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A STUDY OF SOME DISCRETE DIRAC EQUATIONS WITH PRINCIPAL FUNCTIONS

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Abstract. Let L denote the operator generated in $\ell_2(\mathbb{N},\mathbb{C}^2)$ by

$$\begin{cases} a_{n+1}y_{n+1}^{(2)}+b_ny_n^{(2)}+p_ny_n^{(1)}=\lambda y_n^{(1)} \\ a_{n-1}y_{n-1}^{(1)}+b_ny_n^{(1)}+q_ny_n^{(2)}=\lambda y_n^{(2)}, \end{cases} \quad n\in\mathbb{N}$$

and the boundary condition

$$(\gamma_0+\gamma_1\lambda+\gamma_2\lambda^2)y_1^{(2)}+(\beta_0+\beta_1\lambda+\beta_2\lambda^2)y_0^{(1)}=0$$

where (a_n), (b_n), (p_n) and (q_n), $n\in\mathbb{N}$ are complex sequences, $\gamma_i,\beta_i\in\mathbb{C}$, $i=0,1,2$ and λ is a eigenparameter. With respect to the spectral properties of L, we investigate the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of L, if

$$\sum_{n=1}^{\infty}\exp(\varepsilon n^{\delta})\Big(|1-a_n|+|1+b_n|+|p_n|+|q_n|\Big)<\infty$$

holds for some $\varepsilon>0$ and $\delta\in[\frac{1}{2},1]$.

1. Introduction

Consider the discrete boundary value problem (BVP)

$$\begin{cases} y_{n+1}^{(2)}-y_n^{(2)}+p_ny_n^{(1)}=\lambda y_n^{(1)}, \\ -y_n^{(1)}+y_{n-1}^{(1)}+q_ny_n^{(2)}=\lambda y_n^{(2)}, \end{cases} \tag{1.1}$$

$$y_0^{(1)}=0, \tag{1.2}$$

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where (p_n) and (q_n) are complex sequences for $n = 1, 2, \dots$ and λ is a spectral parameter. The spectral analysis of the BVP (1.1)–(1.2) with principal functions has been studied in [5]. In this article it is proved that the spectrum of the BVP (1.1)–(1.2) consists of the continuous spectrum, the eigenvalues and the spectral singularities and they are finite in number of them with finite multiplicities. Moreover the authors in [5] found the integral representation for the Weyl function and the spectral expansion of (1.1)–(1.2) in terms of the principal functions.

The effect of the spectral singularities in the spectral expansion in terms of the principal vectors was investigated in [9] and [1]. In [6] and [3,4,10] some problems related to the spectral analysis of difference equations with spectral singularities were discussed. Also, the spectral analysis of eigenparameter dependent non-selfadjoint difference equation was studied in [2] and the spectral analysis of quadratic eigenparameter dependent nonselfadjoint discrete Sturm–Liouville equation has been studied in [8]. Let us consider the non-selfadjoint BVP for the discrete Dirac equations

$$\begin{cases} a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)}, \\ a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \end{cases} \quad n \in \mathbb{N}, \tag{1.3}$$

$$(\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2)y_1^{(2)} + (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)y_0^{(1)} = 0, \tag{1.4}$$

where $\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $n \in \mathbb{N}$ are vector sequences, $a_n \neq 0$, $b_n \neq 0$ for all n and L denotes the associated operator. Also $\gamma_2 \neq \frac{\beta_1}{a_0}$, $|\gamma_2| + |\beta_2| \neq 0$ and $\gamma_0\beta_1 - \gamma_1\beta_0 \neq 0$ where γ_i, $\beta_i \in \mathbb{C}$, $i = 0, 1, 2$.

It has been shown in [8] that under the condition

$$\sum_{n=1}^{\infty} \left[\, \exp(\varepsilon n^{\delta})\right] \big(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|\big) < \infty, \tag{1.5}$$

the equation (1.3) has the solution

$$f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \alpha_n \left(I + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right) \begin{pmatrix} e^{iz/2} \\ -i \end{pmatrix} e^{inz}, \; n = 1, 2, \dots \tag{1.6}$$

$$f_0^{(1)}(z) = \alpha_0^{11} \left\{ e^{iz/2} \left[1 + \sum_{m=1}^{\infty} A_{0m}^{11} e^{imz} \right] - i \sum_{m=1}^{\infty} A_{0m}^{12} e^{imz} \right\} \tag{1.7}$$

for $\lambda = 2\sin z/2$ and $z \in \overline{\mathbb{C}}_+ := \left\{ z \in \mathbb{C} : \operatorname{Im} z \geq 0 \right\}$, where

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} \; \alpha_n^{12} \\ \alpha_n^{21} \; \alpha_n^{22} \end{pmatrix}, \quad I = \begin{pmatrix} 1 \; 0 \\ 0 \; 1 \end{pmatrix}, \quad A_{nm} = \begin{pmatrix} A_{nm}^{11} \; A_{nm}^{12} \\ A_{nm}^{21} \; A_{nm}^{22} \end{pmatrix}.$$

Note that $\prod_{k=-\infty}^{\infty}(-1)^{-k}b_k a_{k-1}$ is absolutely convergent by (1.5). Therefore, α_n^{ij} and A_{nm}^{ij} $(i,j=1,2)$ are uniquely expressed in terms of (a_n), (b_n), (p_n) and (q_n), $n\in\mathbb{N}$. Moreover

$$|A_{nm}^{ij}| \le C \sum_{k=n+[|m/2|]}^{\infty} \big(|1-a_k| + |1+b_k| + |p_k| + |q_k|\big)\,, \tag{1.8}$$

holds, where $[|m/2|]$ is the integer part of $m/2$ and $C>0$ is a constant. Therefore f_n is a vector valued analytic function with respect to z in $\mathbb{C}_+ := \{z\in\mathbb{C} : \operatorname{Im} z>0\}$ and continuous in $\overline{\mathbb{C}}_+$ ([7]). The solution $f(z) = (f_n(z)) = \binom{f_n^{(1)}(z)}{f_n^{(2)}(z)}$ is called Jost solution of (1.3).

Also, if we define

$$\begin{aligned} f(z) &= \big(\gamma_0 + 2\gamma_1 \sin z/2 + 4\gamma_2 \sin^2 z/2\big)\, f_1^{(2)}(z) \\ &\quad + \big(\beta_0 + 2\beta_1 \sin z/2 + 4\beta_2 \sin^2 z/2\big)\, f_0^{(1)}(z), \end{aligned}$$

then using (1.6) and (1.7) we obtain,

$$\begin{aligned} f(z) &= -\alpha_0^{11}\beta_2 e^{-i\frac{z}{2}} + i(\alpha_1^{22}\gamma_2 + \alpha_0^{11}\beta_1) \\ &\quad + \big[\alpha_1^{22}\gamma_1 - \alpha_1^{21}\gamma_2 + \alpha_0^{11}(\beta_0+2\beta_2)\big]\, e^{iz/2} \\ &\quad + i\big[-\alpha_1^{22}(\gamma_0+2\gamma_2) + \alpha_1^{21}\gamma_1 - \alpha_0^{11}\beta_1\big]\, e^{iz} \\ &\quad + \big[-\alpha_1^{22}\gamma_1 + \alpha_1^{21}(\gamma_0+2\gamma_2) - \alpha_0^{11}\beta_2\big]\, e^{i3z/2} \\ &\quad + i\big(\alpha_1^{22}\gamma_2 - \alpha_1^{21}\gamma_1\big)\, e^{2iz} - \alpha_1^{21}\gamma_2 e^{i5z/2} \\ &\quad + i\sum_{m=1}^{\infty} \alpha_0^{11}\beta_2 A_{0m}^{12} e^{i(m-1)z} + \sum_{m=1}^{\infty} \alpha_0^{11}\big(\beta_1 A_{0m}^{12} - \beta_2 A_{0m}^{11}\big)\, e^{i(m-1/2)z} \\ &\quad + i\sum_{m=1}^{\infty} \Big\{\gamma_2\big(\alpha_1^{22}A_{1m}^{22} + \alpha_1^{21}A_{1m}^{12}\big) + \alpha_0^{11}\big[\beta_1 A_{0m}^{11} - (\beta_0+2\beta_2)A_{0m}^{12}\big]\Big\} e^{imz} \\ &\quad + \sum_{m=1}^{\infty} \Big\{\alpha_1^{22}(\gamma_1 A_{1m}^{22} - \gamma_2 A_{1m}^{21}) + \alpha_1^{21}(\gamma_1 A_{1m}^{12} - \gamma_2 A_{1m}^{11}) \\ &\qquad + \alpha_0^{11}\big[(\beta_0+2\beta_2)A_{0m}^{11} - \beta_1 A_{0m}^{12}\big]\Big\} e^{i(m+1/2)z} \end{aligned}$$

$$+ i \sum_{m=1}^{\infty} \Big\{ - \alpha_1^{22} \big[(\gamma_0 + 2\gamma_2) A_{1m}^{22} - \gamma_1 A_{1m}^{21} \big] - \alpha_1^{21} \big[(\gamma_0 + 2\gamma_2) A_{1m}^{12} - \gamma_1 A_{1m}^{11} \big]$$

$$+ \alpha_0^{11} (\beta_2 A_{0m}^{12} - \beta_1 A_{0m}^{11}) \Big\} e^{i(m+1)z} + \sum_{m=1}^{\infty} \Big\{ - \alpha_1^{22} \big[\gamma_1 A_{1m}^{22} - (\gamma_0 + 2\gamma_2) A_{1m}^{21} \big]$$

$$+ \alpha_1^{21} \big[(\gamma_0 + 2\gamma_2) A_{1m}^{11} - \gamma_1 A_{1m}^{12} \big] - \alpha_0^{11} \beta_2 A_{0m}^{11} \Big\} e^{i(m+3/2)z}$$

$$+ i \sum_{m=1}^{\infty} \big[\alpha_1^{22} (\gamma_2 A_{1m}^{22} - \gamma_1 A_{1m}^{21}) + \alpha_1^{21} (\gamma_2 A_{1m}^{12} - \gamma_1 A_{1m}^{11}) \big] e^{i(m+2)z}$$

$$+ \sum_{m=1}^{\infty} - \gamma_2 (\alpha_1^{22} A_{1m}^{21} + \alpha_1^{21} \gamma_2 A_{1m}^{11}) e^{i(m+5/2)z}.$$

However, because of the unboundedness of f, we take

$$F(z) := f(z) e^{iz/2},$$

and hence,

$$F(z) = \big\{ \gamma_0 + \gamma_1 \big[(-i)(e^{iz/2} - e^{-iz/2}) \big] \big\} f_1^{(2)}(z)$$

$$+ \big\{ \beta_0 + \beta_1 \big[(-i)(e^{iz/2} - e^{-iz/2}) \big] \big\} f_0^{(1)}(z)$$

$$= i \alpha_0^{11} \beta_1 + \big(\gamma_1 \alpha_1^{22} + \alpha_0^{11} \beta_0 \big) e^{iz/2} + i \big(- \gamma_0 \alpha_1^{22} + \gamma_1 \alpha_1^{22} - \alpha_0^{11} \beta_1 \big) e^{iz}$$

$$+ \big(\gamma_0 \alpha_1^{21} - \gamma_1 \alpha_1^{22} \big) e^{i3z/2} - i \gamma_1 \alpha_1^{21} e^{2iz} + \sum_{m=1}^{\infty} \alpha_0^{11} \beta_1 A_{0m}^{12} e^{i(m-1/2)z}$$

$$+ i \sum_{m=1}^{\infty} \big(- \alpha_0^{11} \beta_0 A_{0m}^{12} + \alpha_0^{11} \beta_1 A_{0m}^{11} \big) e^{imz}$$

$$+ \sum_{m=1}^{\infty} \big(\gamma_1 \alpha_1^{21} A_{1m}^{12} + \gamma_1 \alpha_1^{22} A_{1m}^{22} + \alpha_0^{11} \beta_0 A_{0m}^{11} - \alpha_0^{11} \beta_1 A_{0m}^{12} \big) e^{i(m+1/2)z}$$

$$+ i \sum_{m=1}^{\infty} \Big(- \gamma_0 \alpha_1^{21} A_{1m}^{12} - \gamma_0 \alpha_1^{22} A_{1m}^{22} + \gamma_1 \alpha_1^{21} A_{1m}^{11}$$

$$+ \gamma_1 \alpha_1^{22} A_{1m}^{21} - \alpha_0^{11} \beta_1 A_{0m}^{11} \Big) e^{i(m+1)z}$$

$$+ \sum_{m=1}^{\infty} \big(\gamma_0 \alpha_1^{21} A_{1m}^{11} + \gamma_0 \alpha_1^{22} A_{1m}^{21} - \gamma_1 \alpha_1^{21} A_{1m}^{12} - \gamma_1 \alpha_1^{22} A_{1m}^{22} \big) e^{i(m+3/2)z}$$

$$+ i \sum_{m=1}^{\infty} \left(-\gamma_1 \alpha_1^{21} A_{1m}^{11} - \gamma_1 \alpha_1^{22} A_{1m}^{21} \right) e^{i(m+2)z}.$$

It follows that the function F is analytic in $\mathbb{C}_+$, continuous on the real axis and

$$F(z+4\pi) = F(z).$$

We denote the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$ respectively. From the definition of the eigenvalues and spectral singularities we have

$$\sigma_d = \Big\{ \lambda : \lambda = 2\sin\frac{z}{2},\ z \in P_0,\ F(z) = 0 \Big\},$$

$$\sigma_{ss} = \Big\{ \lambda : \lambda = 2\sin\frac{z}{2},\ z \in [-2\pi, 2\pi],\ F(z) = 0 \Big\}$$

where $P_0 := \{ z : z \in \mathbb{C},\ z = x + iy,\ -2\pi \le x \le 2\pi,\ y > 0 \}$.

In [7] the author proved that eigenvalues and spectral singularities of L are finite and have finite multiplicities under the condition (1.5).

In this paper, we aim to investigate the principal functions corresponding to the eigenvalues and the spectral singularities of L.

2. Principal functions of L

Throughout this section we assume that (1.5) holds.

Definition 2.1 [7]. The multiplicity of a zero of F in $P := P_0 \cup [-2\pi, 2\pi]$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ and $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_\nu$ denote the zeros of F in P_0 and $[-2\pi, 2\pi]$ with multiplicities $m_1, m_2, \dots, m_k$ and $m_{k+1}, m_{k+2}, \dots, m_\nu$, respectively.

Let us define $\ell := \binom{\ell^{(1)}}{\ell^{(2)}}$ where

$$(\ell^{(1)} y)_n = a_{n+1} y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)}, \quad n \in \mathbb{N}$$

and

$$(\ell^{(2)} y)_n = a_{n-1} y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)}, \quad n \in \mathbb{N}.$$

Definition 2.2. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the vectors y_n, $\frac{d}{d\lambda} y_n$, $\frac{d^2}{d\lambda^2} y_n$, $\dots$, $\frac{d^\nu}{d\lambda^\nu} y_n$;

$$\frac{d^j}{d\lambda^j}\, y := \Big\{ \frac{d^j}{d\lambda^j}\, y_n \Big\}_{n\in\mathbb{N}}, \quad j = 0, 1, \dots, \nu,\ n \in \mathbb{N}$$

satisfy the conditions

$$(\ell y)_n - \lambda_0 y_n = 0,$$

$$\Big(\ell\Big(\frac{d^j}{d\lambda^j}\,y\Big)\Big)_n - \lambda_0 \frac{d^j}{d\lambda^j} y_n - \frac{d^{j-1}}{d\lambda^{j-1}} y_n = 0, \quad j = 1, 2, \dots, \nu;\ n \in \mathbb{N}$$

then y_n is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The vectors $\frac{d}{d\lambda} y_n$, $\frac{d^2}{d\lambda^2} y_n$, …, $\frac{d^\nu}{d\lambda^\nu} y_n$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$. The principal vectors of the spectral singularities of L are defined analogously.

We define the vectors

$$\frac{d^j}{d\lambda^j}\, V_n(\lambda_i) = \begin{pmatrix} \frac{1}{j!}\big\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \big\}_{\lambda=\lambda_i} \\ \frac{1}{j!}\big\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \big\}_{\lambda=\lambda_i} \end{pmatrix}, \tag{2.1}$$

$n \in \mathbb{N}$, $j = 0, 1, \dots, m_i - 1$; $i = 1, 2, \dots, k, k+1, \dots, \nu$, where $\lambda = 2 \sin z/2$ and

$$E_n(\lambda) = \begin{pmatrix} E_n^{(1)}(\lambda) \\ E_n^{(2)}(\lambda) \end{pmatrix} := f_n(2 \arcsin \lambda/2) = \begin{pmatrix} f_n^{(1)}(2 \arcsin \lambda/2) \\ f_n^{(2)}(2 \arcsin \lambda/2) \end{pmatrix}. \tag{2.2}$$

If

$$y(\lambda) = \{y_n(\lambda)\} := \begin{pmatrix} y_n^{(1)}(\lambda) \\ y_n^{(2)}(\lambda) \end{pmatrix}_{n \in \mathbb{N}}$$

is a solution of (1.3), then

$$\frac{d^j}{d\lambda^j} y(\lambda) = \Big\{ \Big(\frac{d^j}{d\lambda^j}\Big) y_n(\lambda) \Big\}_{n \in \mathbb{N}} := \begin{Bmatrix} (\frac{d^j}{d\lambda^j}) y_n^{(1)}(\lambda) \\ (\frac{d^j}{d\lambda^j}) y_n^{(2)}(\lambda) \end{Bmatrix}$$

satisfies

$$\begin{pmatrix} a_{n+1} \frac{d^j}{d\lambda^j} y_{n+1}^{(2)}(\lambda) + b_n \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + p_n \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) \\ a_{n-1} \frac{d^j}{d\lambda^j} y_{n-1}^{(1)}(\lambda) + b_n \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + q_n \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + j \frac{d^{j-1}}{d\lambda^{j-1}} y_n^{(1)}(\lambda) \\ \lambda \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + j \frac{d^{j-1}}{d\lambda^{j-1}} y_n^{(2)}(\lambda) \end{pmatrix}. \tag{2.3}$$

From (2.1)–(2.3) we get that

$$(\ell V(\lambda_i))_n - \lambda_0 V_n(\lambda_i) = 0,$$

$$\Big(\ell\Big(\frac{d^j}{d\lambda^j}V(\lambda_i)\Big)\Big)_n-\lambda_0\frac{d^j}{d\lambda^j}V_n(\lambda_i)-\frac{d^{j-1}}{d\lambda^{j-1}}V_n(\lambda_i)=0,$$
$$n\in\mathbb{N},\ j=1,2,\dots,m_i-1;\ i=1,2,\dots,\nu.$$

The vectors $\frac{d^j}{d\lambda^j}V_n(\lambda_i)$ for $j=0,1,2,\dots,m_i-1$; $i=1,2,\dots,k$ and $\frac{d^j}{d\lambda^j}V_n(\lambda_i)$ for $j=0,1,2,\dots,m_i-1$; $i=k+1,k+2,\dots,\nu$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

Theorem 2.1. *We have*

$$\frac{d^j}{d\lambda^j}V_n(\lambda_i)\in\ell_2(\mathbb{N},\mathbb{C}^2),\quad j=0,1,2,\dots,m_i-1;\ i=1,2,\dots,k$$

and

$$\frac{d^j}{d\lambda^j}V_n(\lambda_i)\notin\ell_2(\mathbb{N},\mathbb{C}^2),\quad j=0,1,2,\dots,m_i-1;\ i=k+1,k+2,\dots,\nu.$$

Proof. Using (2.2) we get that

$$\Big\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\Big\}_{\lambda=\lambda_i}=\sum_{t=0}^{j}C_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z)\Big\}_{z=z_i},\quad n\in\mathbb{N}$$

and

$$\Big\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\Big\}_{\lambda=\lambda_i}=\sum_{t=0}^{j}D_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z)\Big\}_{z=z_i},\quad n\in\mathbb{N}$$

where $\lambda_i=2\sin z_i/2$, $z_i\in P=P_0\cup[-2\pi,2\pi]$ for $i=1,2,\dots,k$ and C_t, D_t are constants depending on λ. From (1.6) we obtain that

$$\Big\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z)\Big\}_{z=z_i}=\alpha_n^{11}i^t(n+1/2)^te^{iz_i(n+1/2)} \tag{2.4}$$
$$+\sum_{m=1}^{\infty}\alpha_n^{11}\Big\{A_{nm}^{11}i^t(m+n+1/2)^te^{i(m+n+1/2)z_i}-A_{nm}^{12}i^{t+1}(m+n)^te^{i(m+n)z_i}\Big\}$$

and

$$\Big\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z)\Big\}_{z=z_i}=\alpha_n^{21}i^t(n+1/2)^te^{iz_i(n+1/2)}-i(in)^t\alpha_n^{22}e^{inz_i} \tag{2.5}$$
$$+\sum_{m=1}^{\infty}\alpha_n^{21}\Big\{A_{nm}^{11}i^t(m+n+1/2)^te^{i(m+n+1/2)z_i}-A_{nm}^{12}i^{t+1}(m+n)^te^{i(m+n)z_i}\Big\}$$

$$+\sum_{m=1}^{\infty}\alpha_n^{22}\Big\{A_{nm}^{21}i^t(m+n+1/2)^t e^{i(m+n+1/2)z_i}-A_{nm}^{22}i^{t+1}(m+n)^t e^{i(m+n)z_i}\Big\}.$$

For the principal vectors $\frac{d^j}{d\lambda^j}V_n(\lambda_i)=\{\frac{d^j}{d\lambda^j}V_n(\lambda_i)\}_{n\in\mathbb{N}}$ for $j=0,1,\dots,m_i-1$; $i=1,2,\dots,k$ corresponding to the eigenvalues of L we get

$$\frac{1}{j!}\Big\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\Big\}_{\lambda=\lambda_i}=\frac{1}{j!}\sum_{t=0}^{j}C_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\Big\}, \tag{2.6}$$

and

$$\frac{1}{j!}\Big\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\Big\}_{\lambda=\lambda_i}=\frac{1}{j!}\sum_{t=0}^{j}D_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\Big\}, \tag{2.7}$$

$j=0,1,\dots,m_i-1$; $i=1,2,\dots,k$. Since $\operatorname{Im}\lambda_i>0$ for $i=1,2,\dots,k$ from (2.6) and (2.7) we obtain that

(2.8)

$$\Big\|\frac{d^j}{d\lambda^j}V_n\Big\|^2=\sum_{n=1}^{\infty}\bigg(\Big|\frac{1}{j!}\Big\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\Big\}_{\lambda=\lambda_i}\Big|^2+\Big|\frac{1}{j!}\Big\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\Big\}_{\lambda=\lambda_i}\Big|^2\bigg)$$

$$=\Big(\frac{1}{j!}\Big)^2\sum_{n=1}^{\infty}\bigg(\Big|\sum_{t=0}^{j}C_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\Big\}\Big|^2+\Big|\sum_{t=0}^{j}D_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\Big\}\Big|^2\bigg)$$

$$\le\Big(\frac{1}{j!}\Big)^2\sum_{n=1}^{\infty}\bigg\{\bigg(\sum_{t=0}^{j}\Big|C_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\Big\}\Big|^2\bigg)+\bigg(\sum_{t=0}^{j}\Big|D_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\Big\}\Big|^2\bigg)\bigg\}$$

$$\le\Big(\frac{1}{j!}\Big)^2\bigg[\sum_{n=1}^{\infty}\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\bigg(\Big|\Big\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\Big\}\Big|+\Big|\Big\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\Big\}\Big|\bigg)\bigg]^2$$

or

$$\Big\|\frac{d^j}{d\lambda^j}V_n\Big\|^2\le\Big(\frac{1}{j!}\Big)^2\bigg\{\sum_{n=1}^{\infty}\bigg[\sum_{t=0}^{j}\max\{|C_t|,|D_t|\} \tag{2.9}$$

$$\times\Big\{\big(|\alpha_n^{11}|+|\alpha_n^{21}|\big)\Big(|n+1/2|^t e^{-(n+1/2)\operatorname{Im}z_i}\Big)+|\alpha_n^{22}||n|^t e^{-n\operatorname{Im}z_i}\Big\}$$

$$+\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\bigg\{\sum_{m=1}^{\infty}\big(|\alpha_n^{11}|+|\alpha_n^{21}|\big)$$

$$\times\Big(|A_{nm}^{11}||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im}z_i}\Big)+|A_{nm}^{12}||m+n|^t e^{-(m+n)\operatorname{Im}z_i}\bigg\}$$

$$+\sum_{t=0}^{j}|D_t|\Bigg\{\sum_{m=1}^{\infty}|\alpha_n^{22}|\Big(\big|A_{nm}^{21}\big||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im} z_i} + \big|A_{nm}^{22}\big||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\Big)\Bigg\}\Bigg]\Bigg\}^2$$

From (2.9), if we say

$$Y=\Big(\frac{1}{j!}\Big)^2\sum_{n=1}^{\infty}\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}$$
$$\times\Big\{\big(|\alpha_n^{11}|+|\alpha_n^{21}|\big)\big(|n+1/2|^t e^{-(n+1/2)\operatorname{Im} z_i}\big)+|\alpha_n^{22}||n|^t e^{-n\operatorname{Im} z_i}\Big\}$$

then

$$Y\le\frac{A}{(j!)^2}\sum_{n=1}^{\infty}\Big(1+(n+1/2)+(n+1/2)^2+\ldots+(n+1/2)^j\Big) \tag{2.10}$$
$$\times e^{-(n+1/2)\operatorname{Im} z_i}+(1+n+n^2+\ldots+n^j)e^{-n\operatorname{Im} z_i}$$
$$\le\frac{A(j+1)}{(j!)^2}\sum_{n=1}^{\infty}\Big[(n+1/2)^j e^{-(n+1/2)\operatorname{Im} z_i}+n^j e^{-n\operatorname{Im} z_i}\Big]<\infty$$

holds, where

$$A=\max\big\{|C_t|,|D_t|\big\}\max\big\{(|\alpha_n^{11}|+|\alpha_n^{21}|),|\alpha_n^{22}|\big\}.$$

Now we define the function

$$g_n(z)=\sum_{t=0}^{j}\max\big\{|C_t|,|D_t|\big\}\Big\{\sum_{m=1}^{\infty}\big(|\alpha_n^{11}|+|\alpha_n^{21}|\big) \tag{2.11}$$
$$\times\Big(|A_{nm}^{11}||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im} z_i}+\big|A_{nm}^{12}\big||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\Big)\Big\}$$
$$+\sum_{t=0}^{j}|D_t|\Big\{\sum_{m=1}^{\infty}|\alpha_n^{22}|\Big(|A_{nm}^{21}||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im} z_i}$$
$$+|A_{nm}^{22}||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\Big)\Big\}.$$

So we get,

$$\Big(\frac{1}{j!}\Big)^2\sum_{n=1}^{\infty}\Bigg[\sum_{t=0}^{j}\max\big\{|C_t|,|D_t|\big\}\Big\{\sum_{m=1}^{\infty}\big(|\alpha_n^{11}|+|\alpha_n^{21}|\big)$$

$$\times\Big(|A_{nm}^{11}||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im} z_i}+|A_{nm}^{12}||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\Big)\Big\}$$

$$+\sum_{t=0}^{j}|D_t|\Big\{\sum_{m=1}^{\infty}|\alpha_n^{22}|\Big(|A_{nm}^{21}||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im} z_i}$$

$$+|A_{nm}^{22}||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\Big)\Big\}\Bigg]^2=\Big(\frac{1}{j!}\Big)^2\Big(\sum_{n=1}^{\infty}g_n(z)\Big)^2.$$

Using the boundedness of A_{nm}^{ij} and α_n^{ij} for $i,j=1,2$, we obtain that

$$g_n(z)\le\max\big\{|C_t|,|D_t|\big\}\,M\sum_{t=0}^{j}\sum_{m=1}^{\infty}\Big\{|m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im} z_i}$$

$$+|m+n|^t e^{-(m+n)\operatorname{Im} z_i}\Big\}$$

where

$$M=\max\Big\{(|\alpha_n^{11}|+|\alpha_n^{21}|)|A_{nm}^{11}|,\ |\alpha_n^{22}||A_{nm}^{21}|,$$

$$(|\alpha_n^{11}|+|\alpha_n^{21}|)|A_{nm}^{12}|,\ |\alpha_n^{22}||A_{nm}^{22}|\Big\}.$$

If we take $\max\{|C_t|,|D_t|\}\,M=N$, we can write

$$g_n(z)\le N\sum_{t=0}^{j}e^{-n\operatorname{Im} z_i}\sum_{m=1}^{\infty}\Big\{(m+n+1/2)^t e^{-m\operatorname{Im} z_i}+(m+n)^t e^{-m\operatorname{Im} z_i}\Big\}$$

$$=Ne^{-n\operatorname{Im} z_i}\Big\{\sum_{m=1}^{\infty}2e^{-m\operatorname{Im} z_i}+\sum_{m=1}^{\infty}e^{-m\operatorname{Im} z_i}\big((m+n+1/2)+(m+n)\big)$$

$$+\cdots+\sum_{m=1}^{\infty}e^{-m\operatorname{Im} z_i}\big((m+n+1/2)^j+(m+n)^j\big)\Big\}$$

$$\le Ne^{-n\operatorname{Im} z_i}\sum_{m=1}^{\infty}\sum_{t=0}^{j}e^{-m\operatorname{Im} z_i}\big((m+n+1/2)^t+(m+n)^t\big)\le Be^{-n\operatorname{Im} z_i}$$

where

$$B = N \sum_{t=0}^{j} e^{-m \operatorname{Im} z_i} \big((m+n+1/2)^t + (m+n)^t \big) .$$

Therefore, we have,

$$\left(\frac{1}{j!} \sum_{n=1}^{\infty} g_n(z) \right)^2 \le \left(\frac{1}{j!} \sum_{n=1}^{\infty} B e^{-n \operatorname{Im} z_i} \right)^2 < \infty. \tag{2.12}$$

From (2.10) and (2.12), $\frac{d^j}{d\lambda^j} V_n(\lambda_i) \in \ell_2(\mathbb{N}, \mathbb{C}^2)$ for $j = 0, 1, \ldots, m_i - 1;\ i = 1, 2, \ldots, k$.

On the other hand, since $\operatorname{Im} z_i = 0$ for $j = 0, 1, \ldots, m_i - 1;\ i = k+1, k+2, \ldots, \nu$ using (2.4), we find that

$$\sum_{n=1}^{\infty} \big| \alpha_n^{11} i^t (n+1/2)^t e^{i z_i (n+1/2)} \big|^2 = \infty,$$

but the other terms in (2.4) belong to $\ell_2(\mathbb{N}, \mathbb{C}^2)$, so $\frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$. Similarly, from (2.5), we get $\frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$, then we obtain that $\frac{d^j}{d\lambda^j} V_n(\lambda_i) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$ for $j = 0, 1, \ldots, m_i - 1$; $i = k+1, k+2, \ldots, \nu$. □

Let us introduce the Hilbert space

$$H_{-j}(\mathbb{N}) = \left\{ y = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} : \sum_{n \in \mathbb{N}} (1+|n|)^{-2j} \big(|y_n^{(1)}|^2 + |y_n^{(2)}|^2 \big) < \infty \right\},$$

$j = 0, 1, 2, \ldots,$ with

$$\|y\|_{-j}^2 = \sum_{n \in \mathbb{N}} (1+|n|)^{-2j} \big(|y_n^{(1)}|^2 + |y_n^{(2)}|^2 \big) .$$

Now we have the following result:

THEOREM 2.2. *We have* $\frac{d^j}{d\lambda^j} V_n(\lambda_i) \in H_{-(j+1)}(\mathbb{N})$ *for* $j = 0, 1, \ldots, m_i - 1$; $i = k+1, k+2, \ldots, \nu$.

PROOF. Using (2.1), (2.6) and (2.7) we have

(2.13)

$$\sum_{n \in \mathbb{N}} (1+|n|)^{-2(j+1)} \left(\left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 + \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 \right)$$

$$= \sum_{n\in\mathbb{N}} \frac{(1+|n|)^{-2(j+1)}}{(j!)^2} \left\{ \Big| \sum_{t=0}^{j} C_t \Big\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \Big\} \Big|^2 + \Big| \sum_{t=0}^{j} D_t \Big\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \Big\} \Big|^2 \right\}$$

$$\le \frac{1}{(j!)^2} \sum_{n=1}^{\infty} (1+|n|)^{-2(j+1)}$$

$$\times \left\{ \Big(\sum_{t=0}^{j} \Big| C_t \Big\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \Big\} \Big| \Big)^2 + \Big(\sum_{t=0}^{j} |D_t \Big\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \Big\}| \Big)^2 \right\}$$

for $j=0,1,2,\dots,m_i-1$; $i=k+1,k+2,\dots,\nu$. Since $\operatorname{Im} z_i = 0$, using (2.13) we obtain

$$\text{(2.14)} \qquad \sum_{n\in\mathbb{N}} (1+|n|)^{-2(j+1)} \frac{1}{(j!)^2} \Big(\sum_{t=0}^{j} \Big| C_t \Big\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \Big\} \Big| \Big)^2$$

$$\le \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \Big\{ \sum_{t=0}^{j} (1+|n|)^{-(j+1)} (n+1/2)^t |\alpha_n^{11}| |C_t|$$

$$+ \sum_{t=0}^{j} |C_t| |\alpha_n^{11}| (1+|n|)^{-(j+1)} \sum_{m=1}^{\infty} |A_{nm}^{11}| (m+n+1/2)^t + |A_{nm}^{12}| (m+n)^t \Big\}^2$$

$$= \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \left\{ \Big(\sum_{t=0}^{j} (1+|n|)^{-(j+1)} (n+1/2)^t |\alpha_n^{11}| |C_t| \Big)^2 \right.$$

$$+ 2(1+|n|)^{-2(j+1)} |\alpha_n^{11}|^2 \Big[\sum_{t=0}^{j} (n+1/2)^t |C_t| \Big]$$

$$\times \Big[\sum_{t=0}^{j} |C_t| \sum_{m=1}^{\infty} |A_{nm}^{11}| (m+n+1/2)^t + |A_{nm}^{12}| (m+n)^t \Big]$$

$$+ \Big(\sum_{t=0}^{j} |C_t| (1+|n|)^{-(j+1)} |\alpha_n^{11}| \sum_{m=1}^{\infty} |A_{nm}^{11}| \Big)$$

$$\left. \times \big(m+n+1/2 \big)^t + |A_{nm}^{12}| (m+n)^t \right\}^2.$$

Using (2.14), (1.5) and (1.8) we first obtain that

$$
\begin{aligned}
(2.15)\qquad & \Big(\sum_{t=0}^{j}|C_t||\alpha_n^{11}|(1+|n|)^{-(j+1)} \\
&\times\sum_{m=1}^{\infty}\Big(|A_{nm}^{11}|(m+n+1/2)^t+|A_{nm}^{12}|(m+n)^t\Big)\Big)^2 \\
&\le 4\Big(\sum_{t=0}^{j}|C_t||\alpha_n^{11}|\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}(m+n+1/2)^t \\
&\times C\sum_{j=n+\lceil m/2\rceil}^{\infty}\big(|1-a_j|+|1+b_j|+|p_j|+|q_j|\big)\,e^{-\varepsilon j^\delta}e^{\varepsilon j^\delta}\Big)^2 \\
&\le 4\Big\{\sum_{t=0}^{j}|\alpha_n^{11}|\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}(m+n+1/2)^t C\exp\big(-\varepsilon((n+m)/4)^\delta\big) \\
&\times\sum_{j=n+\lceil m/2\rceil}^{\infty}e^{\varepsilon j^\delta}\big(|1-a_j|+|1+b_j|+|p_j|+|q_j|\big)\Big\}^2 \\
&\le C_1\Big(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}(m+n+1/2)^t\exp\big(-\varepsilon((n+m)/4)^\delta\big)\Big)^2 \\
&\le C_1\Big(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}(m+n+1/2)^t\exp\big(-\varepsilon((n+m)/4)^{1/2}\big)\Big)^2 \\
&\le C_1\Big(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}(m+n+1/2)^t\exp\big(-\varepsilon\sqrt{2}(n^{1/2}+m^{1/2})/4\big)\Big)^2 \\
&= C_1(1+|n|)^{-2(j+1)}\exp\big(-\varepsilon\sqrt{2}n^{1/2}/2\big) \\
&\times\Big(\sum_{t=0}^{j}\sum_{m=1}^{\infty}(m+n+1/2)^t\exp\big(-\varepsilon\sqrt{2}m^{1/2}/4\big)\Big)^2 \\
&= G\exp\big(-\varepsilon\sqrt{2}n^{1/2}/2\big)\,(1+|n|)^{-2(j+1)}
\end{aligned}
$$

where

$$
C_1=\Big(2C|\alpha_n^{11}|\sum_{j=n+\lceil m/2\rceil}^{\infty}e^{\varepsilon j^\delta}\big(|1-a_j|+|1+b_j|+|p_j|+|q_j|\big)\Big)^2
$$

$$G = C_1\Big(\sum_{t=0}^{j}\sum_{m=1}^{\infty}(m+n+1/2)^t \exp\big(-\varepsilon\sqrt{2}m^{1/2}/4\big)\Big)^2.$$

Hence we get from (2.15)

$$\begin{aligned}(2.16)\qquad &\sum_{n=1}^{\infty}\Big(\sum_{t=0}^{j}|C_t|(1+|n|)^{-(j+1)}|\alpha_n^{11}| \\ &\times\sum_{m=1}^{\infty}|A_{nm}^{11}|(m+n+1/2)^t+|A_{nm}^{12}|(m+n)^t\Big)^2 \\ &\le G\sum_{n=1}^{\infty}\exp\big(-\varepsilon\sqrt{2}n^{1/2}/2\big)\,(1+|n|)^{-2(j+1)}<\infty.\end{aligned}$$

Secondly, using (2.14) and (2.15) we obtain that

$$\begin{aligned}(2.17)\qquad &\sum_{n=1}^{\infty}2\Big\{\Big[\sum_{t=0}^{j}|\alpha_n^{11}||C_t|(1+|n|)^{-(j+1)}(n+1/2)^t\Big]\Big[\sum_{t=0}^{j}|C_t||\alpha_n^{11}| \\ &\times\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}((m+n+1/2)^t|A_{nm}^{11}|)+(m+n)^t|A_{nm}^{12}|\Big]\Big\} \\ &\le T\sum_{n=1}^{\infty}\Big[\sum_{t=0}^{j}(1+|n|)^{-2(j+1)}(n+1/2)^t\exp\big(-\varepsilon\sqrt{2}n^{1/2}/4\big)\Big]<\infty,\end{aligned}$$

where

$$T = |\alpha_n^{11}|G^{1/2}\max|C_t|$$

and also expression of the left side of (2.15) is obviously convergent. So, we get from (2.16) and (2.17)

$$\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\Big(\sum_{t=0}^{j}\Big|C_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\Big\}\Big|\Big)^2<\infty$$

and similarly

$$\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\Big(\sum_{t=0}^{j}\Big|D_t\Big\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\Big\}\Big|\Big)^2<\infty.$$

Finally $\frac{d^j}{d\lambda^j}V_n(\lambda_i)\in H_{-(j+1)}(\mathbb{N})$ for $j=0,1,\dots,m_i-1$; $i=k+1,\dots,\nu$. □

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