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ORIGINAL PAPER

Nonlinear wave dynamics: soliton and complexiton solutions of the generalized shallow-water-like equation in coastal and plasma physics

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Abstract: This research specializes in obtaining new solitary wave solutions to a generalized shallow water type equation. This equation is a very important model in fluid dynamics that characterizes different dynamics of shallow water waves under varying conditions. In this work, the Auto-Bäcklund transformation associated with the equation was established by using the extended homogeneous balance technique in conjunction with the symbolic computing powers of Maple. The latter approach facilitated the systematic derivation of several explicit solutions, which serve as a basis for better understanding the dynamics of the equation. Furthermore, by utilizing the extended transformed rational function approach, we explore a broader class of solutions and successfully identify complexiton solutions—special types of wave structures that exhibit both solitary and oscillatory characteristics. To provide deeper insight into the obtained results, we present three-dimensional graphical representations that highlight the physical significance and dynamic behavior of the solutions. These visualizations offer a clearer understanding of the wave phenomena described by the generalized shallow-water-like equation and highlight the effectiveness of our analytical techniques in solving nonlinear partial differential equations.

Keywords: Auto-Bäcklund transformation; Homogeneous balance technique; Extended transformed rational function method; Soliton; Hirota direct method; Complexiton solutions

1. Introduction

Nonlinear partial differential equations (NPDEs) are mathematical models that are used to describe global phenomena. NPDEs appear in a variety of scientific and engineering applications. The same NPDE study has been conducted in a variety of algebraic approaches. Nonlinear evolution equations (NLEEs) are a subtype of NPDEs that explain the evolution of a system over time. NLEEs and their solutions are now broadly acknowledged as the most comprehensive technique to characterize the physical significance of nonlinear phenomena.

Soliton solutions are especially notable in the study of nonlinear physical sciences, optical fibers, chemical kinematics, and other emerging phenomena in science and engineering [1–3]. There are several intriguing techniques for investigating soliton-type exact solutions [4] such as the similarity reduction method [5], the Hirota Bilinear method [6], the Bäcklund Transformation [7, 8], the inverse scattering method [9], the exp-function method [10], the unified and generalized unified method [11], the tanh and extended tanh-function method [12], the Wronskian method [13], the simplified Hirota's method [14], the enhanced modified extended tanh-expansion approach [15], the compact finite difference method [16], the

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modified Kudryashov method [17], the new extended auxiliary equation method [18], the inverse (G'/G) method [20], the Padé approximation method [21], the newly proposed extended generalized approach [22], the simplified Hirota method [23], the unified Riccati equation expansion method [24], the monotone iterative and quasi-linearization method [25], the piecewise calculation scheme for the unconditionally stable Chebyshev finite-difference time-domain method [26], etc. Yuan et al. [27, 28] move the subject area forward by conducting systematic submerged experiments varying jet pressure, nozzle geometry, standoff distance and submergence depth to chart ice-failure regimes and to determine visible failure modes under submerged conditions.

Shallow water equations are a basic yet convincing set of equations that are commonly used in meteorological or ocean modeling applications. In this paper, we will consider a generalized shallow water-like equation. Using Maple, Mathematica and an adaptation of the homogeneous balance technique, we find the Auto-Bäcklund transformation of the governed equation. A variety of solitary wave solutions have been obtained using the transformations outlined in this article. Also, the extended transformed rational function technique, with the aid of Hirota bilinear form, is employed to acquire complexiton solutions. The obtained solutions are suitable for describing characteristic behaviors of shallow-water waves in applied domains such as coastal engineering, tsunami modeling, internal wave dynamics, and channel/reservoir hydrodynamics.

This article is organized as follows. Section 2 presents the Auto-Bäcklund transformation of Eq. (1) by employing the homogeneous balance technique. Several solitary-wave solutions, including rational, trigonometric, and hyperbolic function types, are derived in Section 3. The extended transformed rational function technique is utilized in Section 4, and the conclusions are summarized in Section 5.

2. Auto-bäcklund transformation for the generalized shallow water-like equation

In [29], we find a novel version of the (3+1)-dimensional generalized shallow water (SW) equation:

$$\frac{\partial^4 v}{\partial x^3 \partial y} + 3\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial y} + 3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial t} - \frac{\partial^2 v}{\partial x \partial z} = 0. \tag{1}$$

With logarithmic transformation

$$v = 2(lng)_x, \tag{2}$$

whose bilinear form is

$$(\nabla_x^3 \nabla_y - \nabla_y \nabla_t - \nabla_x \nabla_z) g \cdot g = 0, \tag{3}$$

where ∇ is the Hirota bilinear operator. The SW-like equation, motivated by the traditional bilinear form, is briefly examined in this section.

$$(\nabla_{q,x}^3 \nabla_{q,y} - \nabla_{q,y} \nabla_{q,t} - \nabla_{q,x} \nabla_{q,z}) g \cdot g = 0. \tag{4}$$

Here, ∇_q operator has the form given in [30, 31].

$$\begin{aligned}
&(\nabla_{q,x_1}^{\mu_1} \cdots \nabla_{q,x_n}^{\mu_n} g \cdot h)(x_1, \ldots, x_n) \\
&= \left(\begin{array}{c} \prod_{k=1}^{n} \left(\frac{\partial}{\partial x_k} + \beta_q \frac{\partial}{\partial'_k}\right)^{\mu_k} g(x_1, \ldots, x_n) \\ h(x'_1 x', \ldots, x'_n)) \end{array} \right) \Bigg|_{x'_1 = x_1, \ldots, x'_n = x_n} \\
&= \prod_{k=1}^{n} \sum_{l_k}^{\mu_k} \beta_q^{l_k} \binom{\mu_k}{l_k} \frac{\partial^{\mu_k - l_k}}{\partial x_k^{\mu_k - l_k}} g(x_1, \ldots, x_n) \\
&\quad \times \frac{\partial^{l_k}}{\partial x_k^{l_k}} h(x_1, \ldots, x_n).
\end{aligned} \tag{5}$$

Here, $\mu_1, \mu_2, \ldots \mu_n$ are non-negative values, and the nth power of β is calculated as follows:

$\beta_q^n = (-1)^r(n)$, where $n \equiv r(n) modq$.

By the definition of bilinear operator ∇_q, we can assume that $q = 3$. Then Eq. (4) is equal to the following formula:

$$\begin{aligned}
&(\nabla_{3,x}^3 \nabla_{3,y} - \nabla_{3,y} \nabla_{3,t} - \nabla_{3,x} \nabla_{3,z}) g \cdot g \\
&= 6 g_{xx} g_{xy} + 2 g_y g_t - 2 g_{yt} g + 2 g_x g_z - 2 g_{xz} g.
\end{aligned} \tag{6}$$

According to the generalized bilinear form, we can express Eq. (1) under the transformation Eq. (2):

$$\begin{aligned}
&\frac{(\nabla_{3,x}^3 \nabla_{3,y} - \nabla_{3,y} \nabla_{3,t} - \nabla_{3,x} \nabla_{3,z}) g \cdot g}{g^2} \\
&= \frac{3}{2} v_x v_y + \frac{3}{4} v v_x + \frac{3}{4} v^2 v_y \partial_x^{-1} v_y + \frac{3}{8} v^3 \partial_x^{-1} v_y - \partial_x^{-1} v_{yt} \\
&\quad - v_z = 0.
\end{aligned} \tag{7}$$

When we apply the homogeneous balance method, we can express the solution of Eq. (1) as [32]:

$$v(x, y, z, t) = g'(w) w_x + v_0(x, y, z, t), \tag{8}$$

where g and w are the two unknown functions, and $v_0(x, y, z, t)$ is the seed solution given in Eq. (1). This fundamental solution serves as a starting point for deriving various other solutions. By substituting Eq. (8) into Eq. (1), we obtain the following equation.

$$
\begin{aligned}
&[g^{(5)}+6g''g''']w_yw_x^4+[(6w_yw_x^2w_{xx}+4w_x^3w_{xy})g^{(4)}\\
&\quad+(3w_{xx}w_yw_x^2+3w_{xy}w_x^3)g'g'''\\
&\quad+(6w_x^3w_{xy}+12w_x^2w_{xx}w_y)(g'')^2]\\
&\quad+[(-w_tw_xw_y+6w_x^2w_{xxy}+3w_yw_{xx}^2+3w_x^3u_{0y}\\
&\quad+3w_yw_x^2u_{0x}\\
&\quad+12w_xw_{xx}w_{xy}+4w_yw_xw_{xxx}-w_zw_x^2)g'''\\
&\quad+(3w_x^2w_{xxy}+3w_{xx}^2w_y\\
&\quad+15w_xw_{xx}w_{xy}+3w_xw_yw_{xxx})g'g'']\\
&\quad+[(3w_{xxx}w_{xy}+3w_{xx}w_{xxy})(g')^2\\
&\quad+(-w_zw_{xx}+3w_x^2u_{0xy}+6w_{xx}w_{xxy}+4w_{xy}w_{xxx}\\
&\quad+4w_xw_{xxxy}+w_yw_{xxxx}\\
&\quad+3w_xw_yv_{0xx}+6w_xw_{xy}v_{0x}+3w_yw_{xx}v_{0x}\\
&\quad+9w_xw_{xx}v_{0y}-2w_xw_{xz}-w_tw_x\\
&\quad-w_tw_{xy}-w_yw_{xt})g'']+[w_{xxxxy}-w_{xxz}-w_{xyt}\\
&\quad+3w_{xy}v_{0xx}+3w_{xx}v_{0xy}\\
&\quad+3w_{xxy}v_{0x}+3w_{xxx}v_{0y}]g'\\
&\quad+[v_{0xxy}+3v_{0xx}v_{0y}+3v_{0x}u_{xy}-v_{0yt}-v_{0xz}]=0.
\end{aligned}
\tag{9}
$$

We further assume that

$$
g=m\ln w. \tag{10}
$$

We have the following relations for a fixed parameter m:

$$
\begin{aligned}
&g''g'''=-\frac{m}{12}g^{(5)},\quad g'g'''=-\frac{m}{3}g^{(4)},\\
&g'g''=-\frac{m}{2}g''',\qquad (g')^2=-mg'',\\
&(g'')^2=-\frac{m}{6}g^{(4)}.
\end{aligned}
\tag{11}
$$

Using the relations above and setting the coefficients of the terms $g''', g'', g', g^{(0)}$ to zero, we have the following differential equation system:

$$
\begin{aligned}
&g^{(5)}: 1-\frac{m}{2}=0,\\
&g^{(4)}: 6w_yw_x^2w_{xx}+4w_x^3w_{xy}-mw_{xx}w_yw_x^2\\
&\quad-mw_{xy}w_x^3-mw_x^3w_{xy}-2mw_x^2w_{xx}w_y=0\\
&g''': -w_tw_xw_y+6w_x^2w_{xxy}+3w_yw_{xx}^2+3w_x^3v_{0y}\\
&\quad+3w_yw_x^2v_{0x}+12w_xw_{xx}w_{xy}\\
&\quad+4w_yw_xw_{xxx}-w_zw_x^2+(3w_x^2w_{xxy}+3w_{xx}^2w_y\\
&\quad+15w_xw_{xx}w_{xy}\\
&\quad+3w_xw_yw_{xxx})\left(-\frac{m}{2}\right)=0\\
&g'': -w_zw_{xx}+3w_x^2u_{0xy}+6w_{xx}w_{xxy}+4w_{xy}w_{xxx}\\
&\quad+4w_xw_{xxxy}+w_yw_{xxxx}\\
&\quad+3w_xw_yv_{0xx}+6w_xw_{xy}v_{0x}+3w_yw_{xx}v_{0x}\\
&\quad+9w_xw_{xx}v_{0y}-2w_xw_{xz}-w_tw_x\\
&\quad-w_tw_{xy}-w_yw_{xt}+(3w_{xxx}w_{xy}+3w_{xx}w_{xxy})(-c)=0\\
&g': w_{xxxxy}-w_{xxz}-w_{xyt}+3w_{xy}v_{0xx}+3w_{xx}v_{0xy}\\
&\quad+3w_{xxy}v_{0x}+3w_{xxx}v_{0y}=0\\
&g^{(0)}: v_{0xxy}+3v_{0xx}v_{0y}+3v_{0x}v_{0xy}-v_{0yt}-v_{0xz}=0.
\end{aligned}
\tag{12}
$$

From the initial equation of Eq. (12), we can find that:

$$
m=2. \tag{13}
$$

The equation system Eq. (12) may be expressed using the equalities Eq. (11), as follows:

$$
\begin{aligned}
&[w_x(-w_tw_y-3w_{xx}w_{xy}+w_yw_{xxx}-w_zw_x\\
&\quad+6w_xw_{xxy}+3w_x^2v_{0y}\\
&\quad+3w_xw_yv_{0x}-3w_xw_{xxy})]g'''+[4w_xw_{xxy}\\
&\quad+w_yw_{xxxx}+3w_x^2v_{0xy}\\
&\quad+3w_xw_yu_{0xx}+9w_xw_{xx}v_{0y}+6w_xw_{xx}v_{0x}\\
&\quad+3w_yw_{xx}v_{0x}-2w_{xy}w_{xxx}\\
&\quad-w_xw_{yt}-w_zw_{xx}-2w_xw_{xz}-w_tw_{xy}-w_yw_{xt}]g''\\
&\quad+[w_{xxxxy}-w_{xxz}\\
&\quad-w_{xyt}+3w_{xy}v_{0xx}+3w_{xx}v_{0xy}+3w_{xxy}v_{0x}+3w_{xxx}v_{0y}]g'\\
&\quad+[v_{0xxy}\\
&\quad+3v_{0xx}v_{0y}+3v_{0x}v_{0xy}-v_{0yt}-v_{0xz}]=0.
\end{aligned}
\tag{14}
$$

Setting the coefficients of g^3, g'', g', and $g^{(0)}$ to zero respectively, we obtain the following differential equations:

$$
\begin{aligned}
&-w_tw_y-3w_{xx}w_{xy}+w_yw_{xxx}-w_zw_x+6w_xw_{xxy}+3w_x^2v_{0y}\\
&\quad+3w_xw_yv_{0x}-3w_xw_{xxy}=0,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
4w_xw_{xxy}+w_yw_{xxxx}+3w_x^2v_{0xy}+3w_xw_yu_{0xx}&\\
+9w_xw_{xx}v_{0y}+6w_xw_{xx}v_{0x}+3w_yw_{xx}v_{0x}&\\
-2w_{xy}w_{xxx}-w_xw_{yt}-w_zw_{xx}&\\
-2w_xw_{xz}-w_tw_{xy}-w_yw_{xt}=0&,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&w_{xxxxy}-w_{xxz}-w_{xyt}+3w_{xy}v_{0xx}+3w_{xx}v_{0xy}\\
&\quad+3w_{xxy}v_{0x}+3w_{xxx}v_{0y}=0,
\end{aligned}
\tag{17}
$$

$$
v_{0xxy}+3v_{0xx}v_{0y}+3v_{0x}v_{0xy}-v_{0yt}-v_{0xz}=0. \tag{18}
$$

We will differentiate Eq. (15), and integrate Eq. (17) to deduce the relation between Eqs.(15) - (17).

$$\begin{aligned}&\frac{\partial}{\partial x}(-w_t w_y - 3w_{xx}w_{xy} + w_y w_{xxx} - w_z w_x\\&+6w_x w_{xxy} + 3w_x^2 v_{0y} + 3w_x w_y v_{0x} - 3w_x w_{xxy})\\&+w_x(w_{xxxy} - w_{yt} - w_{xz} + 3w_{xx}v_{0y} + 3w_{xy}v_{0x})\\&= 4w_x w_{xxy} + w_y w_{xxxx} + 3w_x^2 v_{0xy} + 3w_x w_y v_{0xx}\\&+9w_x w_{xx} v_{0y} + 6w_x w_{xx} v_{0x} - 2w_x w_{xz} - w_t w_{xy} - w_y w_{xt}.\end{aligned} \tag{19}$$

If the following equations are satisfied, then Eq. (19) is satisfied as well.

$$w_{xxxy} - w_{yt} - w_{xz} + 3w_{xx}v_{0y} + 3w_{xy}v_{0x} = 0, \tag{20}$$

$$\begin{aligned}&- w_t w_y - 3w_{xx}w_{xy} + w_y w_{xxx} - w_z w_x + 6w_x w_{xxy}\\&+ 3w_x^2 v_{0y} + 3w_x w_y v_{0x} - 3w_x w_{xxy} = 0,\end{aligned} \tag{21}$$

$$v_{0xxy} + 3v_{0xx}v_{0y} + 3v_{0x}v_{0xy} - v_{0yt} - v_{0xz} = 0. \tag{22}$$

Therefore the Auto-Bäcklund transformation for a generalized shallow water-like equation is obtained as follows.

$$v(x,y,z,t) = 2\frac{w_x}{w} + v_0, \tag{23}$$

whereby w and v_0 meet Eqs. (20)-(22).

Then, by evaluating Eq. (20) and Eq. (21) for every solution v_0 of Eq. (1), we can derive several useful approaches to Eq. (1) from Eq. (23).

1. If we consider $v_0 = j$ in Eq. (23), it gives

$$v(x,y,z,t) = 2\frac{w_x}{w} + j, \tag{24}$$

where $w(x, y, z, t)$ satisfies the equations below:

$$w_{xxxy} - w_{yt} - w_{xz} = 0, \tag{25}$$

$$\begin{aligned}&- w_t w_y - 3w_{xx}w_{xy} + w_y w_{xxx}\\&- w_z w_x + 6w_x w_{xxy} - 3w_x w_{xxy} = 0.\end{aligned} \tag{26}$$

2. If we consider $v_0 = 0$ in Eq. (23), we deduce the Cole-Hopf transformation

$$v(x,y,z,t) = 2\frac{w_x}{w}, \tag{27}$$

where $w(x, y, z, t)$ satisfies Eqs. (25) and Eq. (26).

Theorem A *Due to the consistency and explicit solvability of Eqs. (20)-(22) concerning $w(x, y, z, t)$ and the inclusion of $v_0(x, y, z, t)$ as the seed solution, they create an Auto-Bäcklund transformation for Eq. (1).*

2.1. Soliton and complexiton solutions of generalized SW like equation

In this section, we use symbolic computation software Maple to derive the Auto-Bäcklund transformation and thereby generate several solitary-wave solutions of Eq. (1), as established in the preceding section [33]. Let us find the Auto-Bäcklund transformation Eq. (26) and the Eq. (1) solution v_0.

1. Suppose $w(x, y, z, t)$ is of the form:

$$\begin{aligned}w(x,y,z,t) &= p\ \cosh(cx + dy + mz + rt + \xi_0)\\&+ q\ \sinh(cx + dy + mz + rt + \xi_0) + s,\end{aligned} \tag{28}$$

p, q, c, d, m, r and s are parameters to be calculated and ξ_0 is an arbitrary constant [34]. One verifies an equation system by substituting Eq. (28) into Eq. (25) and Eq. (26).

The following cases are derived from the solutions of the system.

Case A:

$$p = q, \quad r = \frac{c(dc^2 - m)}{d},$$

where q, c, d, m, s are arbitrary. Consequently, we receive:

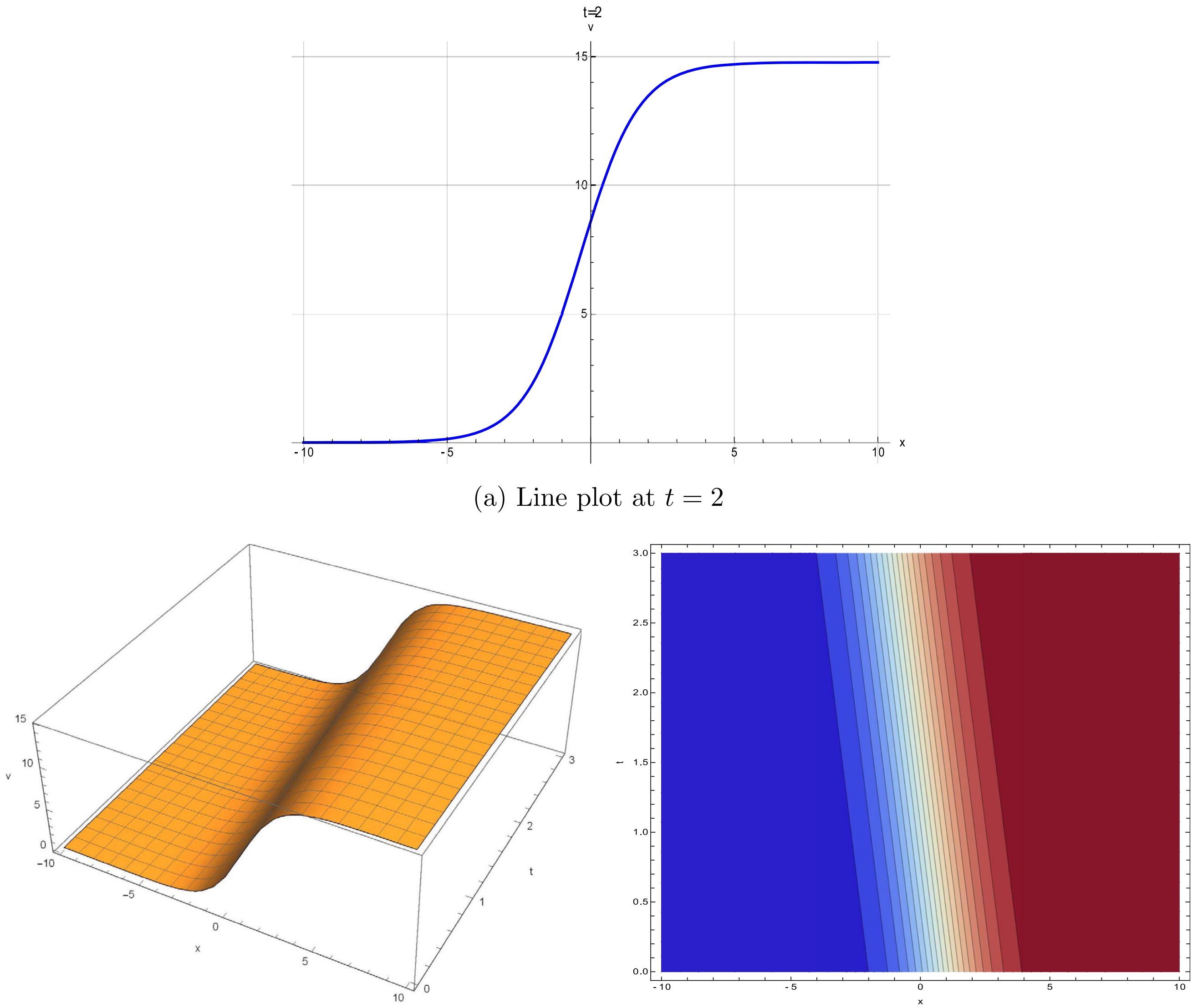
$$\begin{aligned}w_1(x,y,z,t) &= q\cosh\Big(cx + dy + mz\\&+ \frac{c(dc^2 - m)}{d}t + \xi_0\Big)\\&+ q\sinh\left(cx + dy + +mz + \frac{c(dc^2 - m)}{d}t + \xi_0\right) + s.\end{aligned} \tag{29}$$

By substituting Eq. (29) into Eq. (24), hyperbolic solution for Eq. (1) is obtained and shown in Fig. 1.

$$\begin{aligned}&v_1(x,y,z,t)\\&= 2qk\frac{\cosh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t + \xi_0\right) + \sinh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t + \xi_0\right)}{q\left[\cosh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t - \xi_0\right) + \sinh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t - \xi_0\right)\right] + s} + j,\end{aligned} \tag{30}$$

where j is the integration constant.

Case B:



(a) Line plot at $t = 2$

(b) Surface plot

(c) Contour plot

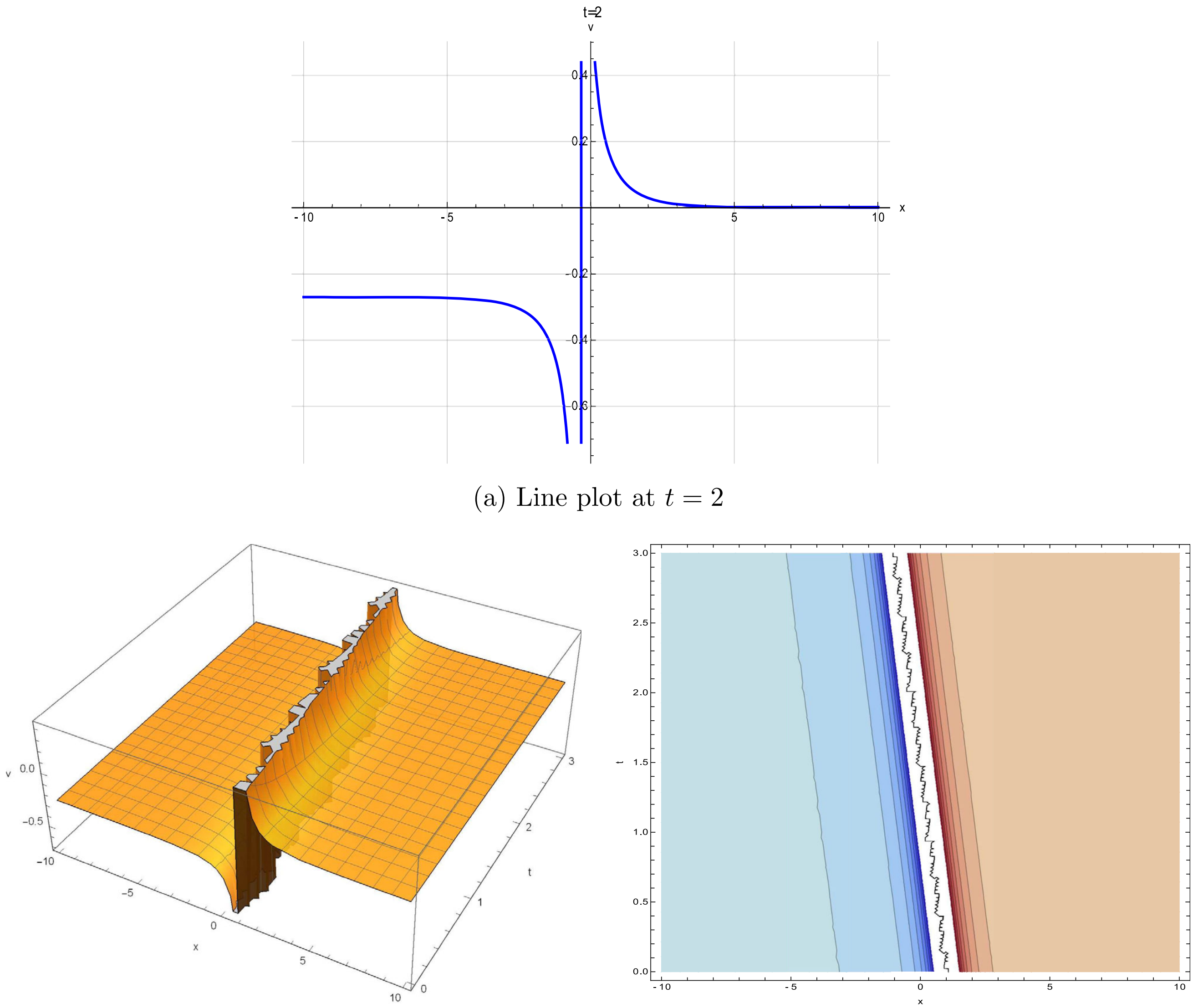
Fig. 1 The graphical representation to $v(x, y, z, t)$ in Case A, when $c = 1$, $d = 3$, $m = 1$, $\xi_0 = 1$, $p = 2$, $q = 2$, $s = 2$, $z = 0$, $j = 0$, $y = 0$.

$$p = -q, \quad r = \frac{c(dc^2 - m)}{d}, \tag{31}$$

where q, c, d, m, s are constants. Consequently, one receives the following solution:

$$\begin{aligned} w_2(x, y, z, t) &= -q \cosh\left(cx + dy + mz + \frac{c(dc^2 - m)}{d} t + \xi_0\right) \\ &\quad + q \sinh\left(cx + dy + mz + \frac{c(dc^2 - m)}{d} t + \xi_0\right) + s. \end{aligned} \tag{32}$$

Substituting Eq. (32) into Eq. (24), we find the desired solution as follows and shown in Fig. 2.



(a) Line plot at $t = 2$

(b) Surface plot

(c) Contour plot

Fig. 2 The graphical representation to $v(x, y, z, t)$ in Case B, when $c = 1$, $d = 3$, $m = 1$, $\xi_0 = 1$, $p = 2$, $q = 2$, $s = 2$, $z = 0$, $j = 0$, $y = 0$

$$v_2(x, y, z, t) = 2qk \frac{\cosh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t + \xi_0\right) - \sinh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t + \xi_0\right)}{-q\left[\cosh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t + \xi_0\right) - \sinh\left(cx + dy + mz + \frac{c(dc^2-m)}{d}t + \xi_0\right)\right] + s} + j, \tag{33}$$

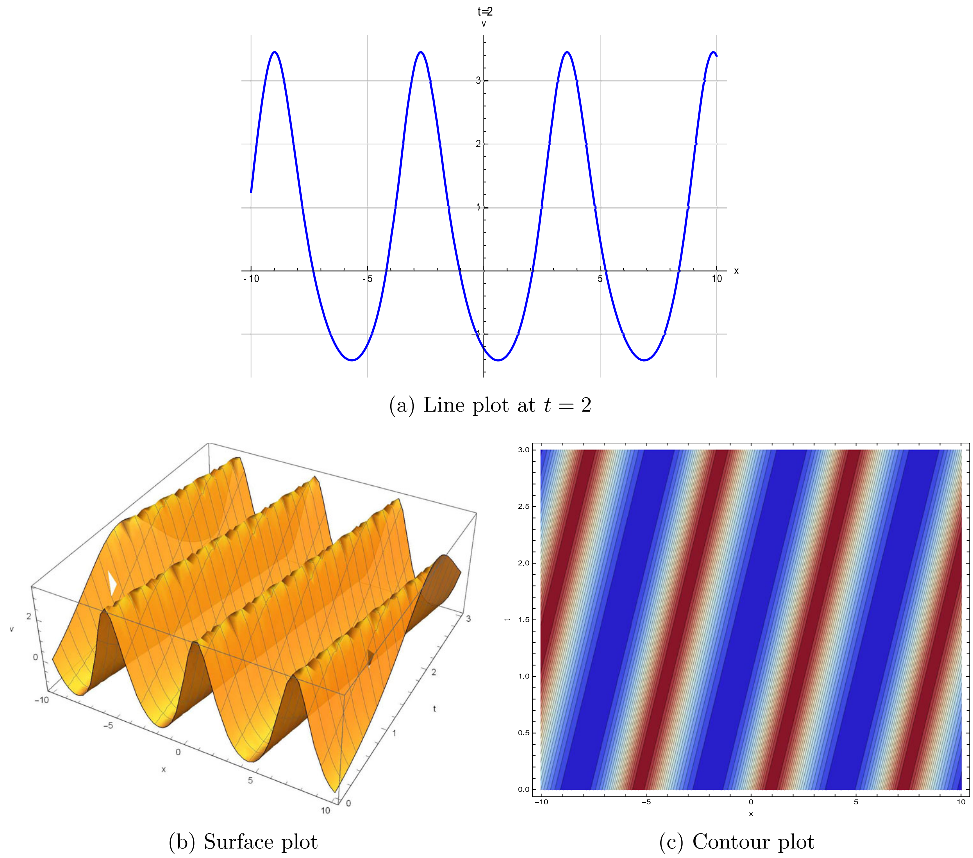
where j is the integration constant.

2. Let us consider $w(x, y, t)$ in the form

$$w(x, y, z, t) = p\ \cos(cx + dy + mz + rt + \xi_0) + q\ \sin(cx + dy + mz + rt + \xi_0) + s, \tag{34}$$

where c, d, m, p, q, s and r are constants and ξ_0 is an arbitrary constant. Substitution of Eq. (34) into algebraic equations Eq. (25) and Eq. (26) has given a system of equations. Then, by solving the equation system, one has obtained:

Fig. 3 The graphical representation to $v(x, y, z, t)$ in Eq. (37), when $c = 1$, $d = 3$, $m = 1$, $\xi_0 = 1$, $p = 2$, $q = 2$, $s = 2$, $z = 0$, j=0, $y = 0$



(a) Line plot at $t = 2$

(b) Surface plot

(c) Contour plot

$$p = q\mathrm{i}, \quad r = -\frac{c(dc^2 + m)}{d}, \tag{35}$$

Following that, we acquire

$$w_3(x, y, z, t) = q\mathrm{i}\cos\left(-cx - dy - mz + \frac{c(dc^2 + m)}{d}t - \xi_0\right) - q\sin\left(-cx - dy - mz + \frac{c(dc^2 + m)}{d}t - \xi_0\right) + s. \tag{36}$$

The trigonometric solution of the equation similar to (3 + 1) SW-like equation is obtained by substituting Eq. (36) into Eq. (24). The obtained solution can be written as follows and can be illustrated in Fig. 3.

where j is the integration constant.

3. Here, we will seek solutions to Eq. (1) as shown below

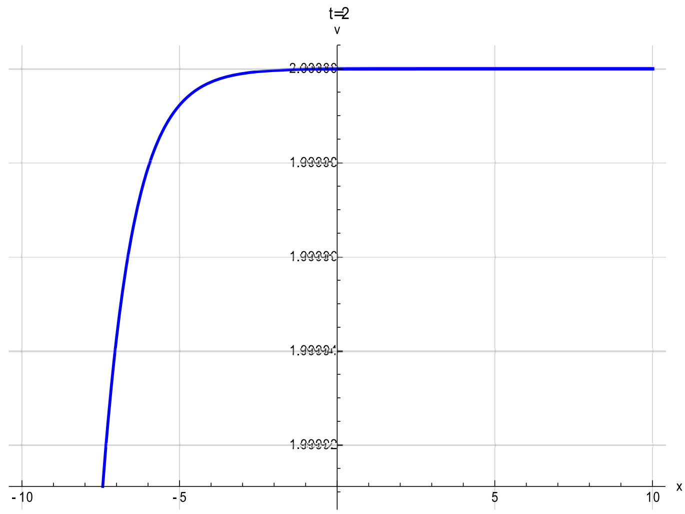
$$\begin{aligned} w(x, y, z, t) &= p\ \exp(c_1 x + d_1 y + m_1 z + r_1 t + \xi_1) \\ &\quad + q\ \exp(c_2 x + d_2 y + m_2 z + r_2 t + \xi_2) + s, \end{aligned} \tag{38}$$

p, q, s, c_1, d_1, r_1, m_1, c_2, d_2, r_2 and m_2 are undetermined constants and ξ_1, ξ_2 are arbitrary constants.

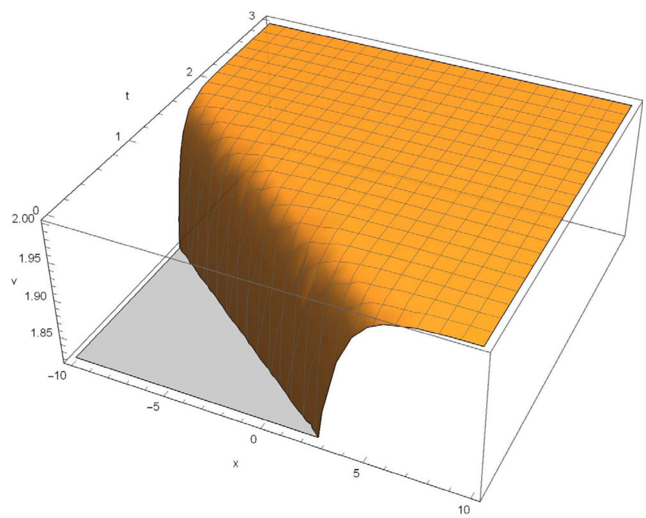
We find Eq. (38) by substituting it into Eq. (25) and Eq. (26),

$$\begin{aligned} &c_1 = 0, \quad c_2 = k_2, \quad d_1 = 0, \quad d_2 = l_2, \\ &r_1 = -\frac{c_2 m_1}{d_2}, \quad r_2 = \frac{c_2(c_2^2 d_2 - m_2)}{d_2}, \end{aligned} \tag{39}$$

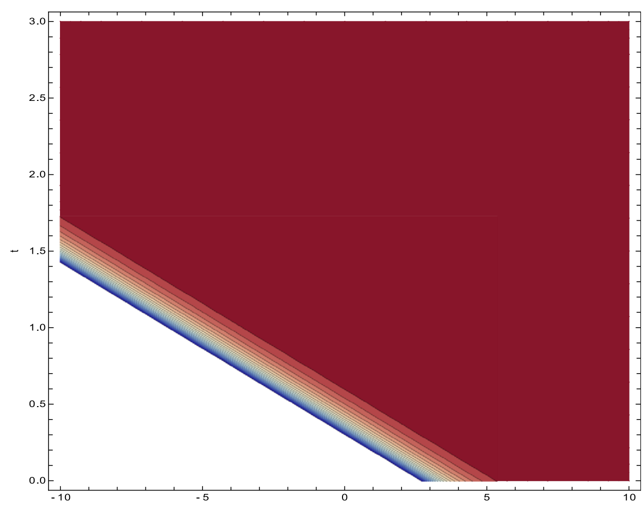
$$v_3(x, y, z, t) = 2qk\frac{\mathrm{i}\sin\left(-cx - dy - mz + \frac{c(dc^2+m)}{d}t - \xi_0\right) + \cos\left(-cx - dy - mz + \frac{c(dc^2+m)}{d}t - \xi_0\right)}{q\left[\cos\left(-cx - dy - mz + \frac{c(dc^2+m)}{d}t - \xi_0\right) - \sin\left(-cx - dy - mz + \frac{c(dc^2+m)}{d}t - \xi_0\right)\right] + s} + j, \tag{37}$$



(a) Line plot at $t = 2$



(b) Surface plot



(c) Contour plot

Fig. 4 The graphical representation to $v(x, y, z, t)$ in Eq.(41), when $c = 1$, $d = 3$, $m = 1$, $\xi_0 = 1$, $p = 2$, $q = 2$, $s = 2$, $z = 0$, j=0, y=0

where c_2, d_2, m_1, m_2 are non-zero parameters. Consequently, we deduce the following expression:

$$w_4(x, y, z, t) = p \exp\left(m_1 z - \frac{c_2 m_1}{d_2} t + \xi_1\right) + q \exp\left(c_2 x + d_2 + m_2 z + \frac{c_2(c_2^2 d_2 - m_2)}{d_2} t + \xi_2\right) + s. \tag{40}$$

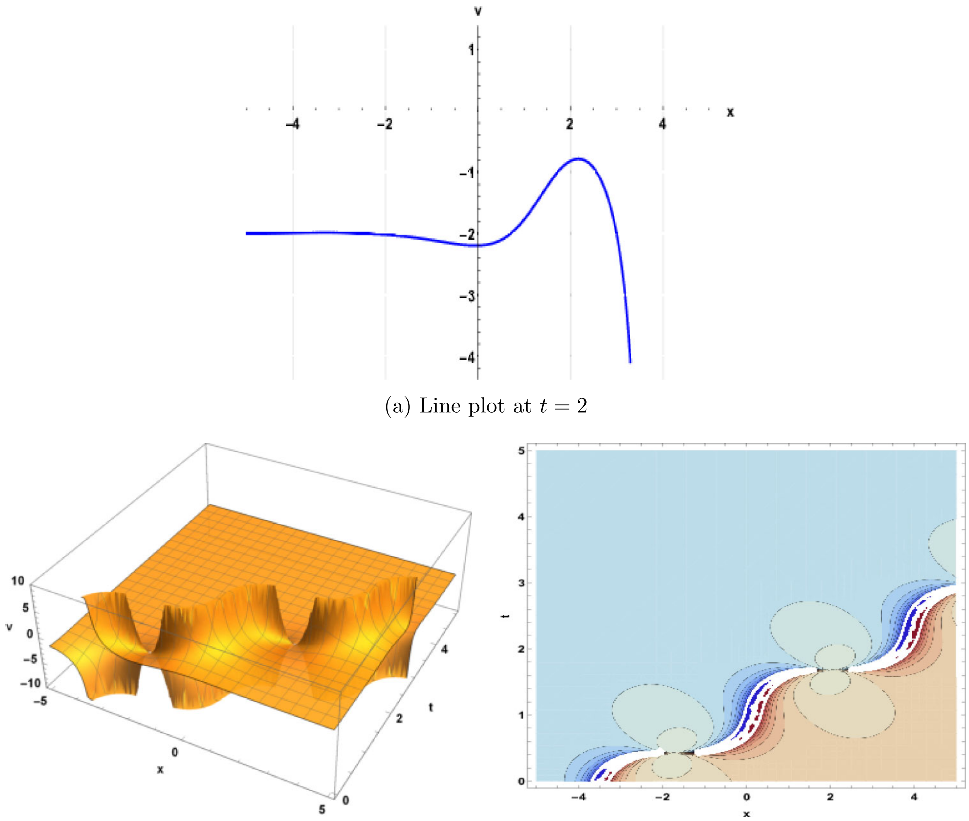
Therefore, the exponential form solution of (3+1)-dimensional SW-like equation Eq. (1) is shown in Fig. 4 which is obtained by inserting Eq. (40) into Eq. (24).

$$v_4(x, y, z, t) = \frac{2qc_2 \exp\left(c_2 x + d_2 + m_2 z + \frac{c_2(c_2^2 d_2 - m_2)}{d_2} t + \xi_2\right)}{p \exp\left(m_1 z - \frac{c_2 m_1}{d_2} t + \xi_1\right) + q \exp\left(c_2 x + d_2 + m_2 z + \frac{c_2(c_2^2 d_2 - m_2)}{d_2} t + \xi_2\right) + s} + j, \tag{41}$$

where j is the integration constant.

It is worth mentioning that we may easily follow the exact explicit solutions of the generalized SW-like equation for Cole-Hopf transformation Eq. (27) by replacing zero for the constant j in the solution.

Theorem B *The hyperbolic, trigonometric, and exponential function waves provide the obtained soliton solutions Eq. (30), Eq. (33), Eq. (37) and Eq. (41) respectively.*



(a) Line plot at $t = 2$

(b) Surface plot

(c) Contour plot

Fig. 5 The graphical representation for $v(x, y, z, t)$, when $k_1 = 1$, $k_2 = 1$, $c = 2$, $b = 2$, $e = 4$, $f = 3$, $z = 0$, $y = 0$

3. Extended transformed rational function technique

In this section, the primary steps of the improved modified rational method are examined. The attention will be shifted to the nonlinear partial differential equation of Hirota's form which has t as the dependent parameter and x, y, z as the independent parameters.

$$R(\nabla_x, \nabla_y, \nabla_z, \ldots)q \cdot q = 0. \tag{42}$$

Here, $\nabla_x, \nabla_y, \nabla_z$,..., denote Hirota differential operators defined by:

$$\begin{aligned}\nabla_v^m q(v) \cdot v(v) &= (\partial_v - \partial_{v'})^m q(y) w(v') \Big|_{v'=v} \\ &= \partial_{\mu'}^m q(v + v') g(v - v') \Big|_{v'=0},\end{aligned} \tag{43}$$

solution of Eq. (42) will be of the following form:

$$q = \frac{m(\mu_1, \mu_2)}{l(\mu_1, \mu_2)}, \tag{44}$$

where $m(\mu_1, \mu_2)$, $l(\mu_1, \mu_2)$ are polynomials, μ_1, and μ_2 yield the differential equations shown below:

$$\mu_1'' = \frac{d^2\mu_1}{d\rho_1^2} = -\mu_1, \tag{45}$$

$$\mu_2'' = \frac{d^2\mu_2}{d\rho_2^2} = \mu_2, \tag{46}$$

where $\rho_1 = k_1 x + bk_1 y + ck_1 z + w_1 t + e, \rho_2 = k_2 x + bk_2 y + ck_2 z + w_2 t + g$.

The solutions of Eq. (45), Eq. (46) can be acquired as follows:

$$\begin{aligned}&\mu_1 = \pm \sin \rho_1 \quad or \quad \mu_1 = \pm \cos \rho_1, \\ &\mu_2 = \pm \sinh \rho_2, \quad or \quad \mu_2 = \pm \cosh \rho_2\end{aligned} \tag{47}$$

Eq. (47) may be used to find the following equalities:

$$\mu_1'^2 = 1 - \mu_1^2, \qquad \mu_2'^2 = 1 + \mu_2^2. \tag{48}$$

Then by selecting appropriate $m(\mu_1, \mu_2)$ and $l(\mu_1, \mu_2)$, Eq. (42) may be turned into an algebraic equation that includes k_i, w_i. The symbolic computation is then used to discover precise complexiton solutions to Eq. (42).

Theorem C *Similarity of the modified rational function method to an exponential function approach is observed* *when $\rho' = \rho$ or $\rho = e^{\mu}$, and when $\rho' = \beta + \rho^2$, where β is a constant, it also becomes the generalized hyperbolic tangent method. However, complexiton solutions are distinct from the other types of traveling wave velocities*; *thus, they are much more intricate and cannot be mathematically derived or modeled using conventional methods in nonlinear equations. Therefore, the modified rational function technique is enhanced to obtain complexiton solutions.*

4. Application of extended transformed rational function procedure

The associated bilinear form, as indicated in Eq. (6), is derived by applying the transformation in Eq. (2).

$$g = C\mu_1 + D\mu_2. \tag{49}$$

We use the method outlined in [35] since Eq. (1) represents a (3+1)-dimensional system.

$$\begin{aligned}\rho_1 &= k_1x + bk_1y + ck_1z + w_1t + e,\\ \rho_2 &= k_2x + bk_2y + ck_2z + w_2t + g.\end{aligned} \tag{50}$$

Here, C, D, k_k and w_k are arbitrary constants. Plugging Eq. (49) into Eq. (6) and using the relations

$$\mu_1'' = \frac{d^2\mu_1}{d\rho_1^2} = -\mu_1, \qquad \mu_2'' = \frac{d^2\mu_2}{d\rho_2^2} = -\mu_2, \tag{51}$$

$$\mu_1'^2 = 1 - \mu_1^2, \qquad \mu_2'^2 = 1 + \mu_1^2. \tag{52}$$

We can rewrite the resultant as a polynomial in terms of $\mu_1^2, \mu_2^2, \mu_1\mu_2, \mu_1'\mu_2'$. By collecting the matching coefficients with respect to $\mu_1^2, \mu_2^2, \mu_1\mu_2, \mu_1'\mu_2'$, we arrive at the following set of algebraic equations. Then we can set these equations to vanish,

$$\begin{aligned}-12CDbk_1^2k_2^2 + 2CDbk_1w_1 - 2CDbk_2w_2&\\ +2CDck_1^2 - 2CDck_2^2 &= 0,\\ 2CDbk_1w_2 + 2CDbk_2w_1 + 4CDck_1k_2 &= 0,\\ 2C^2bk_1w_1 + 2D^2bk_2w_2 + 2C^2ck_1^2 + 2D^2ck_2^2 &= 0.\end{aligned} \tag{53}$$

The result is obtained by solving the system Eq. (53):

$$\begin{aligned}C &= \pm D\frac{k_2}{k_1}, \qquad w_1 = \frac{k_1(6bk_1^2k_2^2 - ck_1^2 - ck_2^2)}{b(k_1^2 + k_2^2)},\\ w_2 &= -\frac{k_2(6bk_1^2k_2^2 + ck_1^2 + ck_2^2)}{b(k_1^2 + k_2^2)}.\end{aligned} \tag{54}$$

By Eq. (49) the solution of Eq. (6) may be obtained as follows:

$$\begin{aligned}g(x,y,z,t) = C\Bigg[&\sin(k_1x + bk_1y + ck_1z\\ &+\frac{k_1(6bk_1^2k_2^2 - ck_1^2 - ck_2^2)}{b(k_1^2 + k_2^2)}t + e)\\ &\pm\frac{k_1}{k_2}\sinh(k_2x + bk_2y + ck_2z\\ &-\frac{k_2(6bk_1^2k_2^2 + ck_1^2 + ck_2^2)}{b(k_1^2 + k_2^2)}t + f)\Bigg],\end{aligned} \tag{55}$$

or

$$\begin{aligned}g(x,y,z,t) = C[&\cos(k_1x + bk_1y + ck_1z\\ &+\frac{k_1(6bk_1^2k_2^2 - ck_1^2 - ck_2^2)}{b(k_1^2 + k_2^2)}t + e)\\ &\pm\frac{k_1}{k_2}\sinh(k_2x + bk_2y + ck_2z\\ &-\frac{k_2(6bk_1^2k_2^2 + ck_1^2 + ck_2^2)}{b(k_1^2 + k_2^2)}t + f)],\end{aligned} \tag{56}$$

where f, e, D, C, k_1, k_2 are arbitrary constants. We can obtain $u(x, y, z, t)$ by inserting Eq. (55) or Eq. (56) into Eq. (1) we get $v(x, y, z, t)$ which is shown in Fig. 5.

5. Results and discussion

For the purposes of the present paper, we developed the Auto-Bäcklund transformation for the generalized shallow-water-like model by employing the extended homogeneous balancing technique with the aid of computer algebra systems, specifically Maple and Mathematica. These useful software programs are used to perform complicated symbolic computations. Through these transformations, we successfully derived new exact and explicit solutions expressed in terms of exponential, hyperbolic, and rational solutions. The graphical representations of these solutions provided in Figs. (1-4) illustrate their distinct characteristics and behaviors. To the best of our knowledge, these graphical representations are fundamental for the study of the complex dynamics of nonlinear waves.

The obtained results of the generalized SW-like model are valuable in various scientific and engineering applications, as they describe the behavior of nonlinear wave phenomena in multiple physical settings. According to the literature, our solutions are distinct and offer a different perspective compared to existing ones [1, 4, 5, 29].

We used the extended transformed rational function method to generate complexiton solutions from the Hirota bilinear form. The complexiton solutions obtained are presented in Figure 5 to facilitate clearer interpretation. Symbolic computation software was also used to verify all solutions and confirm the accuracy and consistency of our results.

6. Conclusions

The present research can be further extended by performing a stability and modulation instability (MI) analysis of the obtained solitary-wave and complexiton solutions. Such an investigation would provide valuable insight into the nonlinear dynamics and robustness of these wave structures under small perturbations.

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Data availability The datasets utilized or analyzed in the current study can be obtained from the corresponding author upon reasonable request.

Declarations

Ethical approval This article does not involve any animal studies conducted by the authors.

Conflict of interest The authors confirm that there are no Conflict of interest related to the publication of this paper.

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