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Helly number, domination and homology

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Abstract
The Helly number $\mathfrak{h}(\mathcal{H})$ of a hypergraph $\mathcal{H}$ whose edges have no vertex in common is the size of its largest inclusion-wise minimal subhypergraph with empty intersection. We prove that the homological connectivity of the independence complex of $\mathcal{H}$ can be bounded in terms of its number of vertices and the Helly number of its complementary hypergraph $\mathcal{H}^c$. Our proof relies on an interconnection between the Helly number and a domination parameter of hypergraphs. Separately, we provide an upper bound to the Leray number of the noncover complex of any hypergraph $\mathcal{H}$ without any isolated vertex involving the strong domination number of $\mathcal{H}$.

Keywords Leray number · Helly number · Strong domination number · Induced matching number · Homological connectivity

1 Introduction and preliminaries

The determination of homological properties of simplicial complexes associated to hypergraphs greatly benefits from the underlying combinatorial structure and invariants of hypergraphs. Early results on this direction are pioneered in a series of collaborative works by Aharoni (see [14] and the references therein). These results were later reformulated and unified in terms of homology by Meshulam [14, Theorem 1.2 and 1.3], where he proves that the homology group of the independence complex

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of a graph must vanish at dimension k, if k is sufficiently smaller than the (total) domination number of the underlying graph.

The next natural level is to consider the same problem for independence complexes of hypergraphs rather than graphs. The right generalization of the (total) domination number from graphs to hypergraphs is introduced recently by Kim and Kim [12] in order to fulfill such an expectation. They prove that a similar conclusion as in the graph's case holds for hypergraphs when we consider the strong total domination number of hypergraphs. In the same work, they also discuss the Lerayness of noncover complexes of hypergraphs. These are nothing but Alexander duals of independence complexes, which play a role on the existence of rainbow covers (transversals) of hypergraphs. However, as they pointed out, Leray number of the noncover complex of a hypergraph need not to be bounded in terms of the strong total domination number in general.

Our aim in this work is two-fold. We prove that the Leray number of an arbitrary simplicial complex can be bounded from above by that of the independence complex of a bipartite graph (the bipartite complement of the facet-vertex bipartite graph). Such a result enables us to construct an upper bound to the Leray number of noncover complexes in terms of the strong domination in hypergraphs. In the other direction, we show that the homological connectivity of independence complexes of hypergraphs can be used to bound the Helly number of hypergraphs and vice versa. Our proof relies on a connection between the Helly number and a new domination parameter of hypergraphs.

We begin with recalling some necessary background terminology together with a detailed analysis of our main results.

A *hypergraph* $\mathcal{H}$ on a (finite) set V is a family of subsets of V. The elements of V and $\mathcal{H}$ are vertices and edges of $\mathcal{H}$. A vertex $v \in V$ is *isolated* in $\mathcal{H}$ if there exists no edge $e \in \mathcal{H}$ with $v \in e$. The *degree* of a vertex is the number of edges in $\mathcal{H}$ containing it. A hypergraph $\mathcal{H}$ is k-*uniform* if $|e| = k$ for every $e \in \mathcal{H}$. For instance, graphs are 2-uniform hypergraphs.

A *simplicial complex* X on a set V is a hypergraph, closed under inclusion, i.e., if $A \subseteq B \in X$, then $A \in X$. The elements of X are its *faces*, and faces which are maximal with respect to the inclusion are *facets* of X. We denote by $\mathcal{F}(X)$, the set of facets of X. Let $\widetilde{H}_j(X)$ denote the j-dimensional reduced homology group of X with coefficients in $\mathbb{Z}_2$. The *Alexander dual* of a simplicial complex is the complex defined by

$$X^\vee := \{A \subseteq V : V \setminus A \notin X\}.$$

The homology groups of a simplicial complex and its Alexander dual are related by the following duality.

Theorem 1 (The (combinatorial) Alexander duality) $\widetilde{H}_j(X) \cong \widetilde{H}_{n-j-3}(X^\vee)$ *for every simplicial complex* X *on* n*-vertices and for all* j.

We refer readers to [5] for an elementary proof of Theorem 1.

The (*homological*) *connectivity* $\mathrm{conn}(X)$ of a simplicial complex X is defined to be the greatest integer k (if exists) such that $\widetilde{H}_j(X) = 0$ for every $-1 \leqslant j \leqslant k$. Throughout, we will use a shifted version, $\eta(X) := \mathrm{conn}(X) + 2$. In the case where

$\widetilde{H}_j(X) = 0$ for all j, i.e., X is homologically trivial, we set $\eta(X) := \infty$. In particular, $\eta(\emptyset) := 0$. We note that $\eta(X) \geqslant d$ if and only if $\widetilde{H}_j(X^\vee) = 0$ for every $j \geqslant n-d-1$ by the Alexander duality.

A simplicial complex X is called *p-Leray* if $\widetilde{H}_i(X[S]) = 0$ for every $i \geqslant p$ and $S \subseteq V$, where $X[S] := \{A \in X \colon A \subseteq S\}$ is the subcomplex of X induced by S. Then, the *Leray number* $\mathrm{L}(X)$ of X is the least integer p such that X is p-Leray. This coincides with the (Castelnuovo-Mumford) *regularity* of the Stanley-Reisner ring of X (see e.g., [4] for details).

When $\mathcal{H}$ is a hypergraph, there are various ways to associate a simplicial complex to $\mathcal{H}$. A subset $I \subseteq V$ is *independent* if $e \not\subset I$ for each $e \in \mathcal{H}$. The family $\mathrm{Ind}(\mathcal{H})$ of all independent sets in $\mathcal{H}$ forms a simplicial complex, the *independence complex* of $\mathcal{H}$. A subset $T \subseteq V$ is said to be a *cover* (or a *transversal*) in $\mathcal{H}$ if $e \cap T \neq \emptyset$ for every $e \in \mathcal{H}$. A subset $U \subseteq V$ is a *noncover* if U is not a cover in $\mathcal{H}$. The family $\mathrm{NC}(\mathcal{H})$ of all noncovers of a hypergraph $\mathcal{H}$ is called the *noncover complex* of $\mathcal{H}$. The independence and noncover complexes of a hypergraph $\mathcal{H}$ are related through the Alexander duality.

Observation 2 $\mathrm{Ind}(\mathcal{H})^\vee = \mathrm{NC}(\mathcal{H})$ *for every hypergraph* $\mathcal{H}$.

1.1 Helly number and domination

The *Helly number* $\mathfrak{h}(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the largest integer h such that there exists a subhypergraph $\{e_1, \ldots, e_h\} \subseteq \mathcal{H}$ for which $\bigcap_{i=1}^h e_i = \emptyset$ while $\bigcap_{j\in\sigma} e_j \neq \emptyset$ for every nonempty subset $\sigma \subsetneq [h] := \{1, 2, \ldots, h\}$. In particular, we set $\mathfrak{h}(\mathcal{H}) := \infty$ if $\mathcal{H}$ is intersecting, i.e., $\bigcap_{e\in\mathcal{H}} e \neq \emptyset$. It is customary to say that $\mathcal{H}$ is p-Helly whenever $\mathfrak{h}(\mathcal{H}) \leqslant p$.

The Helly number of a family of sets originates from the celebrated Helly's theorem [10], stating that any finite family of convex sets in $\mathbb{R}^d$ with an empty intersection has Helly number at most $d+1$ (see also [7] for a dynamic survey).

The concept of a dominating set in hypergraphs varies in the literature. The following (the strong total domination) seems to first appear in [12].

Let $\mathcal{H}$ be a hypergraph on V without any isolated vertex. A subset $D \subseteq V$ *strongly dominates* a vertex v if there exists a subset $D_v \subseteq D \setminus \{v\}$ (possibly empty) such that $D_v \cup \{v\} \in \mathcal{H}$. If every vertex in $V \setminus D$ is strongly dominated by D, then D is called a *strongly dominating set* for $\mathcal{H}$. The least cardinality $\gamma_s(\mathcal{H})$ of a strongly dominating set in $\mathcal{H}$ is called the *strong domination number* of $\mathcal{H}$.

Similarly, if a set $D \subseteq V$ strongly dominates every vertex in V, then D is said to be a *strongly totally dominating set*, and the smallest cardinality $\gamma_{st}(\mathcal{H})$ of such a set is called the *strong total domination number* of $\mathcal{H}$.[1]

If $\mathcal{H}$ contains an isolated vertex, then we define $\gamma_s(\mathcal{H}) = \gamma_{st}(\mathcal{H}) := \infty$ by convention.

[1] Kim and Kim [12] denote the strong total domination number of a hypergraph $\mathcal{H}$ by $\tilde{\gamma}(\mathcal{H})$, and use the term "domination number" instead.

1.2 Main results

We first prove that Leray numbers of the noncover complex of a hypergraph $\mathcal{H}$ and that of the independence complex of its incidence bipartite graph are closely related. Recall that the *incidence graph* $\mathrm{Inci}(\mathcal{H})$ of $\mathcal{H}$ is the bipartite graph on $V \cup \mathcal{H}$ such that $(v, e) \in E(\mathrm{Inci}(\mathcal{H}))$ if and only if $v \in e$.

Theorem 3 $\mathrm{L}(\mathrm{NC}(\mathcal{H})) + 1 \leqslant \mathrm{L}(\mathrm{Ind}(\mathrm{Inci}(\mathcal{H})))$ *for every hypergraph* $\mathcal{H}$ *without any isolated vertex.*

We then provide an upper bound to Leray number of the noncover complexes of hypergraphs in terms of their strong domination numbers.

Theorem 4 $\mathrm{L}(\mathrm{NC}(\mathcal{H})) \leqslant n - \frac{\gamma_s(\mathcal{H})}{2} - 1$ *for every* n*-vertex hypergraph* $\mathcal{H}$ *without any isolated vertex.*

We remark that as pointed by Kim and Kim [12, Example 4.3], the noncover complex of a hypergraph $\mathcal{H}$ does not need to be $(n - \frac{\gamma_{st}(\mathcal{H})}{2} - 1)$-Leray unless $|e| \leqslant 3$ for every edge $e \in \mathcal{H}$.

Our next result concerns the connectivity of the independence complexes of hypergraphs. Let $\mathcal{H}$ be a hypergraph on V. The *complementary hypergraph* of $\mathcal{H}$ is the hypergraph $\mathcal{H}^c$ on the same set V such that $e \in \mathcal{H}^c$ if and only if $V \setminus e \in \mathcal{H}$. In particular, the duality $(\mathcal{H}^c)^c = \mathcal{H}$ holds. We note that the complement graph and the complementary hypergraph of a graph are two distinct notions. Notice that $\mathcal{H}$ contains no isolated vertex if and only if $\mathfrak{h}(\mathcal{H}^c) < \infty$.

We give a lower bound to the (shifted) connectivity of independence complexes involving the Helly number of the complementary hypergraph.

Theorem 5 $\eta(\mathrm{Ind}(\mathcal{H}))) \geqslant \frac{n - \mathfrak{h}(\mathcal{H}^c)}{2}$ *for every* n*-vertex hypergraph* $\mathcal{H}$ *without any isolated vertex.*

We note that the result of Theorem 5 is more interesting than its proof. Put it other way around. Let $\mathcal{H}$ be an n-vertex hypergraph. If $\eta(\mathrm{Ind}(\mathcal{H}^c)) < \infty$, then $\mathfrak{h}(\mathcal{H}) \geqslant n - 2\eta(\mathrm{Ind}(\mathcal{H}^c))$. In other words, if $\widetilde{H}_k(\mathrm{Ind}(\mathcal{H}^c)) \neq 0$ for some $k \geqslant 1$, then $\mathfrak{h}(\mathcal{H}) \geqslant n - 2k - 3$.

The proof of Theorem 5 readily follows from an earlier result of Kim and Kim by the right adjustment of the relevant domination parameter.

Theorem 6 ([12, Theorem 1.4]) $\eta(\mathrm{Ind}(\mathcal{H})) \geqslant \frac{\gamma_{st}(\mathcal{H})}{2}$ *for every hypergraph* $\mathcal{H}$.

For our purposes, we need a slightly weaker version of the strong total domination in hypergraphs.

Definition 7 We call a subset $D \subseteq V$ a *canonical dominating set* for $\mathcal{H}$ if for every vertex $x \in V$, there exists an edge $e_x \in \mathcal{H}$ such that $x \in e_x$ and $|e_x \cap (V \setminus D)| \leqslant 1$. The minimum cardianlity $\gamma_c(\mathcal{H})$ of a canonical dominating set in $\mathcal{H}$ is called the *canonical domination number* of $\mathcal{H}$. We set $\gamma_c(\mathcal{H}) := \infty$ whenever $\mathcal{H}$ contains an isolated vertex.

Observe that the inequalities $\gamma_s(\mathcal{H}) \leqslant \gamma_c(\mathcal{H}) \leqslant \gamma_{st}(\mathcal{H})$ always hold, while the gap between each pair of these numbers could be arbitrarily large (see Sect. 5). Furthermore, if $\mathcal{H} = G$ is just a graph without any isolated vertex, we have that $\gamma_c(G) = \gamma_s(G) = \gamma(G)$ and $\gamma_{st}(G) = \gamma_t(G)$, where $\gamma(G)$ and $\gamma_t(G)$ denote the (ordinary) domination and total domination numbers of G.

As in the case of the strong total domination number, we verify that we cannot replace the strong domination number in Theorem 4 by the canonical domination number (see Example 32).

Notice that by Theorem 6, the proof of Theorem 5 would be completed once we prove the following.

Lemma 8 *$\mathfrak{h}(\mathcal{H}^c) + \gamma_c(\mathcal{H}) = n$ for every n-vertex hypergraph $\mathcal{H}$ without any isolated vertex.*

2 Proof of Theorem 3

In this section, we give the proof of Theorem 3. In fact, we prove a stronger statement than what Theorem 3 claims. Beforehand, we provide some relevant terminology first. Some of the results of this section have appeared in the unpublished manuscript [3].

Let $G = (V, E)$ be a graph. For a given vertex $x \in V$, the open and closed neighborhoods of x are defined by $N_G(x) := \{y \in V : xy \in E\}$ and $N_G[x] := N_G(x) \cup \{x\}$, respectively. In particular, we let $N_G(S) := \bigcup_{s \in S} N_G(s)$ and $N_G[S] := N_G(S) \cup S$ for every subset $S \subseteq V$. Throughout K_n will denote the complete graph on n vertices for each $n \geqslant 1$.

For $U \subset V$, the graph induced by U is written $G[U]$, and in particular, we abbreviate $G[V \setminus U]$ to $G - U$, and write $G - x$ whenever $U = \{x\}$.

The existence of vertices satisfying some extra properties is useful when dealing with the homotopy type of independence complexes of graphs.

Lemma 9 (The fold lemma [8]) *If $N_G(u) \subseteq N_G(v)$, then there is a homotopy equivalence* $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - v)$.

A bipartite graph $B = (V_0, V_1; E)$ is called a *Sperner bipartite graph* if $V(B) = V_0 \cup V_1$ contains no pair of vertices u and v such that $N_B(u) \subseteq N_B(v)$. The *bipartite complement* B^{bc} of B is the bipartite graph on the same set of vertices such that $v_0v_1 \in E(B^{\mathrm{bc}})$ if and only if $v_0v_1 \notin E(B)$ for $v_i \in V_i$ and $i \in \{0, 1\}$.

Definition 10 Let X be a simplicial complex on V. We define a bipartite graph $B_S(X)$ for each $S \subseteq V$ by $V(B_S(X)) := S \cup \mathcal{F}(X)$ and $(s, F) \in E(B_S(X))$ if and only if $s \notin F$ for any $s \in S$ and $F \in \mathcal{F}(X)$. In particular, we abbreviate $B_V(X)$ to $B(X)$.

We note that if we associate to X, the hypergraph $\mathcal{F}(X)$ of facets of X on V, the graph $B(X)$ is just the bipartite complement of the incidence bipartite graph of $\mathcal{F}(X)$, i.e., $B(X) \cong (\mathrm{Inci}(\mathcal{F}(X)))^{\mathrm{bc}}$.

The following theorem of Nagel and Reiner [15] is our main motivation for the definition of $B(X)$.

Theorem 11 [15] *The homotopy equivalence* $\mathrm{Ind}(B(X)) \simeq \Sigma(X)$ *holds for any simplicial complex* X*, where* $\Sigma(X)$ *is the (unreduced) suspension of* X*.*

We first provide a local version of Theorem 11.

Lemma 12 *Let* X *be a simplicial complex. Then* $\mathrm{Ind}(B_S(X)) \simeq \mathrm{Ind}(B(X[S]))$ *for any* $S \subseteq V$*.*

Proof Assume that $\mathcal{F}(X[S]) = \{H_1, \ldots, H_k\}$. We decompose the set $\mathcal{F}(X)$ into subfamilies as follows.

$$\begin{aligned}
\mathcal{F}_i &:= \{F \in \mathcal{F}(X) \colon H_i \subseteq F\} \text{ for each } i \in [k],\\
\mathcal{A}_S &:= \{C \in \mathcal{F}(X) \colon \emptyset \neq (C \cap S) \notin \mathcal{F}(X[S])\},\\
\mathcal{B}_S &:= \{D \in \mathcal{F}(X) \colon D \cap S = \emptyset\}.
\end{aligned}$$

Note that if $(s, H_i) \in E(B(X[S]))$ for some $s \in S$ and $i \in [k]$, then $(s, F) \in E(B_S(X))$ for any $F \in \mathcal{F}_i$. Indeed, if $H_i \cup \{s\} \in X$, then $H_i \cup \{s\} \in X[S]$. However, the latter is impossible due to the fact that H_i is a facet of $X[S]$. Then, it follows that

$$N_{B_S(X)}(F) = N_{B_S(X)}(F') = N_{B_S(X[S])}(H_i)$$

for any $F, F' \in \mathcal{F}_i$ and $i \in [k]$. If we fix a facet, say $F_i \in \mathcal{F}_i$ for each $i \in [k]$, the homotopy equivalence $\mathrm{Ind}(B_S(X)) \simeq \mathrm{Ind}(B_S(X) - (\mathcal{F}_i \setminus F_i))$ holds by Lemma 9. In other words, we can remove the facets in $\mathcal{F}_i$ one by one until there remains only a single facet without altering the homotopy type.

On the other hand, if $C \in \mathcal{A}_S$, then $C \cap S$ is a nonempty face of $X[S]$ which is not maximal. So, it is contained by a facet, say H_i. However, the containment $C \cap S \subseteq H_i$ forces that $N_{B_S(X)}(F_i) \subseteq N_{B_S(X)}(C)$, where $F_i \in \mathcal{F}_i$ is the facet that we fixed earlier. Therefore, the removal of any such vertex from $\mathrm{Ind}(B_S(X))$ does not alter the homotopy type. Similarly, if $D \in \mathcal{B}_S$, then we have $N_{B_S(X)}(F_i) \subseteq N_{B_S(X)}(D)$ for any $i \in [k]$ so that $\mathrm{Ind}(B_S(X)) \simeq \mathrm{Ind}(B_S(X) - \mathcal{B}_S)$.

Finally, if we define $\mathcal{C}_S := \bigcup_{i=1}^{k} (\mathcal{F}_i \setminus F_i) \cup \mathcal{A}_S \cup \mathcal{B}_S$, we conclude that

$$\mathrm{Ind}(B_S(X)) \simeq \mathrm{Ind}(B_S(X) - \mathcal{C}_S) \cong \mathrm{Ind}(B_S(X)[S \cup \{F_1, \ldots, F_k\}]).$$

However, the graph $B_S(X)[S \cup \{F_1, \ldots, F_k\}]$ is isomorphic to $B_S(X[S])$ that proves the claim. □

Corollary 13 $\mathrm{Ind}(B_S(X)) \simeq \Sigma(X[S])$ *for any* $S \subseteq V$*.*

Proof Since $\Sigma(X[S]) \simeq \mathrm{Ind}(B_S(X[S]))$ by Theorem 11, the result follows from Lemma 12. □

Theorem 14 *The inequality* $\mathrm{L}(X) + 1 \leqslant \mathrm{L}(\mathrm{Ind}(B(X)))$ *holds for any simplicial complex* X*.*

Proof Suppose that $\mathrm{L}(X) = n$. If $S \subseteq V$ is a subset such that $\widetilde{H}_{n-1}(X[S]) \neq 0$, then we have $\widetilde{H}_n(\mathrm{Ind}(B_S(X))) \neq 0$ by Corollary 13 so that $\mathrm{L}(\mathrm{Ind}(B(X))) \geqslant n+1$. □

We remark that the gap between $\mathrm{L}(\mathrm{Ind}(B(X)))$ and $\mathrm{L}(X)$ could be arbitrarily large. For instance, let $W(K_n)$ be the *whisker* of the complete graph K_n for some $n \geqslant 3$, which is obtained by attaching a pendant vertex to each vertex of K_n. Then, it follows that $\mathrm{L}(\mathrm{Ind}(B(X(n))) \geqslant n$ and $\mathrm{L}(X(n)) = 1$, where $X(n) := \mathrm{Ind}(W(K_n))$. Indeed, the bipartite graph $B(X(n))$ satisfies that $\mathrm{im}(B(X_n))) \geqslant n$, while $W(K_n)$ is a cochordal graph.

The relation between a simplicial complex X and its bipartite graph $B(X)$ can be reversible over the class of Sperner bipartite graphs.

Proposition 15 *Let $B = (V_0, V_1; E)$ be a Sperner bipartite graph. The family*

$$X_i(B) := \{F \subseteq V_i : F \subseteq V_i \setminus N_B(v) \text{ for some } v \in V_{1-i}\}$$

is a simplicial complex on V_i for $i = 0, 1$. In particular,

$$\max\{\mathrm{L}(X_0(B)), \mathrm{L}(X_1(B))\} + 1 \leqslant \mathrm{L}(\mathrm{Ind}(B)).$$

Proof The first claim is obvious. For the second, observe that the facets of $X_i(B)$ are of the form $V_i \setminus N_B(v)$ for $v \in V_{1-i}$; hence, we have $B(X_i(B)) \cong B$ so that the stated inequality follows from Theorem 14. □

We are ready to complete the proof of Theorem 3.

Proof of Theorem 3 For the noncover complex $\mathrm{NC}(\mathcal{H})$, we have $B(\mathrm{NC}(\mathcal{H})) \cong \mathrm{Inci}(\mathcal{H})$, since any facet of $\mathrm{NC}(\mathcal{H})$ is of the form $V \setminus e$ for some $e \in \mathcal{H}$. Thus, the claim follows from Theorem 14. □

3 Proof of Theorem 4 and its consequences

This section is devoted to the proof of Theorem 4. We begin with recalling some necessary terminology. Let G be a graph. A *matching* in G is a subset M of edges no two of which share a vertex. An *induced matching* is a matching M if no two vertices belonging to different edges of M are adjacent. The maximum size of an induced matching of G is known as the *induced matching number* $\mathrm{im}(G)$ of G.

Proposition 16 $\mathrm{im}(\mathrm{Inci}(\mathcal{H})) + \gamma_s(\mathcal{H}) = n$ *for every n-vertex hypergraph $\mathcal{H}$ without any isolated vertex.*

Proof Let $M = \{(a_1, e_1), \ldots, (a_k, e_k)\}$ be an induced matching in $\mathrm{Inci}(\mathcal{H})$. We claim that the set $D(M) := V \setminus \{a_1, \ldots, a_k\}$ is a strongly dominating set for $\mathcal{H}$. Indeed, if $a_i \in V \setminus D(M) = \{a_1, \ldots, a_k\}$, then $a_i \notin e_i \setminus \{a_i\} \subseteq D(M)$ and $(e_i \setminus \{a_i\}) \cup \{a_i\} \in \mathcal{H}$. This proves that $\mathrm{im}(\mathrm{Inci}(\mathcal{H})) \leqslant n - \gamma_s(\mathcal{H})$.

For the converse, let $D \subseteq V$ be a strongly dominating set for which $\gamma_s(\mathcal{H}) = |D|$. By the definition, for every $x \in V \setminus D$, there exists a subset $A_x \subseteq D \setminus \{x\}$ such that

$A_x \cup \{x\} \in \mathcal{H}$. However, it follows that the set $M(D) := \{(x, A_x \cup \{x\}) : x \in V \setminus D\}$ forms an induced matching in $\mathrm{Inci}(\mathcal{H})$. In fact, if $x, y \in V \setminus D$ are two distinct vertices, then $y \notin A_x \subseteq D$ and $x \notin A_y \subseteq D$. This shows that $\mathrm{im}(\mathrm{Inci}(\mathcal{H})) \geqslant n - \gamma_s(\mathcal{H})$, which completes the proof. □

Proposition 17 [6] *Let G be a graph and let $v \in V$ be given. Then*

$$\mathrm{L}(\mathrm{Ind}(G)) \leqslant \max\{\mathrm{L}(\mathrm{Ind}(G - v)), \mathrm{L}(\mathrm{Ind}(G - N_G[v])) + 1\}.$$

Moreover, $\mathrm{L}(\mathrm{Ind}(G))$ *always equals to one of* $\mathrm{L}(\mathrm{Ind}(G-v))$ *or* $\mathrm{L}(\mathrm{Ind}(G-N_G[v]))+1$.

Theorem 18 *If $B = (U, W; E)$ is a bipartite graph, then*

$$\mathrm{L}(\mathrm{Ind}(B)) \leqslant \frac{1}{2}(\mathrm{im}(B) + \min\{|U|, |W|\}).$$

Proof We only prove the inequality $\mathrm{L}(\mathrm{Ind}(B)) \leqslant \frac{1}{2}(\mathrm{im}(B) + |W|)$ from which the general claim follows.

We start with the graph $B_0 := B$ having the bipartition $U_0 := U$ and $W_0 := W$. We introduce a reduction process that creates a bipartite graph B_i at each state $i \geqslant 0$, and associate an integer prc_i to B_i by letting $\mathrm{prc}_0 := 0$ at the bottom.

Now, pick a vertex $u_i \in U_i$ for $i \geqslant 0$ such that $\deg_{B_i}(u_i) \geqslant 2$. Assume first that $\mathrm{L}(\mathrm{Ind}(B_i - u_i)) = \mathrm{L}(\mathrm{Ind}(B_i))$. We define B_{i+1} to be the bipartite graph having the bipartition $V(B_{i+1}) = U_{i+1} \cup W_{i+1}$, where $U_{i+1} := U_i - u_i$, $W_{i+1} := W_i$ together with $\mathrm{prc}_{i+1} := \mathrm{prc}_i$. Assume next that $\mathrm{L}(\mathrm{Ind}(B_i - u_i)) < \mathrm{L}(\mathrm{Ind}(B_i))$. Then, we have $\mathrm{L}(\mathrm{Ind}(B_i)) = \mathrm{L}(\mathrm{Ind}(B_i - N_{B_i}[u_i])) + 1$ by Proposition 17. In such a case, we take B_{i+1} to be the bipartite graph with $U_{i+1} := U_i - u_i$, $W_{i+1} := W_i - N_{B_i}(u_i)$, and set $\mathrm{prc}_{i+1} := \mathrm{prc}_i + 1$. We successively apply this process until there exists a $j \geqslant 0$ such that either one of the sets U_j and W_j is empty or the degrees of vertices in U_j is at most one.

If $U_j = \emptyset$ or $W_j = \emptyset$, then $\mathrm{L}(\mathrm{Ind}(B)) = \mathrm{prc}_j \leqslant \frac{|W|}{2} \leqslant \frac{1}{2}(\mathrm{im}(B) + |W|)$. On the other hand, if $\deg_{B_j}(u) \leqslant 1$ for each vertex $u \in U_j$, then B_j is a collection of stars together with some isolated vertices. It follows that $\mathrm{L}(\mathrm{Ind}(B_j)) = k \leqslant \mathrm{im}(B)$, where k is the number of stars in B_j.

Observe that $\mathrm{L}(\mathrm{Ind}(B)) = \mathrm{prc}_j + k$ and that the inequality $\mathrm{prc}_j \leqslant \frac{1}{2}(|W| - k)$ holds. However, we have

$$\mathrm{L}(\mathrm{Ind}(B)) = \mathrm{prc}_j + k \leqslant \frac{1}{2}(|W| - k) + k = \frac{1}{2}(|W| + k) \leqslant \frac{1}{2}(|W| + \mathrm{im}(B)).$$

□

We are now ready to complete the proof of Theorem 4.

Proof of Theorem 4 Since $\mathrm{im}(\mathrm{Inci}(\mathcal{H})) = n - \gamma_s(\mathcal{H})$ by Proposition 16, it follows from Theorem 3 together with Theorem 18 that

$$\mathrm{L}(\mathrm{NC}(\mathcal{H})) + 1 \leqslant \mathrm{L}(\mathrm{Ind}(\mathrm{Inci}(\mathcal{H}))) \leqslant \frac{1}{2}(\mathrm{im}(\mathrm{Inci}(\mathcal{H})) + n) = \frac{1}{2}(n - \gamma_s(\mathcal{H}) + n) = n - \frac{\gamma_s(\mathcal{H})}{2}.$$

□

3.1 Consequences of Theorem 4

Let $\mathcal{H}$ be a hypergraph on a set V. A cover $T \subseteq V$ of $\mathcal{H}$ is *minimal* if none of its proper subset is a cover for $\mathcal{H}$. The *transversal hypergraph* $\mathcal{H}^t$ of $\mathcal{H}$ is the hypergraph on V whose edges are minimal covers in $\mathcal{H}$ (see for example [2]). Observe that the operation of mapping a hypergraph to its transversal is not always an involution unless it is simple, where a hypergraph $\mathcal{H}$ is *simple* if $e, f \in \mathcal{H}$ with $e \subseteq f$, then $e = f$. It is obvious that $(\mathcal{H}^t)^t = \mathcal{H}$ and $\mathrm{NC}(\mathcal{H}) = \mathrm{Ind}(\mathcal{H}^t)$ for every simple hypergraph $\mathcal{H}$; hence, the following is immediate by Theorem 4.

Corollary 19 *The independence complex of an n-vertex simple hypergraph $\mathcal{H}$ without any isolated vertex is $(n - \frac{\gamma_s(\mathcal{H}^t)}{2} - 1)$-Leray.*

Lerayness of a simplicial complex turns out to be a key property for the existence of rainbow sets. Recall that if $R_1, \ldots, R_k$ are subsets of a set V each having a property P, a *rainbow set* is a subset of the form $R = \{r_{i_1}, \ldots, r_{i_l}\} \subseteq V$ consisting of l distinct elements with $1 \leqslant i_1 < \cdots < i_l \leqslant k$ such that R has the property P and $r_{i_j} \in R_{i_j}$ for each $j \in [l]$. In our case, the property that we consider is to be a cover (transversal) of a hypergraph.

The following is a special case of the celebrated topological and matroidal extension of the colorful Helly theorem due to Kalai and Meshulam [11].

Theorem 20 (Topological colorful Helly theorem) *If X is a d-Leray simplicial complex, every collection of $d + 1$ sets in $X' := \{T \subset V : T \notin X\}$ has a rainbow set contained in X'.*

By Theorem 4 together with Theorem 20, the following is immediate.

Corollary 21 *If $\mathcal{H}$ is an n-vertex hypergraph without any isolated vertex, then for every $(n - \lceil \frac{\gamma_s(\mathcal{H})}{2} \rceil)$-cover of $\mathcal{H}$, there exists a rainbow cover.*

Leray number of noncover complexes of hypergraphs is also closely related to projective dimension of their independence complexes. Recall that if X is a simplicial complex on a set V, the *projective dimension* $\text{proj-dim}(X)$ of X is defined to the least integer i such that

$$\widetilde{H}_{|S|-i-j-1}(X[S]) = 0$$

for all $j > 0$ and $S \subseteq V$. In particular, $\mathrm{L}(X^\vee) + 1 = \text{proj-dim}(X)$ by the Terai's duality [16]. In the case of hypergraphs, this means that $\mathrm{L}(\mathrm{NC}(\mathcal{H}))+1 = \text{proj-dim}(\mathrm{Ind}(\mathcal{H}))$ for every hypergraph $\mathcal{H}$. Therefore, Theorem 4 translates into the following.

Corollary 22 $\mathrm{proj\text{-}dim}(\mathrm{Ind}(\mathcal{H})) \leqslant n - \frac{\gamma_s(\mathcal{H})}{2}$ *for every n-vertex hypergraph without any isolated vertex.*

Let $\mathcal{H}$ be a simple hypergraph on V containing no isolated vertex. We define a simplicial complex $X(\mathcal{H})$ on V by

$$X(\mathcal{H}) := \{F \subseteq V : F \subseteq f \text{ for some } f \in \mathcal{H}\}.$$

Fact 23 $X(\mathcal{H}) \cong \mathrm{NC}(\mathcal{H}^c)$ *for every simple hypergraph $\mathcal{H}$ without any isolated vertex.*

Proof Let $A \in \mathrm{NC}(\mathcal{H}^c)$ be given. Since A is not a cover in $\mathcal{H}^c$, there exists an edge $e \in \mathcal{H}^c$ such that $A \cap e = \emptyset$. In other words, $A \subseteq V \setminus e \in \mathcal{H}$, i.e., $A \in X(\mathcal{H})$.

For the converse, let $B \in X(\mathcal{H})$ be a face. Then, there exists an edge $f \in \mathcal{H}$ such that $B \subseteq f$. However, it follows that $\mathcal{H}^c \ni V \setminus f \subseteq V \setminus B$; hence, $B \cap (V \setminus f) = \emptyset$, that is, $B \in \mathrm{NC}(\mathcal{H}^c)$. □

Recall that the hypergraph $\mathcal{F}(X)$ of all facets of a simplicial complex X forms a simple hypergraph with $X = X(\mathcal{F}(X))$. Therefore, the following is immediate from Theorem 4 and Fact 23.

Corollary 24 $\mathrm{L}(X) \leqslant n - \frac{\gamma_s(\mathcal{F}(X)^c)}{2} - 1$ *for every simplicial complex X with n-vertices.*

We end this subsection by another application of Theorem 4 combined with Proposition 15.

Let $\mathcal{H} = \{e_1, \ldots, e_m\}$ be a simple hypergraph on V without any isolated vertex. Then, the *nerve complex* of $\mathcal{H}$ is the simplicial complex on $[m]$ defined by

$$\mathcal{N}(\mathcal{H}) := \{\sigma \subseteq [m] : \bigcap_{i \in \sigma} e_i \neq \emptyset\}.$$

Moreover, a hypergraph $\mathcal{H}$ is said to be a *separating hypergraph* if for every pair $\{u_0, u_1\}$ of vertices in V, there exist edges e_0 and e_1 such that $u_i \in e_i$ and $u_{1-i} \notin e_i$ for $i \in \{0, 1\}$.

Corollary 25 *If $\mathcal{H}$ is a simple and separating n-vertex hypergraph without any isolated vertex, then*

$$\max\{\mathrm{L}(\mathcal{N}(\mathcal{H})), \mathrm{L}(X(\mathcal{H}))\} + 1 \leqslant \mathrm{L}(\mathrm{Ind}(\mathrm{Inci}(\mathcal{H}^c))).$$

In particular,

$$\mathrm{L}(\mathcal{N}(\mathcal{H})) \leqslant n - \frac{\gamma_s(\mathcal{H}^c)}{2} - 1.$$

Proof Since $\mathcal{H}$ is simple and separating, the bipartite graph $\mathrm{Inci}(\mathcal{H}^c)$ is Sperner. Furthermore, if we write $V_0 := V$ and $V_1 := \mathcal{H}^c$ so that $V(\mathrm{Inci}(\mathcal{H}^c)) = V_0 \cup V_1$, it follows that $X_0(\mathrm{Inci}(\mathcal{H}^c)) \cong X(\mathcal{H})$ and $X_1(\mathrm{Inci}(\mathcal{H}^c)) \cong \mathcal{N}(\mathcal{H})$. Therefore, the claims follow from Proposition 15 together with the fact that $\mathrm{L}(\mathrm{Ind}(\mathrm{Inci}(\mathcal{H}^c))) \leqslant n - \frac{\gamma_s(\mathcal{H}^c)}{2}$ by the proof of Theorem 4. □

4 Proof of Theorem 5

In this section, we provide the proof of Theorem 5. As we mention earlier, the proof will be completed by Theorem 6 once we prove Lemma 8 whose proof will be divided into several steps.

Let $\mathcal{H}$ be a hypergraph without any isolated vertex. We call a subhypergraph $\mathcal{G} \subseteq \mathcal{H}$ an *edge-cover* if $\bigcup \mathcal{G} = V$. An edge-cover is *minimal* if none of its proper subset is an edge-cover for $\mathcal{H}$. We call

$$\epsilon(\mathcal{H}) := \max\{|\mathcal{G}| \colon \mathcal{G} \text{ is a minimal edge-cover for } \mathcal{H}\}$$

the *edge-covering number* of $\mathcal{H}$. If $\mathcal{H}$ contains an isolated vertex, we set $\epsilon(\mathcal{H}) := \infty$.

Proposition 26 *$\mathfrak{h}(\mathcal{H}) = \epsilon(\mathcal{H}^c)$ for every hypergraph $\mathcal{H}$.*

Proof Observe first that $\mathcal{H}$ is intersecting if and only if $\mathcal{H}^c$ contains an isolated vertex. So, assume that $\mathfrak{h}(\mathcal{H}) = h < \infty$ and let $\mathcal{Y} = \{Y_1, \ldots, Y_h\} \subseteq \mathcal{H}$ be a subhypergraph witnessing $\mathfrak{h}(\mathcal{H})$. Observe that the family $\{V \setminus Y_i \colon 1 \leqslant i \leqslant h\}$ is an edge-cover for $\mathcal{H}^c$; otherwise, $\mathcal{H}$ must be intersecting. Furthermore, for each $i \in [h]$, there exists a vertex $x_i \in \bigcap_{\substack{j=1 \\ i \neq j}}^{h} Y_j$ such that $x_i \notin Y_i$. In other words, there exists a vertex x_i such that $x_i \in V \setminus Y_j$ if and only if $i = j$. This shows that the constructed edge-cover is minimal; hence, $\mathfrak{h}(\mathcal{H}) \leqslant \epsilon(\mathcal{H}^c)$.

For the converse, suppose that $\epsilon(\mathcal{H}^c) = k < \infty$, and let $\mathcal{A} = \{A_1, \ldots, A_k\} \subseteq \mathcal{H}^c$ be a minimal edge-cover. We claim that $\{V \setminus A_i \colon 1 \leqslant i \leqslant k\}$ is k-Helly. Indeed, we have $\bigcap_{i=1}^{k} V \setminus A_i = \emptyset$ as $\mathcal{A}$ is an edge-cover of $\mathcal{H}^c$. Furthermore, the minimality of $\mathcal{A}$ implies that $\bigcap_{i \in \sigma} V \setminus A_i \neq \emptyset$ for every nonempty subset $\sigma \subsetneq [k]$. This proves that $\epsilon(\mathcal{H}^c) \leqslant \mathfrak{h}(\mathcal{H})$, which completes the proof. □

Lemma 27 *Let $\mathcal{H}$ be a hypergraph without any isolated vertex. Then, there exists a canonical dominating set D of $\mathcal{H}$ of minimum cardinality such that for each $y \in V \setminus D$, there exists a subset $D_y \subset D$ with $D_y \cup \{y\} \in \mathcal{H}$ satisfying the following.*

(i) $\displaystyle\bigcup_{y \in V \setminus D} D_y = D$,

(ii) $\mathcal{G} = \{D_y \cup \{y\} : y \in V \setminus D\}$ *is a minimal edge-cover for $\mathcal{H}$.*

Proof Let D be a minimum canonical dominating set for which the number of edges contained in D is maximum. For each $y \in V \setminus D$, pick an edge $D_y \cup \{y\}$ with $D_y \subseteq D$. We shall first show that (i) holds. For a contradiction, suppose that $D' := \displaystyle\bigcup_{y \in V \setminus D} D_y \subsetneq D$, i.e., $D \setminus D' \neq \emptyset$.

Claim 1. There exists a vertex $x \in D \setminus D'$ for which there is no edge e containing x with $e \subset D$.

Proof of Claim 1: Assume that the claim is wrong so that for every vertex $w \in D \setminus D'$, there exists an edge $e_w \in \mathcal{H}$ with $w \in e_w$ such that $e_w \subset D$. Pick a vertex $x \in D \setminus D'$ and consider the set $D \setminus \{x\}$. Notice that any vertex in $V \setminus D$ is canonically dominated

by $D \setminus \{x\}$, as D is a canonical dominating set and $x \notin D'$. Moreover, since the edge e_x satisfies $|e_x \cap (V \setminus (D \setminus \{x\})| \leqslant 1$, the vertex x is itself also canonically dominated by $D \setminus \{x\}$. Furthermore, suppose that there exists a vertex $t \in D \setminus D'$ with $t \neq x$. The existence of the edge e_t with $t \in e_t$ and $e_t \subset D$ implies that $|e_t \cap (V \setminus (D \setminus \{x\})| \leqslant 1$. Therefore, every vertex in $D \setminus D'$ is also canonically dominated by the set $D \setminus \{x\}$. In other words, $D \setminus \{x\}$ would be a canonical dominating set for $\mathcal{H}$ with $|D \setminus \{x\}| < |D|$, a contradiction to the choice of D. This completes the proof of Claim 1.

Now, let $R \subseteq D \setminus D'$ be the subset consisting of $x \in D \setminus D'$ such that there exists no edge f with $x \in f$ and $f \subset D$. By Claim 1, we know that $R \neq \emptyset$. Let $\mathcal{R} \subseteq \mathcal{H}$ be a subset of edges such that for each $x \in R$, there exists an edge $e \in \mathcal{R}$ with $x \in e$ and $|e \cap (V \setminus D)| = 1$. In particular, denote by $\mathcal{R}_x$, the subset consisting of those edges $e \in \mathcal{R}$ with $x \in e$ and $|e \cap (V \setminus D)| = 1$. Since D is a canonical dominating set, we have that $\mathcal{R}_x \neq \emptyset$ for every $x \in R$. We next introduce an equivalence relation on R by $x \sim y$ if and only if $\mathcal{R}_x = \mathcal{R}_y$. We then order the equivalence classes by $[z] \leqslant [w]$ if and only if $\mathcal{R}_w \subset \mathcal{R}_z$. Furthermore, let $x \in R$ be a vertex such that $[x]$ is a maximal element with respect to the given order, and fix an edge $e_x \in \mathcal{R}_x$. If u is the unique vertex in $e_x \cap (V \setminus D)$, we claim that $T := D \setminus \{x\} \cup \{u\}$ is a minimum canonical dominating set of $\mathcal{H}$.

Indeed, if $y \in R$ is a vertex with $x \sim y$ or $x \neq y$ and $[y] < [x]$, then for the edge $e_x \in \mathcal{R}_x \subseteq \mathcal{R}_y$, we have that $y \in e_x$ and $e_x \cap (V \setminus T) = \{x\}$. Thus, such a vertex y is canonically dominated by T. We note that the edge e_x also witnesses the canonical domination of the vertex u by T. If $t \in R \setminus \{x\}$ is a vertex which is neither equivalent to x nor $[t]$ is comparable to $[x]$, then any edge $e_t \in \mathcal{R}_t \setminus \mathcal{R}_x$ satisfies $t \in e_t$ and $|e_t \cap (V \setminus T)| \leqslant 1$. If $w \in D \setminus D'$ is a vertex with $w \notin R$, then there exists an edge e_w with $e_w \subset D$ so that $|e_w \cap (V \setminus T)| \leqslant 1$. We next consider vertices contained in D', and let $v \in D'$ be such a vertex. By the definition of the set D', there exists a vertex $s \in V \setminus D$ such that $v \in D_s$. We then conclude that the edge $e_s := D_s \cup \{s\}$ satisfies $v \in e_s$ and $|e_s \cap (V \setminus T)| \leqslant 1$, since $x \notin D_s \subseteq D'$. Overall, these imply that any vertex in $T \cup \{x\}$ is canonically dominated by T.

Finally, let $z \in V \setminus T$ be a vertex with $z \neq x$ so that $z \notin D$. Then for the edge $e_z = D_z \cup \{z\}$ with $D_z \subseteq D$, we have that $e_z \cap (V \setminus T) = \{z\}$, since $x \notin D_z \subseteq D'$.

Therefore, T is a canonical dominating set for $\mathcal{H}$ with $|T| = |D|$ as claimed. However, T contains at least one more edge (namely the edge $D_u \cup \{u\}$) than D, contradicting the choice of D. This completes the proof of (i).

We next show (ii) holds. By (i), the family $\mathcal{G}$ is an edge-cover of $\mathcal{H}$. The minimality follows since for an arbitrary edge $e_y := D_y \cup \{y\}$, the family $\mathcal{G} \setminus \{e_y\}$ does not cover the vertex y. □

Lemma 27 gives the following immediate corollary as $|\mathcal{G}| = |V \setminus D|$.

Corollary 28 *$\epsilon(\mathcal{H}) + \gamma_c(\mathcal{H}) \geqslant n$ for every n-vertex hypergraph $\mathcal{H}$ without any isolated vertex.*

Proposition 29 *$\epsilon(\mathcal{H}) + \gamma_c(\mathcal{H}) = n$ for every n-vertex hypergraph $\mathcal{H}$ without any isolated vertex.*

Proof Let $\mathcal{H}$ be a hypergraph and $\mathcal{Y}$ be a minimal edge-cover of $\mathcal{H}$ for which $|\mathcal{Y}|$ is maximum. Consider the hypergraph $\mathcal{Y}$ on V. The minimality of $\mathcal{Y}$ implies that every

edge in $\mathcal{Y}$ contains at least one vertex of degree one in $\mathcal{Y}$. For each edge $e \in \mathcal{Y}$, pick a degree one vertex $u_e \in e$ and put all these vertices in a set U. We claim that the set $D = V \setminus U$ is a canonical dominating set. Indeed, let $v \in V$ be given. Then, there exists an edge $e_v \in \mathcal{Y}$ with $v \in e_v$ since $\mathcal{Y}$ is a cover. The fact that every edge in $\mathcal{Y}$ contains exactly one vertex in U implies that $|e_v \cap U| = 1$. This proves that D is a canonical dominating set; hence, we conclude that

$$|D| + |\mathcal{Y}| = |V \setminus U| + |U| = |V| = n.$$

Since $\gamma_c(\mathcal{H}) \leqslant |D|$ and $|\mathcal{Y}| = \epsilon(\mathcal{H})$, this gives that

$$\gamma_c(\mathcal{H}) + \epsilon(\mathcal{H}) \leqslant n.$$

Now, the claim follows from Corollary 28. □

Finally, the proof of Lemma 8 is a direct consequence of Propositions 26 and 29 together with the fact that $(\mathcal{H}^c)^c = \mathcal{H}$. This together with Theorem 6 in turn completes the proof of Theorem 5, since $\gamma_c(\mathcal{H}) \leqslant \gamma_{st}(\mathcal{H})$ for every hypergraph $\mathcal{H}$.

5 Examples

In this section, we provide various examples in order to compare relevant domination parameters of hypergraphs. Beforehand, we state the following easy fact.

Fact 30 *Let $\mathcal{H}_i$ be a hypergraph on a set V_i without any isolated vertex for each $1 \leqslant i \leqslant k$ such that the sets $V_1, V_2, \ldots, V_k$ are pairwise disjoint. Then*

$$\mu(\bigcup_{i=1}^{k} \mathcal{H}_i) = \sum_{i=1}^{k} \mu(\mathcal{H}_i)$$

for every $\mu \in \{\gamma_s, \gamma_c, \gamma_{st}\}$.

We recall that the *independent domination number* $i(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the minimum size of a maximal independent set in $\mathcal{H}$. In particular, it provides a lower bound to Leray number of the noncover complex of $\mathcal{H}$, that is, the inequality $|V| - i(\mathcal{H}) - 1 \leqslant \mathrm{L}(\mathrm{NC}(\mathcal{H}))$ always holds. Indeed, if $F \in \mathcal{F}(\mathrm{Ind}(\mathcal{H}))$ is a facet, then $\mathrm{NC}(\mathcal{H})[V \setminus F]$ is isomorphic to the boundary of the $(|V| - |F| - 1)$-dimensional simplex. Therefore, $\widetilde{H}_{|V|-|F|-2}(\mathrm{NC}(\mathcal{H})[V \setminus F]) \neq 0$; hence, $|V| - |F| - 1 \leqslant \mathrm{L}(\mathrm{NC}(\mathcal{H}))$. As a consequence, the inequality $\gamma_s(\mathcal{H}) \leqslant 2i(\mathcal{H})$ holds for every hypergraph $\mathcal{H}$ without any isolated vertex.

Remark 31 We cannot replace the strong domination number in Theorem 4 neither by the canonical domination number nor by the strong total domination number. For the latter, Kim and Kim provide the following example that also works for the canonical domination (see also Example 34).

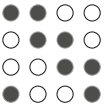


Fig. 1 The set D_4 (black vertices) in Example 33

Example 32 *[12, Example 4.3]* Let $\mathcal{H}_r$ be the hypergraph on $V_r = \{v_1, v_2, \ldots, v_{2r+1}\}$, whose edges are

$$e_1 := \{v_1, \ldots, v_r\}, \quad e_t := \{v_t, v_{r+1}\} \quad \text{and} \quad e_{2r+1} := \{v_{r+2}, v_{r+3}, \ldots, v_{2r+1}\}$$

for each $t \in \{2, 3, \ldots, 2r\} \setminus \{r+1\}$. Now, the set $\{v_1, v_{r+1}, v_{2r+1}\}$ is both a strongly dominating set and a maximal independent set of minimum size in $\mathcal{H}_r$. In other words, we have that $\gamma_s(\mathcal{H}_r) = i(\mathcal{H}_r) = 3$. It follows that

$$2r - 3 \leqslant \mathrm{L}(\mathrm{NC}(\mathcal{H}_r)) \leqslant 2r - 2$$

by Theorem 4. On the other hand, $D_r := \{v_2, v_3, \ldots, v_{2r}\} \setminus \{v_{r+1}\}$ is a canonical dominating set of minimum order so that $\gamma_c(\mathcal{H}_r) = 2r - 2 < 2r - 1 = \gamma_{st}(\mathcal{H}_r)$. Thus, we conclude that

$$(2r+1) - \frac{\gamma_c(\mathcal{H}_r)}{2} - 1 = r + 1 < 2r - 3 \leqslant \mathrm{L}(\mathrm{NC}(\mathcal{H}_r))$$

for every $r \geqslant 5$. This proves that $\mathrm{NC}(\mathcal{H}_r)$ cannot be $((2r+1) - \frac{\gamma_c(\mathcal{H}_r)}{2} - 1)$-Leray whenever $r \geqslant 5$.

Example 32 also shows that the gap among strong total, canonical and strong domination numbers could be arbitrarily large. Now, let $\mathcal{H}(r)$ be the disjoint union of r copies of the hypergraph $\mathcal{H}_r$ given in Example 32 for some $r \geqslant 2$. Then $\gamma_s(\mathcal{H}(r)) = 3r$, $\gamma_c(\mathcal{H}(r)) = 2r(r-2)$ and $\gamma_{st}(\mathcal{H}(r)) = 2r(r-1)$ by Fact 30.

Our next example shows the possibility that all these domination parameters could be equal.

Example 33 *(Non-attacking rooks)* Let $n \geqslant 4$ be an integer, and consider the set $[n] \times [n]$ as an n by n chessboard. We define $\mathcal{R}_n$ to be the hypergraph on $[n] \times [n]$ whose edges correspond to placements of n rooks on the board none of which attacks others. We claim that $\gamma_s(\mathcal{R}_n) = \gamma_c(\mathcal{R}_n) = \gamma_{st}(\mathcal{R}_n) = 2n$.

We let

$$D_n := \{(i,i) \colon 1 \leqslant i \leqslant n\} \cup \{(i, i+1) \colon 1 \leqslant i \leqslant n-1\} \cup \{(n, 1)\},$$

and verify first that D_n is a strongly totally dominating set in $\mathcal{R}_n$ (see Fig. 1 for $n = 4$).

Let (i, j) be a vertex. Consider the subboard obtained from the original chessboard by removing the i^{th}-row and j^{th}-column, which is an $(n-1)$ by $(n-1)$ board. We may consider the hypergraph $\mathcal{R}_{n-1}$ on that board after possibly relabeling. Let D_n^{ij} denote the set of vertices of D_n contained in this subboard. Note that $|D_n \setminus D_n^{ij}| \leqslant 4$

and D_n contains exactly two vertices in each row and each column of the original board. Since, we have one less row and column, we can choose from vertices of D_n^{ij} in order to form an edge $e \in \mathcal{R}_{n-1}$. However, this means that D_n strongly dominates (i, j) as $e \cup \{(i, j)\} \in \mathcal{R}_n$.

We next show that there is no strongly dominating set of $\mathcal{R}_n$ of size less than $2n$. For a contradiction suppose D' is a strongly dominating set of $\mathcal{R}_n$ with $|D'| < 2n$. Then by the pigeonhole principle, the chessboard has a row and a column, say i^{th}-row and j^{th}-column such that D' contains at most one vertex of i^{th}-row and j^{th}-column. Indeed, suppose D' does not contain any vertex in the i^{th}-row. Then D' must contain all vertices in other rows. To see this let us assume $(i', j') \notin D'$ with $i' \neq i$. The fact that every edge in $\mathcal{R}_n$ has exactly one vertex in each row implies that D' cannot strongly dominate (i', j') since D' has no vertex in the i^{th}-row with $i' \neq i$. This gives $|D'| = n(n-1)$, a contradiction since $|D'| < 2n$. Thus D' contains exactly one vertex in the i^{th}-row. We can use similar arguments to deduce that D' contains exactly one vertex in the j^{th}-column. Let (i, k) and (l, j) with $k, l \in [n]$ be the unique vertices of D' in the i^{th}-row and j^{th}-column, respectively.

Consider a vertex (h, k) with $h \neq i$. The facts that every edge in $\mathcal{R}_n$ has exactly one vertex in each row and each column, and (i, k) and (h, k) belong to the same column imply that D' cannot strongly dominate (h, k). Thus we have $(h, k) \in D'$ for all $h \in [n]$, that is, all vertices in the k^{th}-column are contained in D'. Similarly, D' contains all vertices of the l^{th}-row. In total there are $2n - 1$ vertices in D' by this argument. Let us assume there is no other vertex in D'. Then for an arbitrary edge $e \in \mathcal{R}_n$, we have $|e \cap D'| \leqslant 2$, as every edge in $\mathcal{R}_n$ contains exactly one vertex in each row and each column. Thus, D' cannot strongly dominate the vertices in $V \setminus D'$ since $n \geqslant 4$ and $\mathcal{R}_n$ is n-uniform. This gives us the desired contradiction; hence, $\gamma_s(\mathcal{R}_n) \geqslant 2n$ as claimed.

Finally, since D_n is a strongly totally dominating set for $\mathcal{R}_n$ and

$$2n \leqslant \gamma_s(\mathcal{R}_n) \leqslant \gamma_c(\mathcal{R}_n) \leqslant \gamma_{st}(\mathcal{R}_n) \leqslant |D_n| = 2n,$$

we conclude that $\gamma_{st}(\mathcal{R}_n) = \gamma_c(\mathcal{R}_n) = \gamma_s(\mathcal{R}_n) = 2n$.

5.1 Strong independence domination number

There is another domination parameter introduced by Kim and Kim [12] that we have not yet considered. Let $\mathcal{H}$ be a hypergraph without any isolated vertex. A subset $D \subseteq V$ *strongly dominates* a set $A \subseteq V$ if D strongly dominates every vertex in A. The least cardinality of a strongly dominating set of A in $\mathcal{H}$ is denoted by $\gamma_s(\mathcal{H}; A)$. Moreover, an independent set $I \subseteq V$ is called *strongly independent* if $|e \cap I| \leqslant 1$ for each $e \in \mathcal{H}$. Then the *strong independence domination number* $\gamma_{si}(\mathcal{H})$ is defined to be the maximum of $\gamma_s(\mathcal{H}; I)$ where I ranges over all strongly independent sets of $\mathcal{H}$. For graphs, the strong independence domination number $\gamma_{si}(G)$ coincides with the independence domination number $\gamma^i(G)$ introduced by Aharoni [1].

In general, the strong domination and strong independence domination numbers are incomparable. There are graphs G_k with $\gamma_{si}(G_k) + k \leqslant \gamma_s(G_k)$ for every $k \geqslant$

1 (compare to [1]). We also provide examples of hypergraphs satisfying such an inequality in Example 39.

Example 34 For every integer $k \geqslant 3$, consider the hypergraph $\mathcal{G}_k$ on $V_k := \{x_1, \ldots, x_k\}$ whose edges are

$$\{x_1, x_2, \ldots, x_k\} \quad \text{and} \quad \{x_j\}$$

for each $2 \leqslant j \leqslant k$. Observe that $\{x_1\}$ is a strongly independent set of $\mathcal{G}_k$ which is uniquely strongly dominated by the set $\{x_2, \ldots, x_k\}$; hence, $\gamma_{si}(\mathcal{G}_k) = k - 1$. On the other hand, since the set $\{x_1\}$ strongly dominates $V_k \setminus \{x_1\}$, we have that $\gamma_s(\mathcal{G}_k) = 1$. Moreover, $\gamma_c(\mathcal{G}_k) = k - 1$ since $\epsilon(\mathcal{G}_k) = 1$. We further note that $\mathrm{L}(\mathrm{NC}(\mathcal{G}_k)) = k - 2$. Indeed, $\{x_1\}$ is a maximal independent so that

$$k - i(\mathcal{G}_k) - 1 = k - 2 \leqslant \mathrm{L}(\mathrm{NC}(\mathcal{G}_k)) \leqslant k - 2,$$

which shows that $\mathrm{L}(\mathrm{NC}(\mathcal{G}_k)) = k - 2$.

A class of hypergraphs introduced by Kim and Kim [12] also shows that the gap $\gamma_{si}(\mathcal{H}) - \gamma_s(\mathcal{H})$ could be arbitrarily large.

Example 35 *[12, Example* 4.5*]* Consider a $k \times k$ matrix $M = [a_i^j]$ for each $k \geqslant 3$, whose entries are all distinct. Let $\mathcal{M}_k$ be the hypergraph on $V_k = \{a_i^j : 1 \leqslant i, j \leqslant k\}$ whose edges correspond to columns except the first one and rows of M, that is, edges of $\mathcal{M}_k$ are

$$C_j := \{a_1^j, a_2^j, \ldots, a_k^j\} \quad \text{and} \quad R_i := \{a_i^1, a_i^2, \ldots, a_i^k\}$$

for $1 < j \leqslant k$ and $1 \leqslant i \leqslant k$. Since $\mathrm{im}(\mathrm{Inci}(\mathcal{M}_k)) = 2k - 2$, we conclude that $\gamma_s(\mathcal{M}_k) = k^2 - 2k + 2$ by Proposition 16. For the induced matching number of the incidence bipartite graph $\mathrm{Inci}(\mathcal{M}_k)$, we note that the family

$$\{(a_k^j, C_j) : 2 \leqslant j \leqslant k\} \cup \{(a_i^1, R_i) : 1 \leqslant i \leqslant k - 1\}$$

forms an induced matching of size $2k - 2$. On the other hand, $\gamma_{si}(\mathcal{M}_k) \geqslant k(k-1)$, since $C_1 = \{a_1^1, a_2^1, \ldots, a_k^1\}$ is a strongly independent set in $\mathcal{M}_k$ and $D_1 := \{a_i^j : 1 \leqslant i \leqslant k \text{ and } 2 \leqslant j \leqslant k\}$ is the only set that strongly dominates C_1. As a consequence, we conclude that

$$\gamma_s(\mathcal{M}_k) = (k-1)^2 + 1 < k(k-1) \leqslant \gamma_{si}(\mathcal{M}_k)$$

since $k \geqslant 3$.

5.2 Closed neighborhood hypergraphs of graphs

We end this section with a brief discussion on the relevant domination numbers of hypergraphs associated to graphs.

Let $G = (V, E)$ be a graph without any isolated vertex. Then, the *closed neighborhood hypergraph* of G is defined by

$$\mathcal{N}[G] := \{N_G[x] \colon x \in V\}$$

We remark that the independence complex of the hypergraph $\mathcal{N}[G]$ is known as the *dominance complex* of G. In particular, it follows from [13, Theorem 1] that $\nu(G) \geqslant \eta(\mathrm{Ind}(\mathcal{N}[G]))$ for every graph G, where $\nu(G)$ is the *vertex cover number* $\nu(G)$ of G, the smallest cardinality of a cover of G. Therefore, $\nu(G) \geqslant \eta(\mathrm{Ind}(\mathcal{N}[G])) \geqslant \frac{n-\mathfrak{h}(\mathcal{N}[G]^c)}{2}$ by Theorem 5. However, we note that the bound $\nu(G) \geqslant \frac{n-\mathfrak{h}(\mathcal{N}[G]^c)}{2}$ can be directly proven without appealing to any algebraic/topological methods.

In order to determine the domination parameters of these hypergraphs, we first recall some terminology.

A subset $D \subseteq V$ is a dominating set of G if $N_G[D] = V$. The *upper domination number* $\Gamma(G)$ is the maximum size of a minimal dominating set D (see [9] for details).

Let $R \subseteq V$ be a subset of vertices. Then R is called an *irredundant set* if

$$N_G[x] - N_G[R \setminus \{x\}] \neq \emptyset$$

for each $x \in R$. Let $\mathrm{IR}(G)$ denote the maximum cardinality of an irredundant set in G. We remark that the inequality $\Gamma(G) \leqslant \mathrm{IR}(G)$ holds for every graph G.

The equalities in the following result can be derived by Propositions 16 and 29. We therefore omit its proof.

Lemma 36 *If G is an n-vertex graph without any isolated vertex, then*

$$\gamma_s(\mathcal{N}[G]) = n - \mathrm{IR}(G) \quad \text{and} \quad \gamma_c(\mathcal{N}[G]) = n - \Gamma(G).$$

The strong independence domination numbers of closed neighborhood hypergraphs are more subtle to determine. However, when the underlying graph is regular with small diameter, the picture is familiar.

Even if we could not completely identify the strong independence domination number of closed neighborhood hypergraphs, we can determine which independent sets are strong in general.

The square G^2 of a graph G is the graph on the same set V of vertices such that two distinct vertices u and v are adjacent in G^2 if and only if their distance in G is at most two. The maximum size $\rho(G)$ of an independent set in G^2 is called the 2-*packing number* of G.

Fact 37 *A subset $S \subseteq V$ is strongly independent in $\mathcal{N}[G]$ if and only if it is an independent set of G^2.*

Recall that a graph G is d-regular if the degree of every vertex in G is d. Moreover, G is said to be a *diameter two* graph if the distance between every pair of vertices in G is at most two.

An interesting consequence of Fact 37 is that $\delta(G) \leqslant \gamma_{si}(\mathcal{N}[G]) \leqslant \Delta(G)$ holds for every diameter two graph G without any isolated vertex and having the maximum

and minimum degrees $\Delta(G)$ and $\delta(G)$. Notice that $\rho(G) = 1$ for such graphs. In other words, every nonempty strongly independent set in $\mathcal{N}[G]$ is of order one.

Let $G = (V, E)$ be a graph. The *suspension* $\Sigma(G)$ of G is the graph on $V(\Sigma(G)) := V \cup \{x, y\}$ with $x, y \notin V$ such that $E(\Sigma(G)) := E \cup \{xv, yv : v \in V\}$. Note that suspension of every graph is of diameter two. Therefore the following is immediate.

Corollary 38 *$\gamma_{si}(\mathcal{N}[G]) = d$ for every d-regular graph G of diameter two. In particular, $\gamma_{si}(\mathcal{N}[\Sigma(H)]) = d + 2$ for every d-regular graph H.*

Taking Lemma 36 and Corollary 38 into account, we can construct graphs such that $\gamma_s(\mathcal{N}[G]) - \gamma_{si}(\mathcal{N}[G])$ is arbitrarily large.

Example 39 Let nK_2 denote the disjoint union of n copies of K_2 for some $n \geqslant 3$. Then, $\gamma_s(\mathcal{N}[\Sigma(nK_2)]) = n + 2$ by Lemma 36. On the other hand, since nK_2 is 1-regular, we conclude that $\gamma_{si}(\mathcal{N}[\Sigma(nK_2)]) = 3$ by Corollary 38.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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