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The spectrum of eigenparameter-dependent discrete Sturm–Liouville equations

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ABSTRACT

Let us consider the boundary value problem (BVP) for the discrete Sturm–Liouville equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \tag{0.1}$$

$$(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0, \tag{0.2}$$

where (a_n) and (b_n), $n \in \mathbb{N}$ are complex sequences, $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$, and λ is a eigenparameter. Discussing the point spectrum, we prove that the BVP (0.1), (0.2) has a finite number of eigenvalues and spectral singularities with a finite multiplicities, if

$$\sup_{n\in N} \left[\exp(\varepsilon n^{\delta}) \left(|1 - a_n| + |b_n|\right)\right] < \infty,$$

for some $\varepsilon > 0$ and $\frac{1}{2} \le \delta \le 1$.

1. Introduction

Boundary value problems for difference equations have been intensively studied in the last decade. The modeling of certain problems from engineering, economics, control theory and other areas of study has led to the rapid development of the theory of difference equations. Also, the spectral theory of difference equations has been treated by various authors in connection with the classical moment problem (see the monographs of Agarwal [1], Akhiezer [2], Kelley–Peterson [3] and the references therein). The spectral analysis of discrete equations has also been applied to the solution of classes of nonlinear discrete equations and Toda lattices [4,5]. Let us consider the boundary value problem (BVP)

$$\begin{cases} -y'' + q(x)y = \lambda^2 y, & 0 \le x < \infty, \\ y'(0) - hy(0) = 0 \end{cases} \tag{1.1}$$

where q is a complex valued function, $h \in \mathbb{C}$ and λ is a spectral parameter. The spectral analysis of the BVP (1.1) has been investigated in [6]. In this study, the spectrum of the BVP (1.1) was investigated and it is shown that it is composed of the eigenvalues, the continuous spectrum and spectral singularities. The spectral singularities are poles of the resolvent which are imbedded in the continuous spectrum and are not eigenvalues. The effect of the spectral singularities in the spectral expansion of the BVP (1.1) in terms of the principal functions has been investigated in [7]. The spectral analysis of a non-selfadjoint difference equation of second order has been studied in [8]. In this article, it is proved that the Jost solution of this equation has an analytic continuation to the lower half-plane and the finiteness of the eigenvalues and the spectral singularities of the difference equation is obtained as a result of this analytic continuation. The discrete spectrum of general

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difference equations has been investigated in [9]. Some problems related to the spectral analysis of difference equations with spectral singularities have been discussed in [10–14].

Let us consider the non-selfadjoint BVP for the difference equation of second order

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.2}$$

$$(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0, \quad \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad \gamma_1 \neq \frac{\beta_0}{a_0} \tag{1.3}$$

where (a_n), (b_n), $n \in \mathbb{N}$ are complex sequences, $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$. Note that we can write the difference equation (1.2) in the following Sturm–Liouville form:

$$\nabla(a_n \Delta y_n) + h_n y_n = \lambda y_n, \quad n \in \mathbb{N},$$

where $h_n = a_{n-1} + a_n + b_n$, Δ is the forward difference operator, $\Delta y_n = y_{n+1} - y_n$, and ∇ is the backward difference operator, $\nabla y_n = y_n - y_{n-1}$. The specific feature of this study is the presence of the spectral parameter not only in the difference equation but also in the boundary condition.

In this paper, we investigate the eigenvalues and the spectral singularities of the BVP (1.2), (1.3) and prove that this BVP has a finite number of eigenvalues and spectral singularities with finite multiplicities, if

$$\sup_{n\in\mathbb{N}} \left[\exp(\varepsilon n^{\delta})\left(|1 - a_n| + |b_n|\right)\right] < \infty,$$

for some $\varepsilon > 0$ and $\frac{1}{2} \leq \delta \leq 1$.

2. Jost solution and Jost function of (1.2) and (1.3)

Suppose that the complex sequences (a_n) and (b_n) satisfy

$$\sup_{n\in\mathbb{N}} \left[\exp(\varepsilon n^{\delta})\left(|1 - a_n| + |b_n|\right)\right] < \infty, \tag{2.1}$$

for some $\varepsilon > 0$ and $\frac{1}{2} \leq \delta \leq 1$. The following result is obtained in [15,16]: Under the condition (2.1), Eq. (1.2) has the solution

$$e_n(z) = \alpha_n e^{inz}\left(1 + \sum_{m=1}^{\infty} A_{nm} e^{imz}\right), \quad n \in \mathbb{N} \cup \{0\}, \tag{2.2}$$

for $\lambda = 2\cos z$, where $z \in \overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C}, \operatorname{Im} z \geq 0\}$, and α_n, A_{nm} are expressed in terms of (a_n) and (b_n). Moreover,

$$|A_{nm}| \leq C \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(|1 - a_k| + |b_k|\right), \tag{2.3}$$

holds, where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $C > 0$ is constant. Therefore $e_n(z)$ is analytic with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \operatorname{Im} z > 0\}$ and continuous in $\operatorname{Im} z = 0$.

Using the boundary condition (1.3) and (2.2), we define the function f:

$$f(z) = (\gamma_0 + 2\gamma_1 \cos z)e_1(z) + (\beta_0 + 2\beta_1 \cos z)e_0(z). \tag{2.4}$$

The function f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $f(z) = f(z + 2\pi)$.

Analogously to the Sturm–Liouville differential equation, the solution

$$e(z) = \{e_n(z)\}, \quad n \in \mathbb{N} \cup \{0\},$$

and the function f are called the Jost solution and Jost function of (1.2) and (1.3) [17].

Let $\widehat{\varphi}(\lambda) = \{\widehat{\varphi}_n(\lambda)\}$, $n \in \mathbb{N} \cup \{0\}$ be the solution of (1.2) subject to the initial conditions

$$\widehat{\varphi}_0(\lambda) = \gamma_0 + \gamma_1\lambda, \qquad \widehat{\varphi}_1(\lambda) = -(\beta_0 + \beta_1\lambda).$$

If we define

$$\varphi(z) = \widehat{\varphi}(2\cos z) = \{\widehat{\varphi}_n(2\cos z)\}, \quad n \in \mathbb{N} \cup \{0\},$$

then φ is an entire function and

$$\varphi(z) = \varphi(z + 2\pi).$$

Let us define the semi-strip $P_0 := \left\{z : z \in \mathbb{C}, z = \xi + i\tau, -\frac{\pi}{2} \leq \xi \leq \frac{3\pi}{2}, \tau > 0\right\}$ and $P := P_0 \cup [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

For all $z \in P$ and $f(z) \neq 0$, we define

$$G_{nm}(z) = \begin{cases} -\dfrac{\varphi_m(z)e_n(z)}{a_0 f(z)}, & m \leq n, \\ -\dfrac{\varphi_n(z)e_m(z)}{a_0 f(z)}, & m > n. \end{cases} \tag{2.5}$$

The function $G_{nm}(z)$ is called the Green function of the BVP (1.2) and (1.3). It is obvious that, for $g = (g_m)$, $m \in \mathbb{N} \cup \{0\}$,

$$(Rg)_n := \sum_{m=0}^{\infty} G_{nm}(z) g_m, \quad n \in \mathbb{N} \cup \{0\} \tag{2.6}$$

is the resolvent of the BVP (1.2) and (1.3).

3. Eigenvalues and spectral singularities of (1.2) and (1.3)

We will denote the set of all eigenvalues and spectral singularities of the BVP (1.2) and (1.3) by σ_d and σ_{ss}, respectively. From (2.5) and (2.6) and the definition of the eigenvalues and the spectral singularities, we have [17]

$$\sigma_d = \{\lambda : \lambda = 2\cos z, z \in P_0, f(z) = 0\}, \tag{3.1}$$

$$\sigma_{ss} = \left\{\lambda : \lambda = 2\cos z, z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], f(z) = 0\right\} \setminus \{0\}. \tag{3.2}$$

From (2.2) and (2.4), we get

$$\begin{aligned} f(z) &= \left[\gamma_0 + \gamma_1\left(e^{iz} + e^{-iz}\right)\right]\left[\alpha_1 e^{iz}\left(1 + \sum_{m=1}^{\infty} A_{1m} e^{imz}\right)\right] + \left[\beta_0 + \beta_1\left(e^{iz} + e^{-iz}\right)\right]\left[\alpha_0\left(1 + \sum_{m=1}^{\infty} A_{0m} e^{imz}\right)\right] \\ &= \alpha_0\beta_1 e^{-iz} + \gamma_1\alpha_1 + \alpha_0\beta_0 + (\gamma_0\alpha_1 + \alpha_0\beta_1) e^{iz} + \gamma_1\alpha_1 e^{2iz} + \sum_{m=1}^{\infty} \alpha_0\beta_1 A_{0m} e^{i(m-1)z} \\ &\quad + \sum_{m=1}^{\infty} (\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}) e^{imz} + \sum_{m=1}^{\infty} (\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}) e^{i(m+1)z} + \sum_{m=1}^{\infty} \gamma_1\alpha_1 A_{1m} e^{i(m+2)z}. \end{aligned}$$

Let

$$F(z) := f(z) e^{iz}; \tag{3.3}$$

then, the function F is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$,

$$F(z) = F(z + 2\pi)$$

and

$$\begin{aligned} F(z) &= \alpha_0\beta_1 + (\gamma_1\alpha_1 + \alpha_0\beta_0) e^{iz} + (\gamma_0\alpha_1 + \alpha_0\beta_1) e^{2iz} + \gamma_1\alpha_1 e^{3iz} + \sum_{m=1}^{\infty} \alpha_0\beta_1 A_{0m} e^{imz} \\ &\quad + \sum_{m=1}^{\infty} (\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}) e^{i(m+1)z} + \sum_{m=1}^{\infty} (\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}) e^{i(m+2)z} + \sum_{m=1}^{\infty} \gamma_1\alpha_1 A_{1m} e^{i(m+3)z}. \end{aligned} \tag{3.4}$$

It follows from (3.1)–(3.3) that

$$\sigma_d = \{\lambda : \lambda = 2\cos z, z \in P_0, F(z) = 0\}, \tag{3.5}$$

$$\sigma_{ss} = \left\{\lambda : \lambda = 2\cos z, z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], F(z) = 0\right\} \setminus \{0\}. \tag{3.6}$$

Definition 3.1. The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.2) and (1.3).

From (3.5) and (3.6) we get that, in order to investigate the quantitative properties of the sets σ_d and σ_{ss}, we need to discuss the quantitative properties of the zeros of F in P.

Let

$$A_1 := \{z : z \in P_0, F(z) = 0\}, \qquad A_2 := \left\{z : z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], F(z) = 0\right\}. \tag{3.7}$$

We also denote the set of all limit points of A_1 by A_3 and the set of all zeros of F with infinite multiplicity by A_4.

From (3.5) and (3.7), we find that

$$\sigma_d = \{\lambda : \lambda = 2\cos z, z \in A_1\}, \qquad \sigma_{ss} = \{\lambda : \lambda = 2\cos z, z \in A_2\} \setminus \{0\}. \tag{3.8}$$

Theorem 3.1. *If (2.1) holds, then*

(i) *The set A_1 is bounded and countable.*
(ii) $A_1 \cap A_3 = \varnothing, A_1 \cap A_4 = \varnothing$.
(iii) *The set A_2 is compact and $\mu(A_2) = 0$, where μ denotes the Lebesgue measure in the real axis.*
(iv) $A_3 \subset A_2, A_4 \subset A_2$; $\mu(A_3) = \mu(A_4) = 0$.
(v) $A_3 \subset A_4$.

Proof. From (2.3) and (3.4), we have

$$F(z) = \begin{cases} \alpha_0\beta_1 + O(e^{-\tau}), & \beta_1 \neq 0,\ z \in P,\ \tau \to \infty \\ (\gamma_1\alpha_1 + \alpha_0\beta_0)\, e^{iz} + O(e^{-2\tau}), & \beta_1 = 0,\ z \in P,\ \tau \to \infty. \end{cases} \tag{3.9}$$

(3.9) shows that the set A_1 is bounded. Since F is analytic in $\mathbb{C}_+$ and is a 2π periodic function, we get that A_1 has at most a countable number of elements. This proves (i). From the boundary uniqueness theorems of analytic functions, we obtain (ii)–(iv) [18]. Using the continuity of all derivatives of F on $[-\frac{\pi}{2}, \frac{3\pi}{2}]$, we get (v). □

From Theorem 3.1 and (3.8), we have the following.

Theorem 3.2. *Under the condition (2.1),*

(i) *the set of eigenvalues of the BVP (1.2) and (1.3) is bounded, has at most a countable number of elements, and its limit points can lie only in $[-2, 2]$.*
(ii) $\sigma_{ss} \subset [-2, 2]$ *and* $\mu(\sigma_{ss}) = 0$.

Let us assume that the complex sequences (a_n) and (b_n) satisfy

$$\sup_{n\in\mathbb{N}} [\exp(\varepsilon n)\,(|1 - a_n| + |b_n|)] < \infty, \tag{3.10}$$

for some $\epsilon > 0$. Note that, for $\delta = 1$, the condition (2.1) reduces to (3.10).

Theorem 3.3. *Under the condition (3.10), the BVP (1.2), (1.3) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. From (2.3), we get that

$$|A_{nm}| \le C \exp\left[-\frac{\varepsilon}{2}(n + m)\right], \quad n, m \in \mathbb{N},$$

where $C > 0$ is a constant. Using (3.4), we observe that the function F has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{2}$. Since F is a 2π periodic function, the limit points of its zeros in P cannot lie in $[-\frac{\pi}{2}, \frac{3\pi}{2}]$. From Theorem 3.1 we obtain that the bounded sets A_1 and A_2 have a finite number of elements. Using the analyticity of F in $\operatorname{Im} z > -\frac{\varepsilon}{2}$, we find that all zeros of F in P have a finite multiplicity. Therefore we get the finiteness of the eigenvalues and the spectral singularities of the BVP (1.2) and (1.3). □

Now let us suppose that

$$\sup_{n\in\mathbb{N}} \exp(\varepsilon n^{\delta})\,(|1 - a_n| + |b_n|) < \infty, \quad \varepsilon > 0, \quad \frac{1}{2} \le \delta < 1, \tag{3.11}$$

which is weaker than (3.10). It is seen that the condition (3.10) guarantees the analytic continuation of F from the real axis to the lower half-plane. So the finiteness of the eigenvalues and the spectral singularities of the BVP (1.2), (1.3) is obtained as a result of this analytic continuation. It is evident that, under the condition (3.11), the function F is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis. But F does not have an analytic continuation from the real axis to the lower half-plane. Therefore, under the condition (3.11), the finiteness of the eigenvalues and the spectral singularities of the BVP (1.2), (1.3) cannot be shown in a way similar to Theorem 3.3.

Under the condition (3.11), to prove that the eigenvalues and the spectral singularities of the BVP (1.2), (1.3) are of finite number, we will use the following.

Theorem 3.4 (*[11]*)**.** *Let us assume that the 2π periodic function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous in $\overline{\mathbb{C}}_+$, and*

$$\sup_{z\in P} \left|g^{(k)}(z)\right| \le \eta_k, \quad k \in \mathbb{N} \cup \{0\}.$$

If the set $G \subset [-\frac{\pi}{2}, \frac{3\pi}{2}]$ with Lebesgue measure zero is the set of all zeros the function g with infinite multiplicity in P, and if

$$\int_0^{\omega} \ln t(s) \mathrm{d}\mu(G_s) = -\infty,$$

where $t(s) = \inf_k \frac{\eta_k s^k}{k!}$ and $\mu(G_s)$ is the Lebesgue measure of s-neighborhood of G and $\omega > 0$ is an arbitrary constant, then $g \equiv 0$ in $\overline{\mathbb{C}}_+$.

It follows from (2.3) and (3.4) that

$$\left|F^{(k)}(z)\right| \le \eta_k, \quad k \in \mathbb{N} \cup \{0\},$$

where

$$\eta_k = 2^k C \sum_{n=1}^{\infty} n^k \exp(-\varepsilon n^\delta).$$

We can obtain the estimate for η_k:

$$\eta_k \le 2^k C \int_0^\infty x^k \exp(-\varepsilon x^\delta) \mathrm{d}x \le B b^k k! k^{k\frac{1-\delta}{\delta}}, \tag{3.12}$$

where B and b are constants depending on C, ε and δ.

Theorem 3.5. *If (3.11) holds, then $A_4 = \varnothing$.*

Proof. Using Theorem 3.4, we get that the function satisfies the condition

$$\int_0^\omega \ln t(s) \mathrm{d}\mu(A_{4,s}) > -\infty, \tag{3.13}$$

where $t(s) = \inf_k \frac{\eta_k s^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$, $\mu(A_{4,s})$ is the Lebesgue measure of the s-neighborhood of A_4, and η_k is defined by (3.12). Now we obtain

$$t(s) \le B \exp\left\{-\frac{1-\delta}{\delta} \mathrm{e}^{-1} b^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}}\right\}, \tag{3.14}$$

by (3.12). It follows from (3.13) and (3.14) that

$$\int_0^\omega s^{-\frac{\delta}{1-\delta}} \mathrm{d}\mu(A_{4,s}) < \infty. \tag{3.15}$$

Since $\frac{\delta}{1-\delta} \ge 1$, from (3.15), we get that, for arbitrary s, $\mu(A_{4,s}) = 0$ or $A_4 = \varnothing$. □

Theorem 3.6. *Under the condition (3.11), the BVP (1.2), (1.3) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the function F has a finite number of zeros with finite multiplicities in P.

Using Theorems 3.1 and 3.5, we find that $A_3 = \varnothing$. So the bounded sets A_1 and A_2 have no limit points, i.e., the function F has only a finite number of zeros in P. Since $A_4 = \varnothing$, these zeros are of finite multiplicity. □

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