0.
[7] Tripathy K. Oscillation of fourth order nonlinear neutral difference equations II. Math Slovaca 2008;58(5):581–604. http://dx.doi.org/10.2478/s12175-008-0095-y.
[8] Vimala R, Kodeeswaran R, Cep R, Krishnasamy MJI, Awasthi M, Santhakumar G. Oscillation of nonlinear neutral delay difference equations of fourth order. Mathematics 2023;11:1370. http://dx.doi.org/10.3390/math11061370.
[9] Alatwi M, Moaaz O, Askar SS, Alshamrani AM, Elabbasy EM. Fourth-order Emden fowler neutral differential equations: Investigating some qualitative properties of solutions. Symmetry 2023;15:1446. http://dx.doi.org/10.3390/sym15071446.
[10] Agarwal RP. Difference equation and inequalities. New York: Marker Dekker; 2000.
[11] Baculikova B, Dzurina J. Oscillation theorems for second-order nonlinear neutral differential equations. Comput Math Appl 2011;62:4472–8.